Intelligent Systems (AI-2)

Computer Science cpsc422, Lecture 14

Oct, 11, 2017

Slide credit: some slides adapted from Stuart Russell (Berkeley)

422 big picture: Where are we?

StarAI (statistical relational AI)

Hybrid: Det +Sto

Prob CFG

Prob Relational Models

Markov Logics

Deterministic | Stochastic

Query

Logics

First Order Logics

Ontologies

Temporal rep.

- Full Resolution
- SAT

Belief Nets

Approx. : Gibbs

Markov Chains and HMMs

Forward, Viterbi….

Approx. : Particle Filtering

Undirected Graphical Models

Markov Networks

Conditional Random Fields

Planning

Markov Decision Processes and Partially Observable MDP

- Value Iteration
- Approx. Inference

Reinforcement Learning

Applications of AI

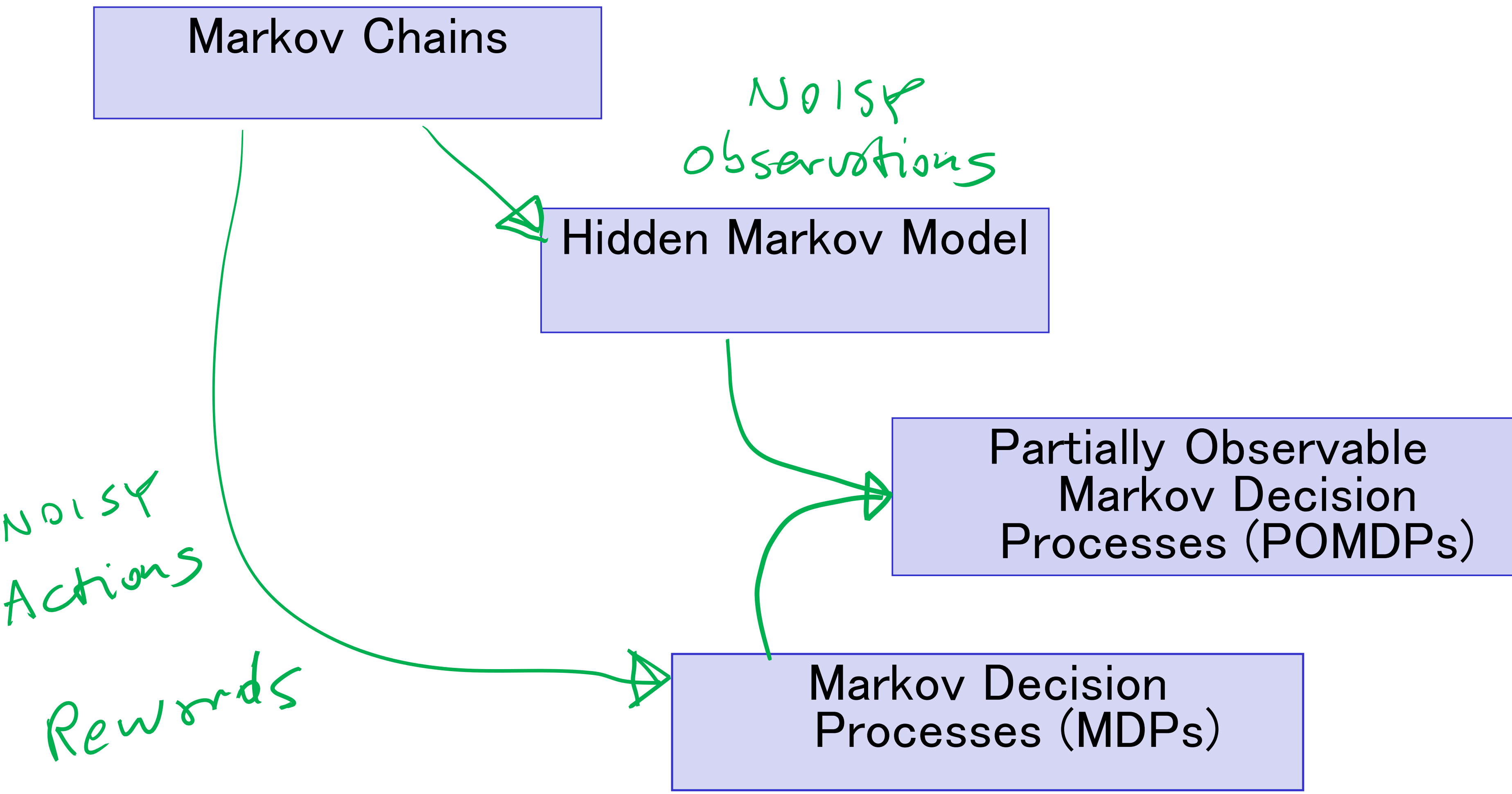
Representation

Reasoning Technique

Lecture Overview (Temporal Inference)

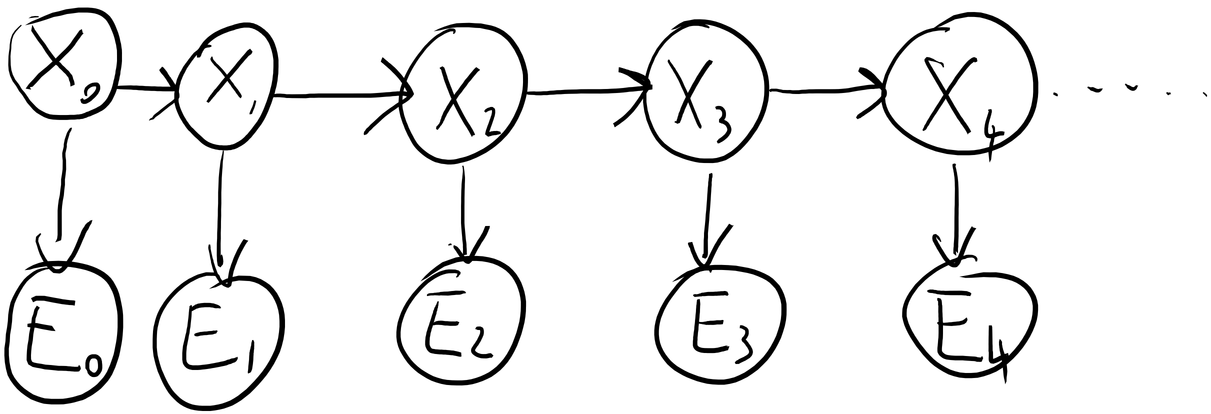
- **Filtering** (posterior distribution over the current state given evidence to date)
  - **From intuitive explanation to formal derivation**
  - **Example**
- **Prediction** (posterior distribution over a future state given evidence to date)
- **(start) Smoothing** (posterior distribution over a *past* state given all evidence to date)

Markov Models



Hidden Markov Model

- A Hidden Markov Model (HMM) starts with a Markov chain, and adds a noisy observation/evidence about the state at each time step:



- $|\text{domain}(X)| = k$
- $|\text{domain}(E)| = h$

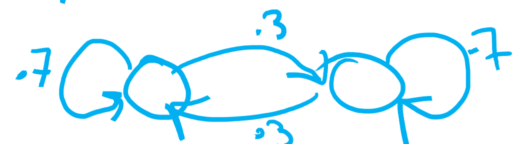
- $P(X_0)$ specifies initial conditions K
- X $P(X_{t+1}|X_t)$ specifies the dynamics K x K
- O $P(E_t|S_t)$ specifies the sensor model K x h $\underbrace{\{\text{K prob. dist. over O}\}}$

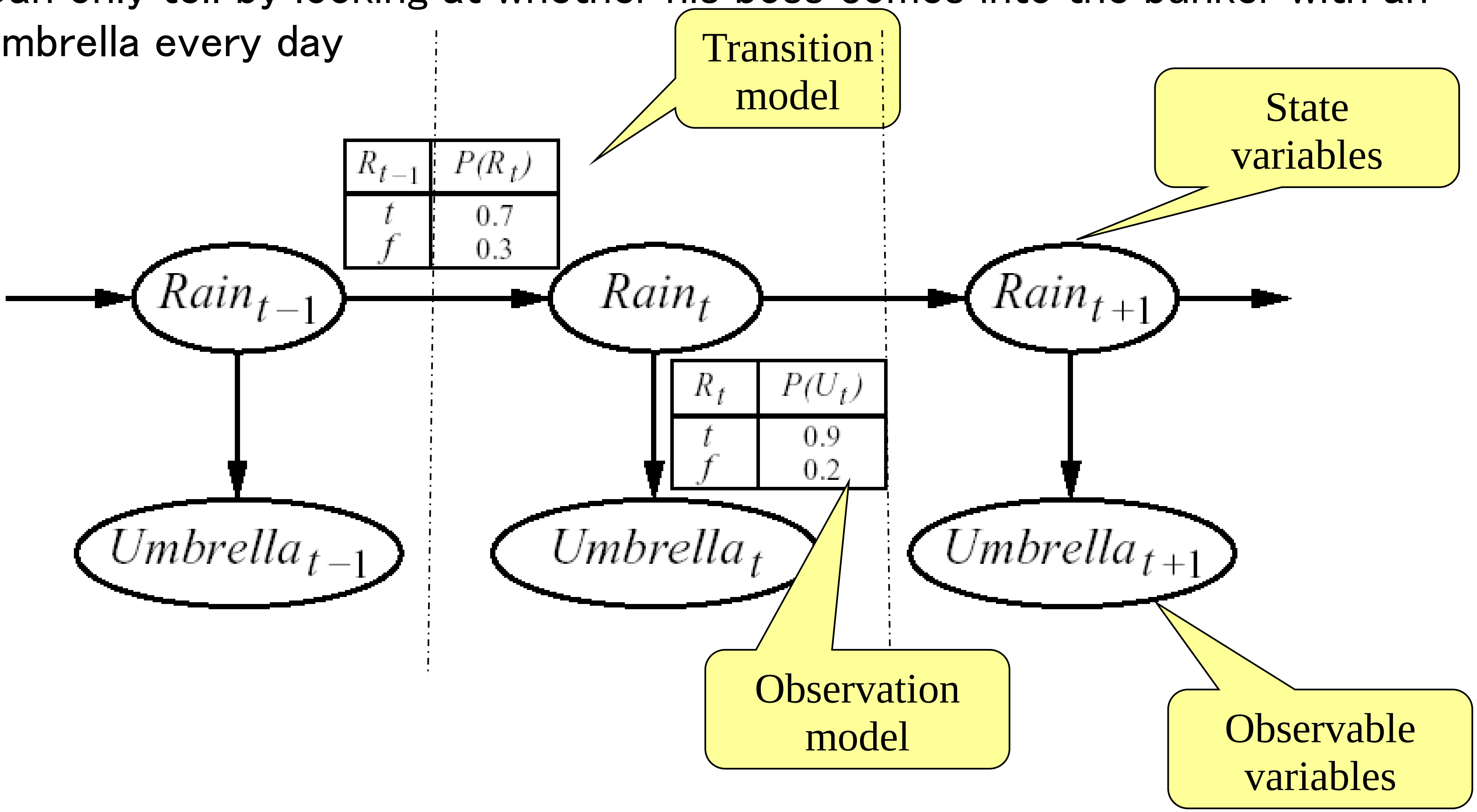
Simple Example

(We'll use this as a running example)

- Guard stuck in a high-security bunker
- Would like to know if it is raining outside
- Can only tell by looking at whether his boss comes into the bunker with an umbrella every day

R_{t-1} \ R_t	T	F
T	.7	.3
F	.3	.7

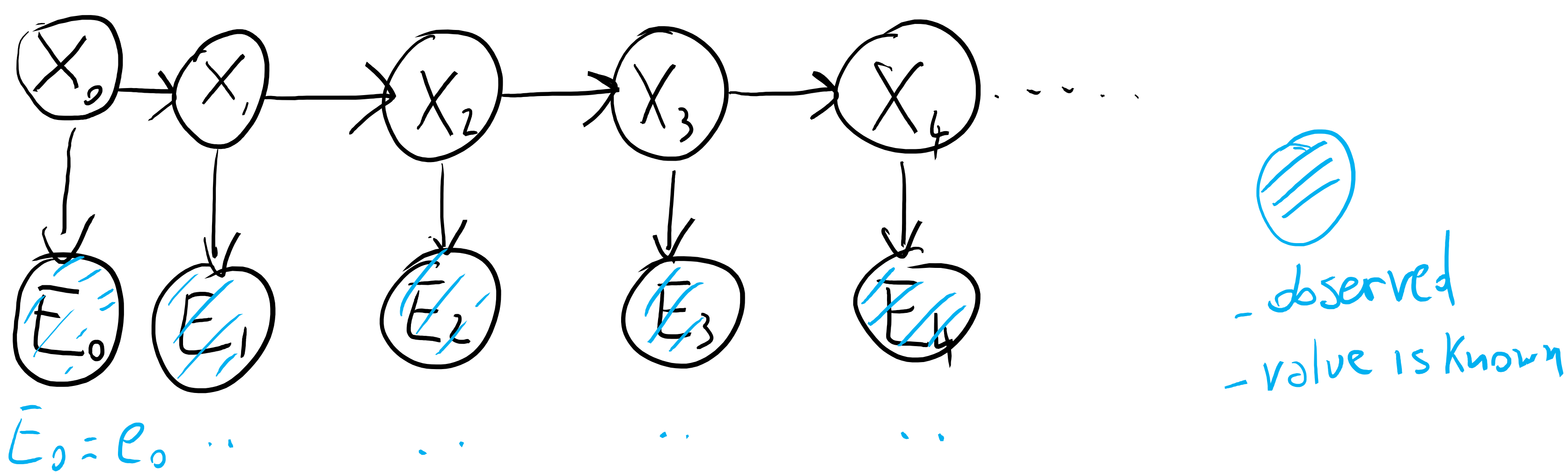




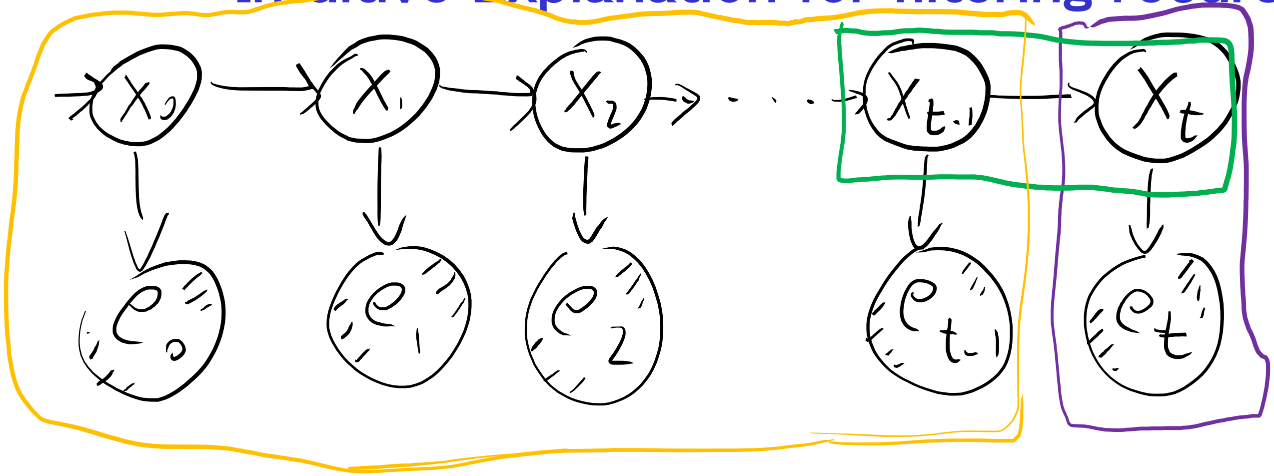
Useful inference in HMMs

- **In general (Filtering):** compute the posterior distribution over the current state given all evidence to date

$$P(X_t \mid e_{0:t})$$



Intuitive Explanation for filtering recursive formula



sequence of evidences $e_0:e_t$

$$P(X_t / e_{0:t}) = \alpha\, P(e_t|X_t) * \sum_{X_{t-1}} P(X_t|x_{t-1}) * P(x_{t-1}|e_0:e_{t-1})$$

X_t generated evidence e_t

whatever x_{t-1} was, X_t was reached from there

and evidence $e_0:e_{t-1}$ must have been generated before getting to x_{t-1}

Filtering

- Idea: recursive approach
  - Compute filtering up to time t-1, and then include the evidence for time t (***recursive estimation***)
- $\mathbf{P}(\mathbf{X}_t \mid \mathbf{e}_{0:t}) = \mathbf{P}(\mathbf{X}_t \mid \mathbf{e}_{0:t-1}, \mathbf{e}_t)$ dividing up the evidence

$= \alpha\, \mathbf{P}(\mathbf{e}_t \mid \mathbf{X}_t, \mathbf{e}_{0:t-1})\, \mathbf{P}(\mathbf{X}_t \mid \mathbf{e}_{0:t-1})$ **WHY?**

$= \alpha\, \mathbf{P}(\mathbf{e}_t \mid \mathbf{X}_t)\, \mathbf{P}(\mathbf{X}_t \mid \mathbf{e}_{0:t-1})$ **WHY?**

iclicker.

A. Bayes Rule

B. Cond. Independence

C. Product Rule

Inclusion of new evidence: **this is available from..**

One step prediction of current state given evidence up to *t-1*

- So we only need to compute $\mathbf{P}(\mathbf{X}_t \mid \mathbf{e}_{0:t-1})$

$$P(X,Y,Z) = P(X|Y,Z)\,P(Y,Z)$$

$$P(X,Y,Z) = P(Y|X,Z)\,P(X,Z)$$

$$P(Y|X,Z) = \frac{P(X|Y,Z)\,P(Y,Z)}{P(X,Z)}$$

$$P(Y,Z) = P(Y|Z)\,P(Z)$$

$$P(X,Z) = P(X|Z)\,P(Z)$$

$$= \frac{P(X|Y,Z)\,P(Y|Z)}{P(X|Z)}$$

$$= \alpha\, P(X|Y,Z)\,P(Y|Z)$$

$$\sum_{y} P(xy|z)$$

$$\Downarrow$$

$$\sum_{y} P(x|yz)\,P(y|z)$$

"moving" the conditioning

$$\boxed{P(AB|C)} = \frac{P(ABC)}{P(C)} * \frac{P(BC)}{P(BC)} =$$

$$= \frac{P(ABC)}{P(BC)} * \frac{P(BC)}{P(C)} =$$

$$= \boxed{P(A|BC) * P(B|C)}$$

Filtering

- Compute $P(X_t \mid e_{0:t-1})$

$$P(X_t \mid e_{0:t-1}) = \Sigma_{x_{t-1}} P(X_t, x_{t-1} / e_{0:t-1}) = \Sigma_{x_{t-1}} P(X_t \mid x_{t-1}, e_{0:t-1}) P(x_{t-1} / e_{0:t-1}) =$$

$$= \Sigma_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} / e_{0:t-1})$$ because of..

Prove it?

Transition model!

Filtering at time *t-1*

- Putting it all together, we have the desired recursive formulation

$$P(X_t / e_{0:t}) = \alpha \, P(e_t \mid X_t) \, \Sigma_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} / e_{0:t-1})$$

Inclusion of new evidence (sensor model)

Propagation to time *t*

Filtering at time *t-1*

- $P(X_{t-1} / e_{0:t-1})$ can be seen as a message $f_{0:t-1}$ that is propagated forward along the sequence, modified by each transition and updated by each observation

Filtering

- Thus, the recursive definition of filtering at time t in terms of filtering at time $t\text{-}1$ can be expressed as a FORWARD procedure
  - $\boldsymbol{f}_{0:t} = \alpha\ FORWARD\ (\mathbf{f}_{0:t-1}, \boldsymbol{e}_t)$
- which implements the update described in

$$\boldsymbol{P}(\boldsymbol{X}_t \mid \boldsymbol{e}_{0:t}) = \alpha\, \boldsymbol{P}(\boldsymbol{e}_t \mid \boldsymbol{X}_t) \sum_{x_{t-1}} \boldsymbol{P}(\boldsymbol{X}_t \mid \boldsymbol{x}_{t-1})\, P(\boldsymbol{x}_{t-1} \mid \boldsymbol{e}_{0:t-1})$$

Filtering at time $t\text{-}1$

Inclusion of new evidence
(sensor model)

Propagation to time t

Analysis of Filtering

- Because of the recursive definition in terms for the forward message, when all variables are discrete the time for each update is constant (i.e. independent of t)

- The constant depends of course on the size of the state space

Rain Example

- Suppose our security guard came with a prior belief of 0.5 that it rained on day 0, just before the observation sequence started.

- Without loss of generality, this can be modelled with a fictitious state R_0 with no associated observation and $P(R_0) = \langle 0.5, 0.5 \rangle$

- **Day 1**: umbrella appears (u_1). Thus

no previous evidence

$$\mathbf{P}(R_1 \mid \mathbf{e}_{0:t-1}) = \mathbf{P}(R_1) = \sum_{r_0} \mathbf{P}(R_1 \mid r_0)\, P(r_0)$$

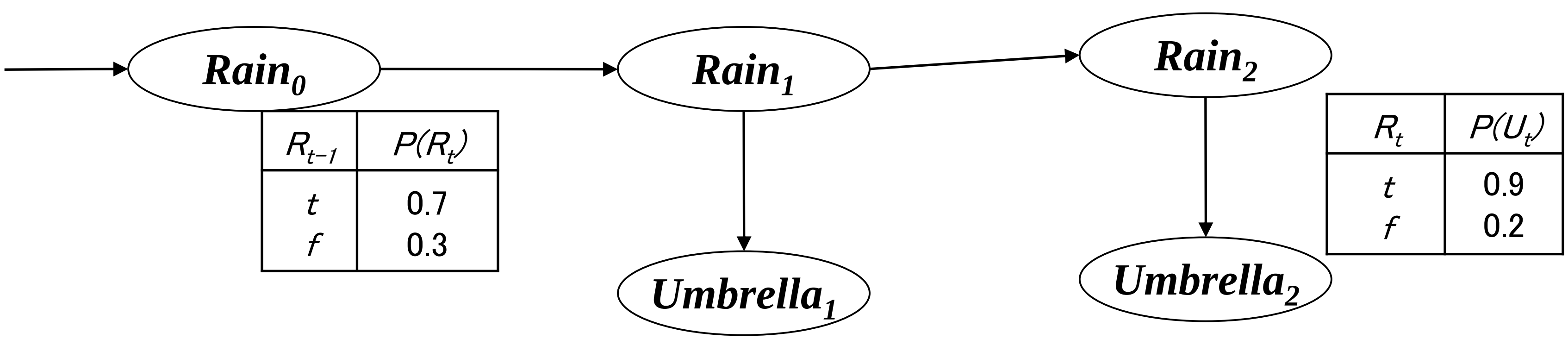
$$= \langle 0.7, 0.3 \rangle * 0.5 + \langle 0.3, 0.7 \rangle * 0.5 = \langle 0.5, 0.5 \rangle$$

T F T T F F

R_{t-1} \ R_t	T	F
T	.7	.3
F	.3	.7

0.5
0.5

TRUE 0.5
FALSE 0.5



R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

Rain Example

- Updating this with evidence from for $t = 1$ (umbrella appeared) gives

  $\mathbf{P}(R_1 | u_1) = \alpha\, \mathbf{P}(u_1 \mid R_1)\, \mathbf{P}(R_1) =$

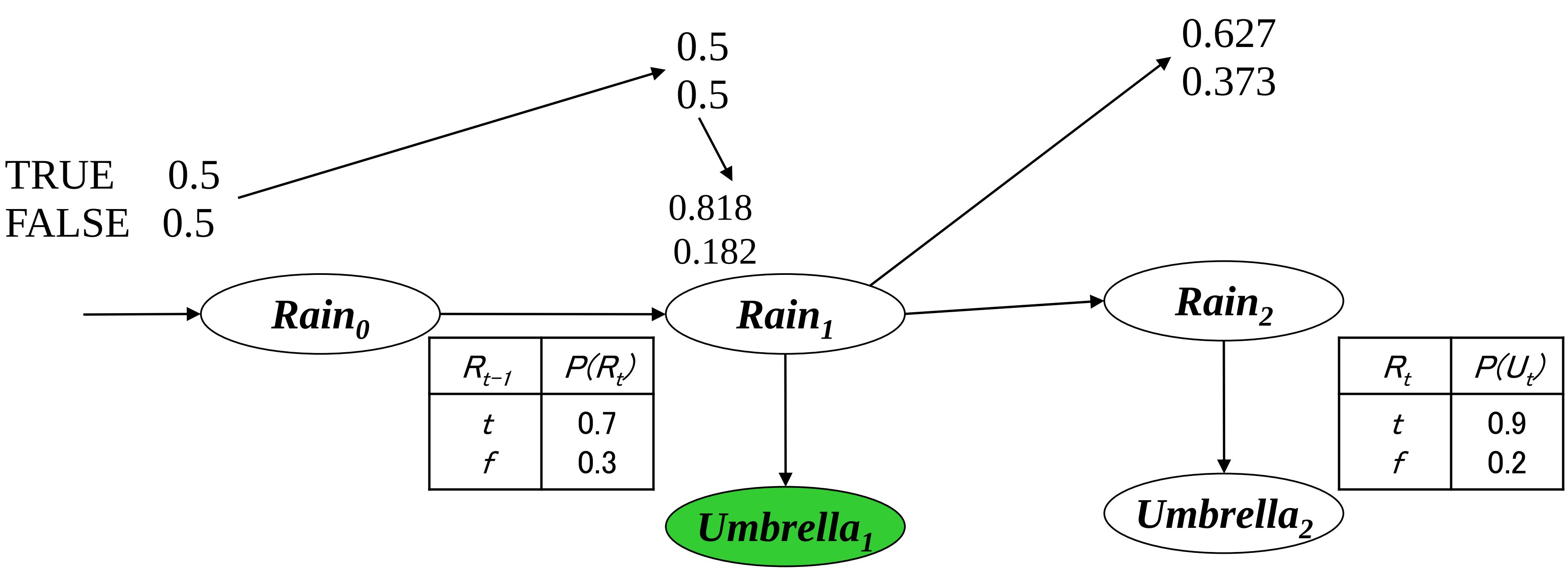
  $\alpha\langle 0.9, 0.2\rangle\langle 0.5, 0.5\rangle = \alpha\langle 0.45, 0.1\rangle \sim \langle 0.818, 0.182\rangle$

- Day 2: umbella appears (u_2). Thus

  $\mathbf{P}(R_2 \mid \mathbf{e}_{0:t-1}) = \mathbf{P}(R_2 \mid u_1) = \sum_{r_1} \mathbf{P}(R_2 \mid r_1)\, P(r_1 \mid u_1) =$

  $= \langle 0.7, 0.3\rangle * 0.818 + \langle 0.3, 0.7\rangle * 0.182 \sim \langle 0.627, 0.373\rangle$

R_{t-1} \ R_t	T	F
T	.7	.3
F	.3	.7



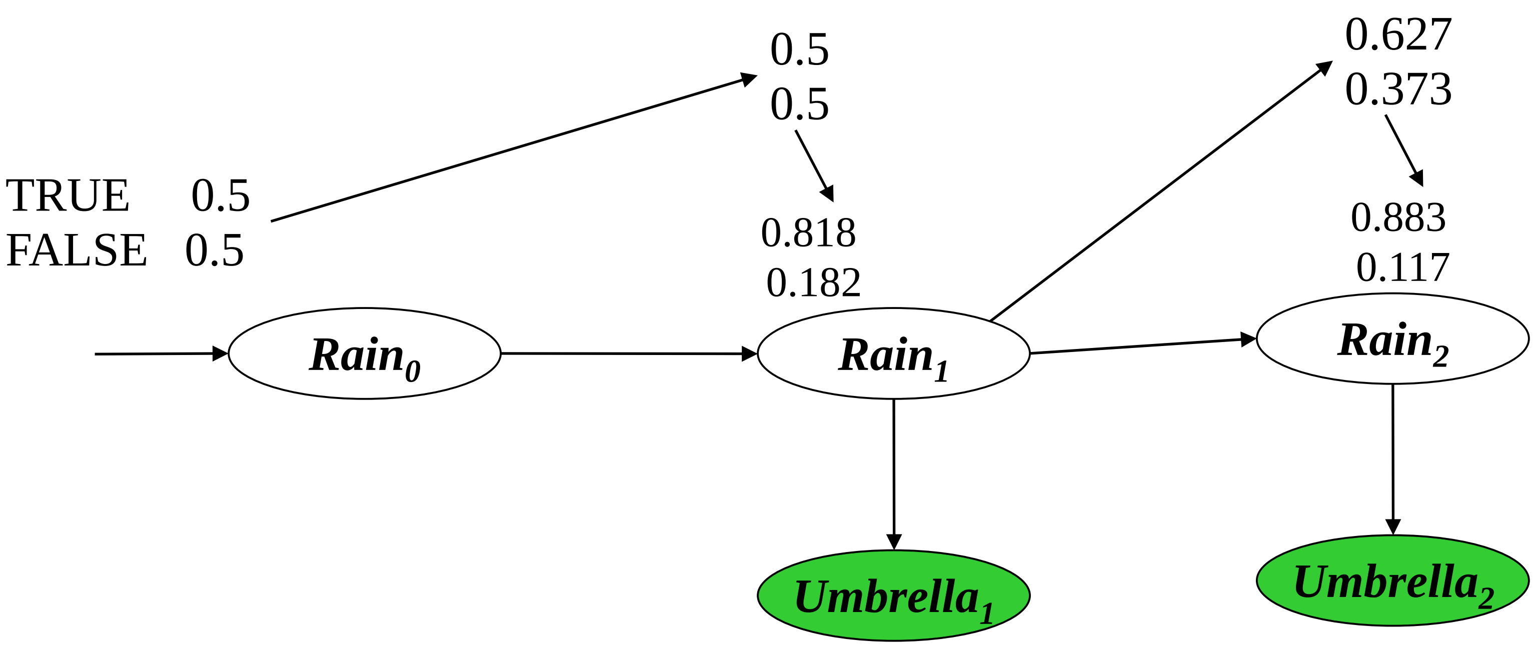
Rain Example

- Updating this with evidence from for $t = 2$ (umbrella appeared) gives

  $\mathbf{P}(R_2 | u_1, u_2) = \alpha\, \mathbf{P}(u_2 | R_2)\, \mathbf{P}(R_2 | u_1) =$

  $\alpha\langle 0.9, 0.2\rangle\langle 0.627, 0.373\rangle = \alpha\langle 0.565, 0.075\rangle \sim \langle 0.883, 0.117\rangle$

- Intuitively, the probability of rain increases, because the umbrella appears twice in a row



Practice exercise (home)

Compute filtering at t_3 if the 3[rd] observation/evidence is *no umbrella* (will put solution on inked slides)

$\langle 0.7, 0.3 \rangle * 0.883 + \langle 0.3, 0.7 \rangle * 0.117$

$\langle 0.618, 0.264 \rangle + \langle 0.035, 0.081 \rangle = \langle 0.653, 0.345 \rangle$

sensor model

$\alpha \langle 0.653, 0.345 \rangle * \langle 0.1, 0.8 \rangle$

$\alpha \langle 0.065, 0.276 \rangle$

normalize / divide by the sum.

$\underline{0.19}$ $\underline{0.81}$

Lecture Overview

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Prediction $P(X_{t+k+1} \mid e_{0:t})$

- Can be seen as filtering without addition of new evidence
- In fact, filtering already contains a one-step prediction

$$\mathbf{P}(\mathbf{X}_t \mid \mathbf{e}_{0:t}) = \alpha\, \mathbf{P}(\mathbf{e}_t \mid \mathbf{X}_t) \sum_{x_{t-1}} \mathbf{P}(\mathbf{X}_t \mid \mathbf{x}_{t-1})\, P(\mathbf{x}_{t-1} \mid \mathbf{e}_{0:t-1})$$

Inclusion of new evidence (sensor model)

Propagation to time *t*

Filtering at time *t-1*

- We need to show how to recursively predict the state at time $t+k+1$ from a prediction for state $t+k$

$$\mathbf{P}(\mathbf{X}_{t+k+1} \mid \mathbf{e}_{0:t}) = \sum_{x_{t+k}} \mathbf{P}(\mathbf{X}_{t+k+1}, \mathbf{x}_{t+k} \mid \mathbf{e}_{0:t}) = \sum_{x_{t+k}} \mathbf{P}(\mathbf{X}_{t+k+1} \mid \mathbf{x}_{t+k}, \mathbf{e}_{0:t})\, P(\mathbf{x}_{t+k} \mid \mathbf{e}_{0:t}) =$$

$$= \sum_{x_{t+k}} \mathbf{P}(\mathbf{X}_{t+k+1} \mid \mathbf{x}_{t+k})\, P(\mathbf{x}_{t+k} \mid \mathbf{e}_{0:t})$$

Prediction for state *t+ k*

Transition model

- Let's continue with the rain example and compute the probability of *Rain* on day four after having seen the umbrella in day one and two: $P(R_4 \mid u_1, u_2)$

Rain Example

- Prediction from day 2 to day 3

$$\mathbf{P}(\mathbf{X}_3 \mid \mathbf{e}_{1:2}) = \sum_{x_2} \mathbf{P}(\mathbf{X}_3 \mid \mathbf{x}_2)\, P(\mathbf{x}_2 \mid \mathbf{e}_{1:2}) = \sum_{r_2} \mathbf{P}(R_3 \mid r_2)\, P(r_2 \mid u_1\, u_2) =$$

$$= \langle 0.7,0.3\rangle * 0.883 + \langle 0.3,0.7\rangle * 0.117 = \langle 0.618,0.265\rangle + \langle 0.035, 0.082\rangle$$

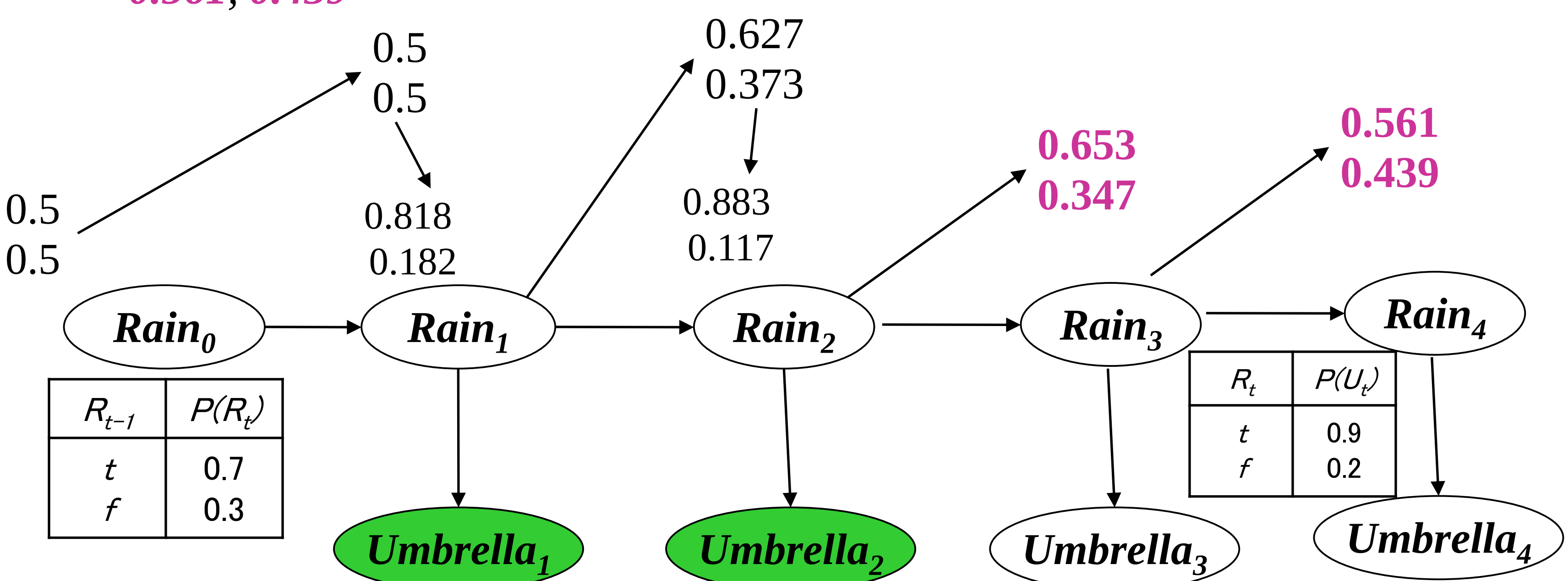
$$= \langle 0.653, 0.347\rangle$$

- Prediction from day 3 to day 4

$$\mathbf{P}(\mathbf{X}_4 \mid \mathbf{e}_{1:2}) = \sum_{x_3} \mathbf{P}(\mathbf{X}_4 \mid \mathbf{x}_3)\, P(\mathbf{x}_3 \mid \mathbf{e}_{1:2}) = \sum_{r_3} \mathbf{P}(R_4 \mid r_3)\, P(r_3 \mid u_1\, u_2) =$$

$$= \langle 0.7,0.3\rangle * \mathbf{0.653} + \langle 0.3,0.7\rangle * \mathbf{0.347} = \langle \mathbf{0.457},\mathbf{0.196}\rangle + \langle \mathbf{0.104}, \mathbf{0.243}\rangle$$

$$= \langle \mathbf{0.561}, \mathbf{0.439}\rangle$$



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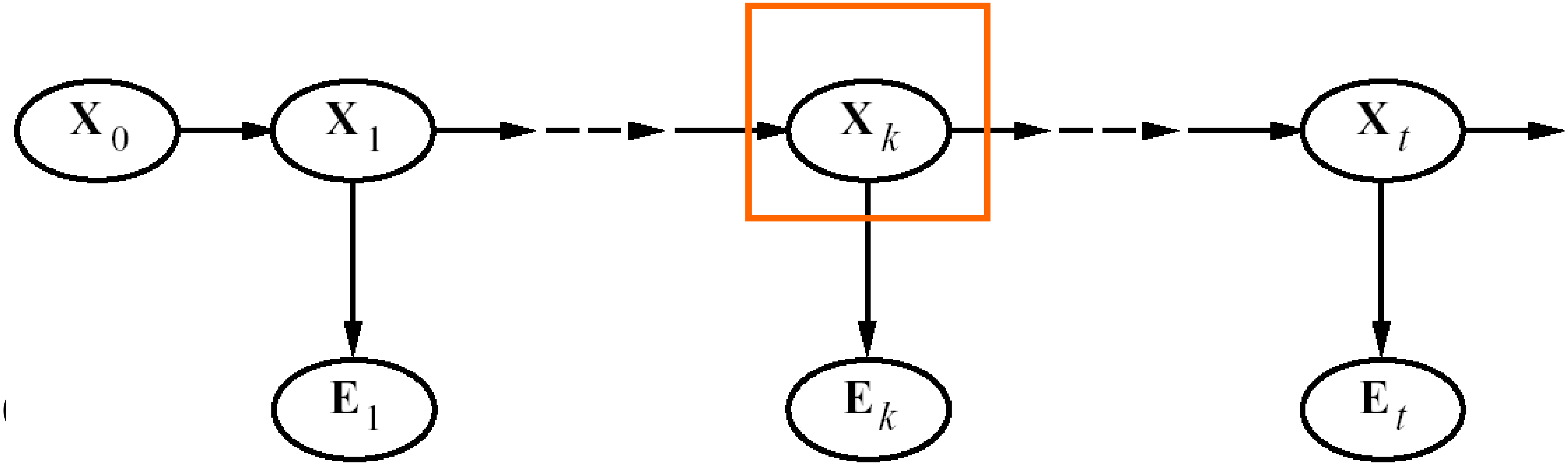
Smoothing

- ***Smoothing***: Compute the posterior distribution over a *past* state given all evidence to date
  - $\boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{0:t})$ for $1 \leq k < t$



Smoothing

- $\boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{0:t}) = \boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{0:k}, \boldsymbol{e}_{k+1:t})$ dividing up the evidence

  $= \alpha\, \boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{0:k})\, \boldsymbol{P}(\boldsymbol{e}_{k+1:t} \mid \boldsymbol{X}_k, \boldsymbol{e}_{0:k})$ using…

  $= \alpha\, \boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{0:k})\, \boldsymbol{P}(\boldsymbol{e}_{k+1:t} \mid \boldsymbol{X}_k)$ using…

forward message from filtering up to state k, $\boldsymbol{f}_{0:k}$

backward message, $\boldsymbol{b}_{k+1:t}$ computed by a recursive process that runs backwards from t

Smoothing

- $\boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{0:t}) = \boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{0:k}, \boldsymbol{e}_{k+1:t})$ dividing up the evidence

$= \alpha\, \boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{0:k})\, \boldsymbol{P}(\boldsymbol{e}_{k+1:t} \mid \boldsymbol{X}_k, \boldsymbol{e}_{0:k})$ using Bayes Rule

$= \alpha\, \boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{0:k})\, \boldsymbol{P}(\boldsymbol{e}_{k+1:t} \mid \boldsymbol{X}_k)$ By Markov assumption on evidence

forward message from filtering up to state k, $\boldsymbol{f}_{0:k}$

backward message, $\boldsymbol{b}_{k+1:t}$ computed by a recursive process that runs backwards from t

Learning Goals for today's class

➢ You can:

- Describe Filtering and derive it by manipulating probabilities
- Describe Prediction and derive it by manipulating probabilities
- Describe Smoothing and derive it by manipulating probabilities

TODO for Fri

- **Keep Reading Textbook Chp 8.5**
- **Keep working on assignment-2 (due on Fri, Oct 20)**