1. Consider the Cauchy problem for the constant coefficient PDE

\[ u_t = P(\partial_x) u, \quad P(\partial_x) = \sum_{j=1}^{m} p_j \frac{\partial^j}{\partial x^j}. \]

(a) Assuming that \( p_m \) is a complex scalar, show that if \( i^m p_m \) has a positive real part, then the problem cannot be well-posed.

(b) Assuming that \( p_j \) are all real, \( m \) is odd, and \( p_{2l} = 0, \ l = 0, 1, \ldots, (m - 1)/2 \), show that

\[ |e^{P(i\xi)t}| = 1, \ -\infty < \xi < \infty. \]

What does this imply regarding the smoothing properties of the solution operator? Does integrating backward in time lead to a well-posed problem?

2. The celebrated Black-Scholes model for the pricing of stock options is central in mathematical finance. The PDE is given by

\[ u_t + \frac{1}{2} \sigma^2 x^2 u_{xx} + r x u_x - r u = 0, \quad 0 < x < \infty, \quad t \leq T. \tag{1} \]

For the sake of completeness let us add that \( u \) is the sought value of the option under consideration, \( t \) is time, \( x \) is the current value of the underlying asset, \( r \) is the interest rate, \( \sigma \) the volatility of the underlying asset, \( T \) the expiry date and \( E \) is the exercise price. In general, \( r \) and \( \sigma \) may vary, but here they are assumed to be known constants, as are \( E \) and \( T \).

For the European call option we have the terminal condition

\[ u(T, x) = \max(x - E, 0), \tag{2a} \]

and the boundary conditions

\[ u(t, 0) = 0, \quad u(t, x) \sim x - E e^{-r(T-t)} \quad \text{as} \ x \to \infty. \tag{2b} \]
(a) Show that the transformation

\[ x = E e^y, \quad t = T - \frac{2s}{\sigma^2}, \quad u = Ev(s, y), \]

results in the initial value PDE

\[ v_s = v_{yy} + (\kappa - 1)v_y - \kappa v, \quad -\infty < y < \infty, \quad v(0, y) = \max(e^y - 1, 0), \]

where \( \kappa = \frac{2r}{\sigma^2} \).

(b) Show further that transforming

\[ v = e^{\gamma y + \beta s} w(s, y), \quad \text{where} \]
\[ \gamma = \frac{(1 - \kappa)}{2}, \quad \beta = -\frac{(\kappa + 1)^2}{4}, \]

yields the PDE problem

\[ w_s = w_{yy}, \quad -\infty < y < \infty, \quad s \geq 0, \quad w(0, y) = \max(e^{\frac{1}{2}(\kappa + 1)y} - e^{\frac{1}{2}(\kappa - 1)y}, 0). \]

(c) Prove that the terminal-value PDE (1)-(2) is well-posed.

[Note that the solution of (4), and therefore also of (3) and (1)-(2), can be specified exactly in terms of the integral

\[ N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\zeta^2/2} d\zeta. \]

However, you don’t need this for the purpose of the present exercise.]

3. Consider the advection equation

\[ u_t + au_x = 0, \]

and recall that the consistent scheme (1.15b) is unconditionally unstable. The Lax-Friedrichs scheme is a variation:

\[ v_j^{n+1} = \frac{1}{2} (v_{j-1}^n + v_{j+1}^n) - \frac{\mu a}{2} (v_{j+1}^n - v_{j-1}^n). \]

Show that the Lax-Friedrichs scheme is stable, provided that the CFL condition holds.

4. Carry out calculations using the three difference schemes (1.15) introduced in class and in the text for the problem

\[ u_t = 2u_x, \quad u(0, x) = u_0(x) = \sin(\eta x), \]
with periodic boundary conditions on $[-\pi, \pi]$. Set $\eta = 2$, $\mu = 0.4$, and employ the three spatial step sizes $h = .1\pi$, $.01\pi$ and $.001\pi$. Record the maximum errors at $t = 1$ using the three schemes. Try also $\eta = 1$ and $\eta = 10$ to see trends, but do not report the obtained errors. What are your observations?

5. The TR-BDF2 is a one-step method for the ODE $y' = f(t, y)$ consisting of applying first the trapezoidal scheme over half a step $k/2$ to approximate the midpoint value, and then the BDF2 scheme over one step:

$$
y_{n+1/2} = y_n + \frac{k}{4}(f(y_n) + f(y_{n+1/2})),
$$
(5a)

$$
y_{n+1} = \frac{1}{3}[4y_{n+1/2} - y_n + kf(y_{n+1})].
$$
(5b)

One advantage is that only two systems of the original size need be solved per time step.

(a) Write the method (5) as a Runge-Kutta method in standard tableau form (i.e. find $A$ and $b$). This is an instance of a diagonally implicit Runge-Kutta (DIRK) method: please explain this name.

(b) Show that both the order and the stage order equal 2.

(c) Show that the stability function satisfies $R(-\infty) = 0$: this method is L-stable and has stiff decay.

(d) Can you construct an example where this method would fail where the BDF2 method would not?