CPSC 403/542
Assignment 1

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1. To draw a circle of radius r on a graphics screen, one may proceed to evaluate pairs of values $x = r\cos\theta$, $y = r\sin\theta$ for a succession of values θ. But this is expensive. A cheaper method may be obtained by considering the ODE

$$\begin{aligned} \dot{x} &= -y \qquad x(0) = r \\ \dot{y} &= x \qquad y(0) = 0 \end{aligned}$$

   where $\dot{x} = \frac{dx}{d\theta}$, and approximating this using a simple discretization scheme. However, care must be taken so as to ensure that the obtained approximate solution looks right, i.e. that the approximate curve closes rather than spirals.

   For each of the three discretization schemes introduced in Chapter 3, namely, forward Euler, backward Euler and trapezoidal schemes, carry out this integration using a uniform step size $h = .02$ for $0 \le \theta \le 120$. Determine if the solution spirals in, spirals out, or forms an approximate circle as desired. Explain the observed results. [Hint: this has to do with a certain invariant function of x and y, rather than with the order of the methods.]

2. When deriving the trapezoidal scheme, we proceeded to replace $\mathbf{y}'(t_{n-1/2})$ in

$$\frac{\mathbf{y}(t_n) - \mathbf{y}(t_{n-1})}{h_n} = \mathbf{y}'(t_{n-1/2}) + \frac{h_n^2}{24}\mathbf{y}'''(\xi_n)$$

   by an average and then use the given ODE

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}).$$

   If instead we first use the ODE, replacing $\mathbf{y}'(t_{n-1/2})$ by $\mathbf{f}(t_{n-1/2}, \mathbf{y}(t_{n-1/2}))$, and then average $\mathbf{y}$, we obtain the important implicit *midpoint scheme*,

$$\mathbf{y}_n = \mathbf{y}_{n-1} + h_n \mathbf{f}(t_{n-1/2}, \frac{1}{2}(\mathbf{y}_n + \mathbf{y}_{n-1}))$$

   (a) Show that this scheme is symmetric, second-order and A-stable. How does it relate to the trapezoidal scheme for the constant coefficient ODE $\mathbf{y}' = A\mathbf{y}$?

(b) Show that even if we allow λ to vary in t, i.e. we consider the scalar ODE

$$y' = \lambda(t)y$$

in place of the test equation, what corresponds to A-stability holds, namely, using the midpoint scheme,

$$|y_n| \le |y_{n-1}| \qquad \text{if} \qquad \mathcal{R}\text{e}(\lambda) \le 0$$

(this property is called *AN-stability*). Show that the same cannot be said about the trapezoidal scheme: the latter is not AN-stable.

3. (a) Show that the trapezoidal step

$$\mathbf{y}_n = \mathbf{y}_{n-1} + \frac{h_n}{2}(\mathbf{f}_{n-1} + \mathbf{f}_n)$$

can be viewed as half a step of forward Euler followed by half a step of backward Euler.

(b) Show that the midpoint step (previous question) can be viewed as half a step of backward Euler followed by half a step of forward Euler.

(c) Consider an autonomous system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ and a fixed step size, $h_n = h$, $n = 1, \ldots, N$. Show that the trapezoidal scheme applied N times is equivalent to applying first half a step of forward Euler (i.e. forward Euler with step size $h/2$), followed by $N-1$ midpoint steps, finishing off with half a step of backward Euler.

Conclude that these two symmetric schemes are *dynamically equivalent*, i.e., for h small enough their performance is very similar independently of N, even over a very long time: $b = Nh \gg 1$.

(d) However, if h is not small enough (compared to the problem's small parameter, say λ^{-1}) then these schemes do not necessarily perform similarly. Construct an example where one of these schemes blows up (error $> 10^5$, say) while the other yields an error below 10^{-5}. [Do not program anything: this is a pen-and-paper question.]

4. Write a short program which uses the forwad Euler, the backward Euler and the trapezoidal *or* midpoint schemes to integrate a linear, scalar ODE with a known solution, using a fixed step size $h = b/N$, and finds the maximum error. Apply your program to the following problem

$$\frac{dy}{dt} = (\cos t)y, \qquad 0 \le t \le b$$

$y(0) = 1$. The exact solution is

$$y(t) = e^{\sin t}$$

b	N	forward Euler	backward Euler	trapezoidal	midpoint
1	10	.35e-1	.36e-1	.29e-2	.22e-2
	20	.18e-1	.18e-1	.61e-3	.51e-3
10	100				
	200				
100	1000	2.46	25.90	.42e-2	.26e-2
	2000				
1000	1000				
	10000	2.72	1.79e+11	.42e-2	.26e-2
	20000				
	100000	2.49	29.77	.42e-4	.26e-4

Table 0.1: Maximum errors for long interval integration of $y' = (\cos t)y$

Verify those entries given in Table 0.1 and complete the missing ones. Make as many (useful) observations as you can on the results in the complete table. Attempt to provide explanations. [Hint: plotting these solution curves for $b = 20$, $N = 10b$, say, may help.]