Department of Computer Science
Undergraduate Events
More details @ https://my.cs.ubc.ca/students/development/events

Co-op Info Session

Thurs., Sept 17

12:30 – 1:30 pm

MCLD 202

Simba Technologies Tech Talk/Info Session

Mon., Sept 21

6 – 7 pm

DMP 310

EA Info Session

Tues., Sept 22

6 – 7 pm

DMP 310

Intelligent Systems (AI-2)

Computer Science cpsc422, Lecture 4

Sep, 16, 2015

More material in this lecture this year because in Lect. 3 the projector did not work

Announcements

Assignment0 / Survey results

- Discussion on **Piazza** (sign up piazza.com/ubc.ca/winterterm12015/cpsc422)
- More than 50% took 322 more than a year ago… so make sure you revise 322 material!

What to do with readings? In a few lectures we will discuss the first research paper. Instructions on what to do are available on the course webpage.

Lecture Overview

Markov Decision Processes

- Some ideas and notation
- Finding the Optimal Policy
  - Value Iteration
- From Values to the Policy
- Rewards and Optimal Policy

Sketch of ideas to find the optimal policy for a MDP (Value Iteration)

We first need a couple of definitions

- $V^{\pi}(s)$: the expected value of following policy π in state s
- $Q^{\pi}(s, a)$, where *a* is an action: expected value of performing *a* in *s*, and then following policy π.

Can we express $Q^{\pi}(s, a)$ in terms of $V^{\pi}(s)$?

$$Q^{\pi}(s, a) = V^{\pi}(s) + R(s)$$

A.

$$Q^{\pi}(s, a) = R(s) + \sum_{s' \in X} P(s' \mid s, a) * V^{\pi}(s')$$

B.

$$Q^{\pi}(s, a) = R(s) + \sum_{s' \in X} V^{\pi}(s')$$

C.

D. None of the above

X: set of states reachable from *s* by doing *a*

Discounted Reward Function

- Suppose the agent goes through states $s_1, s_2, \ldots, s_k$ and receives rewards $r_1, r_2, \ldots, r_k$
- We will look at ***discounted reward*** to define the reward for this sequence, i.e. its *utility* for the agent

$$\gamma \ \textbf{\textit{discount factor}}, \ 0 \leq \gamma \leq 1$$

$$R_{\max} \text{ bound on R(s) for every } s$$

$$U[s_1, s_2, s_3, ..] = r_1 + \gamma r_2 + \gamma^2 r_3 + ...$$

$$= \sum_{i=0}^{\infty} \gamma^i r_{i+1} \leq \sum_{i=0}^{\infty} \gamma^i R_{\max} = \frac{R_{\max}}{1-\gamma}$$

Sketch of ideas to find the optimal policy for a MDP (Value Iteration)

We first need a couple of definitions

- $V^{\pi}(s)$: the expected value of following policy π in state s
- $Q^{\pi}(s, a)$, where *a* is an action: expected value of performing *a* in *s*, and then following policy π.

We have, by definition

$$Q^{\pi}(s, a) = R(s) + \gamma \sum_{s'} P(s' \mid s, a) V^{\pi}(s')$$

reward obtained in *s*

Discount factor

states reachable from *s* by doing *a*

Probability of getting to *s'* from *s* via *a*

expected value of following policy π in *s'*

Value of a policy and Optimal policy

We can also compute $V^{\pi}(s)$ in terms of $Q^{\pi}(s, a)$

$$V^{\pi}(s) = Q^{\pi}(s, \pi(s))$$

Expected value of following π in s

Expected value of performing the action indicated by π in s and following π after that

action indicated by π in s

For the optimal policy π * we also have

$$V^{\pi^*}(s) = Q^{\pi^*}(s, \pi^*(s))$$

Value of Optimal policy

$$V^{\pi^*}(s) = Q^{\pi^*}(s, \pi^*(s))$$

Remember for any policy π

$$Q^{\pi}(s, \pi(s)) = R(s) + \gamma \sum_{s'} P(s' \mid s, \pi(s)) \times V^{\pi}(s'))$$

But the Optimal policy π^* is the one that gives the action that maximizes *the future reward* for each state

$$\overset{V^{\pi^*}(s)}{\cancel{Q^{\pi^*}(s, \pi^*(s))}} = R(s) + \gamma \max_a \sum_{s'} P(s' \mid s, a) \times V^{\pi^*}(s')$$

So…

$$V^{\pi^*}(s) = R(s) + \gamma \max_a \sum_{s'} P(s' \mid s, a) \times V^{\pi^*}(s'))$$

Value Iteration Rationale

- Given *N* states, we can write an equation like the one below for each of them

$$\boxed{V(s_1) \;=\; R(s_1) + \gamma \max_a \sum_{s'} P(s' \mid s_1, a) V(s')}$$

$$\boxed{V(s_2) \;=\; R(s_2) + \gamma \max_a \sum_{s'} P(s' \mid s_2, a) V(s')}$$

- Each equation contains *N* unknowns – the V values for the *N* states
- N equations in N variables (Bellman equations): It can be shown that they have a unique solution: the values for the optimal policy
- Unfortunately the N equations are non-linear, because of the max operator: Cannot be easily solved by using techniques from linear algebra
- **Value Iteration Algorithm**: Iterative approach to find the V values and the corresponding
- optimal policy

Value Iteration in Practice

- Let $V^{(i)}(s)$ be the utility of state *s* at the ith iteration of the algorithm
- Start with arbitrary utilities on each state *s*: $V^{(0)}(s)$
- Repeat simultaneously for every *s* until there is "no change"

$$\boxed{V^{(k+1)}(s) = R(s) + \gamma \max_a \sum_{s'} P(s'|s,a) V^{(k)}(s')}$$

- True "no change" in the values of V(s) from one iteration to the next are guaranteed only if run for infinitely long.
  - In the limit, this process converges to a unique set of solutions for the Bellman equations
  - They are the total expected rewards (utilities) for the optimal policy

Example

(sorry (column, row) to indicate state)

- Suppose, for instance, that we start with values $V^{(0)}(s)$ that are all 0

Iteration 0

3	0	0	0	+1
2	0		0	-1
1	0	0	0	0
	1	2	3	4

Iteration 1

3	0	0	0	+1
2	0		0	-1
1	$\boxed{-0.04}$	0	0	0
	1	2	3	4

$$V^{(1)}(1,1) = -0.04 + \overset{\gamma}{1} * \max \begin{bmatrix} 0.8V^{(0)}(1,2) + 0.1V^{(0)}(2,1) + 0.1V^{(0)}(1,1) & UP \\ 0.9V^{(0)}(1,1) + 0.1V^{(0)}(1,2) & LEFT \\ 0.9V^{(0)}(1,1) + 0.1V^{(0)}(2,1) & DOWN \\ 0.8V^{(0)}(2,1) + 0.1V^{(0)}(1,2) + 0.1V^{(0)}(1,1) & RIGHT \end{bmatrix}$$

$$V^{(1)}(1,1) = -0.04 + \max \begin{bmatrix} 0 & UP \\ 0 & LEFT \\ 0 & DOWN \\ 0 & RIGHT \end{bmatrix}$$

Example (cont'd)

(sorry (column, row) to indicate state)

➢ Let's compute $V^{(1)}(3,3)$

Iteration 0

3	0	0	0	+1
2	0		0	-1
1	0	0	0	0
	1	2	3	4

Iteration 1

3	0	0	$\boxed{0.76}$	+1
2	0		0	-1
1	$\boxed{-0.04}$	0	0	0
	1	2	3	4

$$V^{(1)}(3,3) = -0.04 + 1 * \max \begin{bmatrix} 0.8V^{(0)}(3,3) + 0.1V^{(0)}(2,3) + 0.1V^{(0)}(4,3) & UP \\ 0.8V^{(0)}(2,3) + 0.1V^{(0)}(3,3) + 0.1V^{(0)}(3,2) & LEFT \\ 0.8V^{(0)}(3,2) + 0.1V^{(0)}(2,3) + 0.1V^{(0)}(4,3) & DOWN \\ 0.8V^{(0)}(4,3) + 0.1V^{(0)}(3,3) + 0.1V^{(0)}(3,2) & RIGHT \end{bmatrix}$$

$$V^{(1)}(3,3) = -0.04 + \max \begin{bmatrix} 0.1 & UP \\ 0 & LEFT \\ 0.1 & DOWN \\ 0.8 & RIGHT \end{bmatrix}$$

Example (cont'd)

(sorry (column, row) to indicate state)

- Let's compute $V^{(1)}(4,1)$

Iteration 0

3	0	0	0	+1
2	0		0	-1
1	0	0	0	0
	1	2	3	4

Iteration 1

3	0	0	$\boxed{.76}$	+1
2	0		0	-1
1	$\boxed{-0.04}$	0	0	$\boxed{-0.04}$
	1	2	3	4

$$V^{(1)}(4,1) = -0.04 + \max \begin{bmatrix} 0.8V^{(0)}(4,2) + 0.1V^{(0)}(3,1) + 0.1V^{(0)}(4,1) & UP \\ 0.8V^{(0)}(3,1) + 0.1V^{(0)}(4,2) + 0.1V^{(0)}(4,1) & LEFT \\ 0.9V^{(0)}(4,1) + 0.1V^{(0)}(3,2) & DOWN \\ 0.9V^{(0)}(4,1) + 0.1V^{(0)}(4,2) & RIGHT \end{bmatrix}$$

$$V^{(1)}(4,1) = -0.04 + \max \begin{bmatrix} -0.8 & UP \\ -0.1 & LEFT \\ 0 & DOWN \\ -0.1 & RIGHT \end{bmatrix}$$

After a Full Iteration

Iteration 1

3	-.04	-.04	0.76	+1
2	-.04		-.04	-1
1	-.04	-.04	-.04	-.04
	1	2	3	4

- Only the state one step away from a positive reward (3,3) has gained value, all the others are losing value

Some steps in the second iteration

Iteration 1

	1	2	3	4
3	-.04	-.04	0.76	+1
2	-.04		-.04	-1
1	-.04	-.04	-.04	-.04

Iteration 2

	1	2	3	4
3	-.04	-.04	0.76	+1
2	-.04		-.04	-1
1	-0.08	-.04	-.04	-.04

$$V^{(2)}(1,1) = -0.04 + 1 * \max \begin{bmatrix} 0.8V^{(1)}(1,2) + 0.1V^{(1)}(2,1) + 0.1V^{(1)}(1,1) & UP \\ 0.9V^{(1)}(1,1) + 0.1V^{(1)}(1,2) & LEFT \\ 0.9V^{(1)}(1,1) + 0.1V^{(1)}(2,1) & DOWN \\ 0.8V^{(1)}(2,1) + 0.1V^{(1)}(1,2) + 0.1V^{(1)}(1,1) & RIGHT \end{bmatrix}$$

$$V^{(2)}(1,1) = -0.04 + \max \begin{bmatrix} -.04 & UP \\ -.04 & LEFT \\ -.04 & DOWN \\ -.04 & RIGHT \end{bmatrix} = -0.08$$

Example (cont'd)

- Let's compute $V^{(1)}(2,3)$

Iteration 1

3	-.04	-.04	0.76	+1
2	-.04		-.04	-1
1	-.04	-.04	-.04	-.04
	1	2	3	4

Iteration 2

3	-.04	0.56	0.76	+1
2	-.04		-.04	-1
1	-0.08	-.04	-.04	-.04
	1	2	3	4

$$V^{(1)}(2,3) = -0.04 + 1 * \max \begin{bmatrix} 0.8V^{(0)}(2,3) + 0.1V^{(0)}(1,3) + 0.1V^{(0)}(3,3) & UP \\ 0.8V^{(0)}(1,3) + 0.1V^{(0)}(2,3) + 0.1V^{(0)}(2,3) & LEFT \\ 0.8V^{(0)}(2,3) + 0.1V^{(0)}(1,3) + 0.1V^{(0)}(3,3) & DOWN \\ 0.8V^{(0)}(3,3) + 0.1V^{(0)}(2,3) + 0.1V^{(0)}(2,3) & RIGHT \end{bmatrix}$$

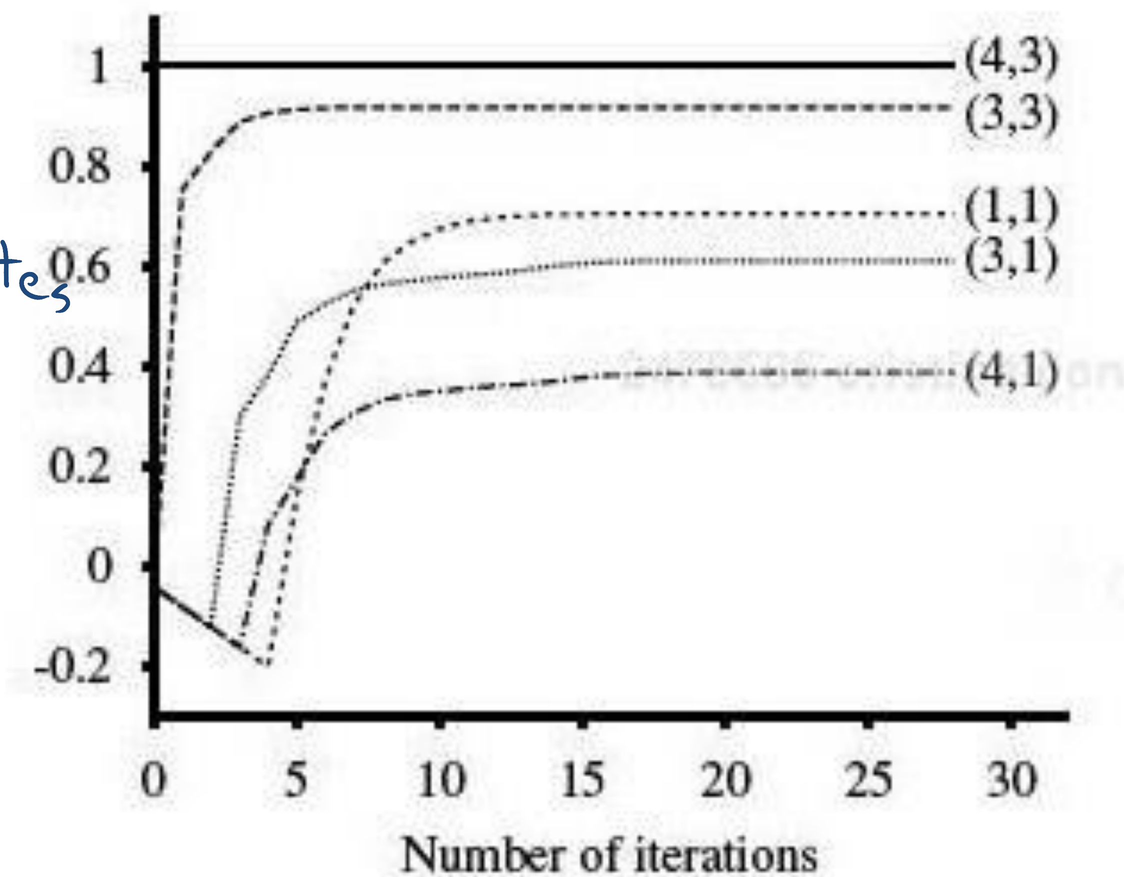
$$V^{(1)}(2,3) = -0.04 + (0.8 * 0.76 + 0.2 * -0.04) = 0.56$$

- Steps two moves away from positive rewards start increasing their value

State Utilities as Function of Iteration

(only for 5 states)

Estimates for V



		(3,3)	(4,3)
			(4,2)
(1,1)		(3,1)	(4,1)

- Note that values of states at different distances from (4,3) accumulate negative rewards until a path to (4,3) is found

Value Iteration: Computational Complexity

Value iteration works by producing successive approximations of the optimal value function.

$$\forall s: \; V^{(k+1)}(s) \;=\; R(s) + \gamma \max_a \sum_{s'} P(s' \mid s, a) V^{(k)}(s')$$

What is the complexity of each iteration?

A. $O(|A|^2|S|)$

B. $O(|A||S|^2)$

C. $O(|A|^2|S|^2)$

…or faster if there is sparsity in the transition function.

small sets

Relevance to state of the art MDPs

FROM : Planning with Markov Decision Processes: An AI Perspective Mausam (UW), Andrey Kolobov (MSResearch) Synthesis Lectures on Artificial Intelligence and Machine Learning Jun 2012

Free online through UBC

Artificial Intelligence and Machine Learning

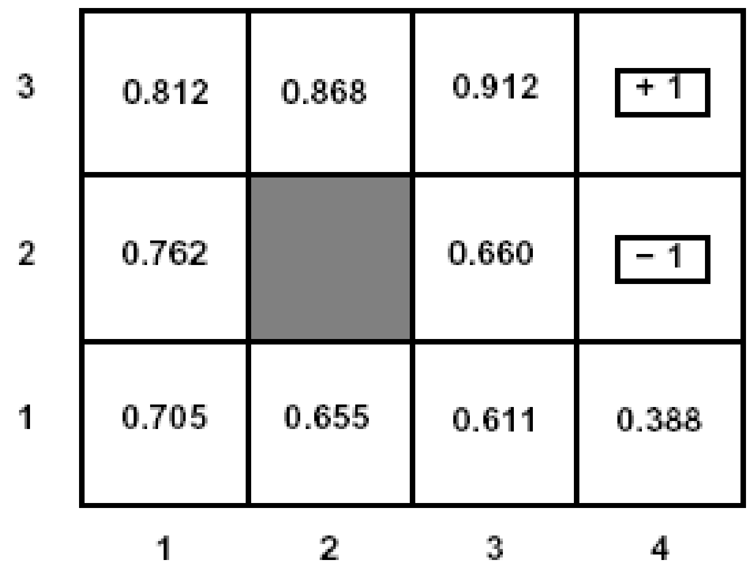
" Value Iteration (VI) forms the basis of most of the advanced MDP algorithms that we discuss in the rest of the book. ……..."

Lecture Overview

Markov Decision Processes

- ……
- Finding the Optimal Policy
  - Value Iteration
- **From Values to the Policy**
- Rewards and Optimal Policy

Value Iteration: from state values V to π*



- Now the agent can chose the action that implements the **MEU principle**: maximize the expected utility of the subsequent state

Value Iteration: from state values V to *л**

- Now the agent can chose the action that implements the MEU principle: maximize the expected utility of the subsequent state

$$\pi^*(s) = \arg\max_a \sum_{s'} P(s'|s,a) V^{\pi^*}(s')$$

expected value of following policy л* in s’

states reachable from s by doing a

Probability of getting to *s’* from *s* via *a*

Example: from state values V to π^*

$$\pi^*(s) = \arg\max_a \sum_{s'} P(s'|s,a)V^{\pi^*}(s')$$

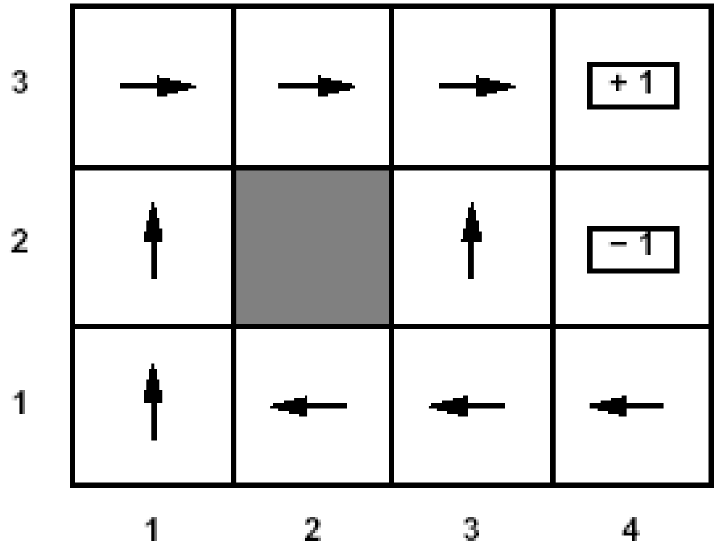
	1	2	3	4
3	0.812	0.868	0.912	+1
2	0.762		0.660	−1
1	0.705	0.655	0.611	0.388

- To find the best action in (1,1)

$$\pi^*(1,1) = \arg\max \begin{bmatrix} 0.8\overset{.762}{\boxed{V(1,2)}} + 0.1\overset{.655}{\boxed{V(2,1)}} + 0.1\overset{.705}{\boxed{V(1,1)}} & UP \; * \\ 0.9\overset{.705}{\boxed{V(1,1)}} + 0.1\overset{.762}{\boxed{V(1,2)}} & LEFT \\ 0.9\overset{.705}{\boxed{V(1,1)}} + 0.1\overset{.655}{\boxed{V(2,1)}} & DOWN \\ 0.8V(2,1) + 0.1V(1,2) + 0.1V(1,1) & RIGHT \end{bmatrix}$$

Optimal policy

- This is the policy that we obtain….



Learning Goals for today's class

You can:

Define/read/write/trace/debug the Value Iteration (VI) algorithm. Compute its complexity.

- Compute the Optimal Policy given the output of VI
- Explain influence of rewards on optimal policy

TODO for Fri

- **Read Textbook** 9.5.6 Partially Observable MDPs

- **Also Do Practice Ex. 9.C**
http://www.aispace.org/exercises.shtml