Bottom Up: Soundness and Completeness

Computer Science cpsc322, Lecture 21

(Textbook Chpt 5.2)

February, 27, 2009

Lecture Overview

- Recap
- Soundness of Bottom-up Proofs
- Completeness of Bottom-up Proofs

(Propositional) Logic: Key ideas

Given a domain that can be represented with **n propositions** you have …2ⁿ… interpretations (possible worlds)

If you do not know anything you can be in any of those

If you know that some **logical formulas** are true (your KB…..). You know that you can be only in ……models…

interpretations in which KB is true

It would be nice to know what else is true in all those… models

what it is logically entailed by the KB

PDCL syntax / semantics / proofs

Domain can be represented by three propositions: *p, q, r*

$$KB = \begin{cases} q. \\ r. \\ \boxed{p \leftarrow \boxed{q \wedge r.}} \end{cases}$$

Interpretations?

r	q	p
T	T	T
~~T~~	~~T~~	~~F~~
~~T~~	~~F~~	~~T~~
~~T~~	~~F~~	~~F~~
~~F~~	~~T~~	~~T~~
~~F~~	~~T~~	~~F~~
~~F~~	~~F~~	~~T~~
~~F~~	~~F~~	~~F~~

Models? $KB \models G$

What is logically entailed ?

$C = \{q, r, p\}$

Prove $G = (q \wedge p)$ BU $KB \vdash_{BU} G$ $G \subseteq C$

PDCL syntax / semantics / proofs

$$KB = \begin{cases} p \leftarrow \boxed{q \wedge r.} \\ q. \end{cases}$$

Interpretations

r	q	p
T	T	T
~~T~~	~~T~~	~~F~~
~~T~~	~~F~~	~~T~~
~~T~~	~~F~~	~~F~~
F	T	T
F	T	F
~~F~~	~~F~~	~~T~~
~~F~~	~~F~~	~~F~~

Models

What is logically entailed?

Prove $G = \overset{q}{\boxed{(q \wedge p)}}$ $KB \nvdash_{BU} G$

$C = \{q\}$

$G \notin C$

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Soundness of bottom-up proof procedure

Generic Soundness of proof procedure:

If G can be proved by the procedure (KB ⊢ G)
then G is logically entailed by the KB (KB ⊨ G)

For Bottom-Up proof

if $G \subseteq C$ at the end of procedure
then G is logically entailed by the KB

So BU is sound, if …….

all the atoms in C
are logically entailed by KB

Soundness of bottom-up proof procedure

Suppose this is not the case.

1. Let *h* be the first atom added to *C* that is not entailed by KB (i.e., that's true in every model of *KB*)
2. Suppose *h* isn't true in model *M* of *KB*.
3. Since *h* was added to C, there must be a clause in KB of form: $h \leftarrow b_1 \wedge \ldots \wedge b_m$
4. Each b_i is true in *M* *(because of 1.)*. *h* is false in *M*. So...... $h \leftarrow b_1 \wedge \ldots \wedge b_m$ is false in M
5. Therefore M is not a model of KB
6. Contradiction! thus no such *h* exists.

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Completeness of Bottom Up

Generic Completeness of proof procedure:
If G is logically entailed by the KB (KB ⊧ G)
then G can be proved by the procedure (KB ⊢ G)

Sketch of our proof:

1. Suppose $KB \models G$. Then G is true in all models of *KB*.
2. Thus *G* is true in any particular model of KB
3. We will define a model so that if G is true in that model, G is ~~generated~~ proved by the bottom up algorithm.
4. Thus $KB \vdash G$.

$G \subseteq C$

Let's work on step 3

3. We will define a model so that if G is true in that model, G is generated by the bottom up algorithm.

$G \subseteq C$

3.1 We will define an interpretation $\mathcal{I}$ so that if G is true in $\mathcal{I}$, G is generated by the bottom up algorithm. $G \subseteq C$

3.2 We will then show that ……… I is a model of KB

Let's work on step 3.1

3.1 Define interpretation $\mathcal{I}$ so that if G is true in $\mathcal{I}$,
Then $G \subseteq C$.

> Let I be the interpretation in which every element of C is true and every other atom is false.

KB

$a \leftarrow e \wedge g.$
$b \leftarrow f \wedge g.$
$c \leftarrow e.$
$f \leftarrow c \wedge e.$
$e.$
$d.$

F	F	T	T	T	T	F
$\{a,$	$b,$	$c,$	$d,$	$e,$	$f,$	$g\}$

B

$\{\,\} \leftarrow C$
$\{e\} \leftarrow$
$\{e, d\} \leftarrow$
$\{e, d, c\} \leftarrow$
$\{e, d, c, f\} \leftarrow = C$

Let's work on step 3.2

Claim: I is a model of *KB*. (we will call it the minimal model)

Proof: **Assume** that I is not a model of *KB*.

- **Then** there must exist some clause $h \leftarrow b_1 \wedge \ldots \wedge b_m$ in *KB* (having zero or more b_i's) which is false in I.
- The only way this can occur is if $b_1 \ldots b_m$ are true in I (i.e., are in C) and h is false in I (i.e., is not in C)
- **But** if each b_i belonged to C, Bottom Up would have added h to C as well.
- So, there can be no clause in the KB that is false in interpretation I (which implies the claim :-)

Completeness of Bottom Up

(complete proof)

If $KB \models G$ then $KB \vdash_{BU} G$

- Suppose $KB \models G$.
- Then G is true all models KB
- Thus G is true minimal model
- Thus $G \subseteq C$
- Thus G is ~~generated~~ proved by BU
- Thus $KB \vdash_{BU} G$

Learning Goals for today's class

You can:

- Prove that BU proof procedure is sound
- Prove that BU proof procedure is complete

Next class

(still section 5.2)

- Using PDC Logic to model the electrical domain
- Reasoning in the electrical domain
- Top-down proof procedure (as Search)

Study for midterm (Wed March 4)

Midterm: ~10 short questions + 2 problems

6 pts each

~ 20 pts each

- Study: textbook and **inked** slides
- Work on **all** practice exercises
- While you revise the **learning goals**, work on **review questions** - I may even reuse some verbatim ☺
- Will post a **couple of problems** from previous offering (maybe slightly more difficult /inappropriate for you because they were not informed by the learning goals) … but I'll give you the solutions ☺