Lecture Stat 302
Introduction to Probability - Slides 9

AD

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Example: Optimal Stock

- Vancouver 2010 Olympic Torchbearer red mittens are currently sold at a net profit of b\$ for each unit sold and will bring a net loss of l\$ for each unit left unsold after the Olympics. The number of units of the product sold at a specific store is a random variable X with pmf $p(i)$, $i \geq 0$. If the store must stock this product in advance, compute the expected profit for a determined number of units s. Find $\widehat{s}$ which optimizes the maximum expected profit.
- The profit is equal to

$$\begin{aligned} Y(s) &= bX - l(s - X) \text{ if } X \leq s \\ &= sb \qquad\qquad\quad \text{if } X > s \end{aligned}$$

- Hence the expected profit is

$$\begin{aligned} E[Y(s)] &= \sum_{i=0}^{s} (bi - l(s-i))\, p(i) + \sum_{i=s+1}^{\infty} sbp(i) \\ &= sb + (b+l) \sum_{i=0}^{s} (i-s)\, p(i) \end{aligned}$$

Example: Optimal Stock

- We want to maximize the profit. We note that

$$E\left[Y\left(s+1\right)\right]-E\left[Y\left(s\right)\right]=b-\left(b+l\right)\sum_{i=0}^{s}p\left(i\right)$$

so we have

$$E\left[Y\left(s+1\right)\right]-E\left[Y\left(s\right)\right]\geq 0$$

is equivalent to

$$\sum_{i=0}^{s}p\left(i\right)<\frac{b}{b+l}.$$

- So we should consider s^{*} the largest value such that $\sum_{i=0}^{s^{*}}p\left(i\right)<\frac{b}{b+l}$ then

$$E\left(Y\left(0\right)\right)<\cdots<E\left(Y\left(s^{*}\right)\right)<E\left(Y\left(s^{*}+1\right)\right)>E\left(Y\left(s^{*}+2\right)\right)>...$$

so the optimal stock is $\widehat{s}=s^{*}+1$.

Variance

- A r.v. X is entirely defined by its p.m.f. but it is very useful to be able to summarize the essential properties of it through a few suitably defined measures.
- We have defined previously the expected value $E(X)$. However this measure tells us nothing about the variations of X; e.g. consider $X = 0$ with proba. 1, $Y = 1$ w.p. 0.5 and $Y = -1$ w.p. 0.5 and finally $Z = 100$ w.p. and $Z = -100$ w.p. 0.5 then

$$E(X) = E(Y) = E(Z) = 0$$

  whereas there is a much greater spread/variability in the values of Y than in the deterministic value of X and the variability of Z than Y.
- We expect X to be around its expected value/mean $\mu = E(X)$ so it would make sense to quantify the average deviations from it. This is what the variance achieves.

Variance

- If X is a r.v. with expected value/mean $\mu = E(X)$ then the variance is
$$Var(X) = E\left((X-\mu)^2\right)$$
and $Std(X) = \sqrt{Var(X)}$ is called standard deviation.
- We have
$$Var(X) = E(X^2) - \mu^2$$
- **Example**: Consider a fair die with $X = i$ is "number i rolled" then we have seen that $E(X) = \frac{7}{2}$ and
$$\begin{aligned} E(X^2) &= 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} \\ &= \frac{1+4+9+16+25+36}{6} = \frac{91}{6} \end{aligned}$$
so
$$Var(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

Example: Variance of Payout of Roulette Wheel

- *Red/Black*: $P(X_1 = 1) = \frac{18}{38}$, $P(X_1 = -1) = \frac{20}{38}$ and $E(X_1) = -\frac{1}{19}$ so

$$Var(X_1) = 1^2 \times \frac{18}{38} + (-1)^2 \times \frac{20}{38} - \left(-\frac{1}{19}\right)^2 = 0.99$$

- *Straight up*: $P(X_2 = 35) = \frac{1}{38}$, $P(X_2 = -1) = \frac{37}{38}$ and $E(X_2) = -\frac{1}{19}$ so

$$Var(X_2) = 35^2 \times \frac{1}{38} + (-1)^2 \times \frac{37}{38} - \left(-\frac{1}{19}\right)^2 = 33.21.$$

- *Split bet*: $P(X_3 = 17) = \frac{2}{38}$, $P(X_3 = -1) = \frac{36}{38}$, $E(X_3) = -\frac{1}{19}$ so

$$Var(X_3) = 17^2 \times \frac{2}{38} + (-1)^2 \times \frac{36}{38} = 16.15.$$

- *Street bet*: $P(X_4 = 11) = \frac{3}{38}$, $P(X_4 = -1) = \frac{35}{38}$, $E(X_4) = -\frac{1}{19}$ so

$$Var(X_4) = 11^2 \times \frac{3}{38} + (-1)^2 \times \frac{35}{38} - \left(-\frac{1}{19}\right) = 10.47.$$

Properties

- Consider $Y = aX + b$ then

$$Var(Y) = a^2 Var(X)$$

- We have

$$\begin{aligned} Var(Y) &= E\left((Y - E(Y))^2\right) \\ &= E\left[\left(aX + b - \underbrace{(aE(X) + b)}_{=E(Y)}\right)^2\right] \\ &= E\left[a(X - E(X))^2\right] \\ &= a^2 E\left[(X - E(X))^2\right] = a^2 Var(X) \end{aligned}$$

Bernoulli Distribution

- Assume you have an experiment which can be classified as either success or failure.
- If you associate a random variable X such that $X = 1$ corresponds to a success and $X = 0$ corresponds to a failure then

$$
\begin{aligned}
P(X = 1) &= p(1) = p \\
P(X = 0) &= p(0) = 1 - p.
\end{aligned}
$$

- Such a random variable is a called a *Bernoulli* random variable.
- We have

$$E(X) = p \text{ and } Var(X) = p(1 - p).$$

Binomial Distribution

- Consider a sequence of n independent success/failure experiments, each of which yields success with probability p.
- Let X denote the number of successes then we have $X = X_1 + X_2 + \cdots + X_n$ where X_i is a Bernoulli random variable with proba. p. It is a binomial random variable of parameters (n, p).
- X is a discrete random variables in $\{0, 1, ..., n\}$ such that

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

- This expression follows from the fact that any specific sequence with exactly k successes (hence $n-k$ failures) has a proba $p^k (1-p)^{n-k}$. There are $\binom{n}{k}$ such sequences.

Example

- *Example*: Everyday one of your classmates is taking the tram twice without any ticket. There is a 5% chance of meeting inspectors each time one boards the tram. What is the proba that he is going to be caught at least twice in a 30 day month?
- *Answer*: Let X be the number of times he will meet inspectors per year. X is a binomial random variable of parameters $n = 2 \times 30$, $p = 0.05$ and

$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= \binom{n}{0} p^0 (1-p)^n + \binom{n}{1} p^1 (1-p)^{n-1} \\ &= (1-p)^n + np(1-p)^{n-1} \\ &= 0.04671 + 0.1455 = 0.4174 \end{aligned}$$

so

$$P(X \geq 2) = 1 - P(X \leq 1) = 0.728$$

Example

- *Example*: Your friend pretends that he is able to distinguish pepsi from coke. You give him a test. Out of 5 trials, he gets the right answer 4 times. What is the probability that he just got lucky?
- *Answer*: Let X be the number of right answers. If he answers completely randomly, then X is a binomial of parameters $n = 5$ and $p = \frac{1}{2}$. We have

$$\begin{aligned} P(X \geq 4) &= P(X = 4) + P(X = 5) \\ &= \binom{5}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{5-4} + \binom{5}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{5-0} \\ &= 0.1875. \end{aligned}$$

  This is no convincing evidence that your friend can distinguish the two beverages.
- This is related to what is called p-value in statistics: the probability that you would have observed such results just by pure luck.

Example

- *Example*: A survey from Teenage Research Unlimited (Northbrook, Ill.) found that 30% of teenage consumers receive their spending money from part-time jobs. If five teenagers are selected at random, find the probability that at least three of them will have part-time jobs.
- *Answer*: Let X be the number of teenagers having a part-time job among 5 teenagers, then X is Binomial of parameters $n = 5$, $p = 0.3$ so

$$\begin{aligned} P(X \geq 3) &= P(X = 3) + P(X = 4) + P(X = 5) \\ &= 0.132 + 0.028 + 0.002 = 0.162 \end{aligned}$$

- *Example*: What is the probability of obtaining 45 or fewer heads in 100 tosses of a fair coin?
- *Solution*: Let X be the number of heads, then X is binomial of parameters $n = 100$, $p = 0.5$ and

$P(X \leq 45) = \sum_{k=0}^{45} \binom{n}{k} p^k (1-p)^k = 0.184.$

Mean of the Binomial Random Variable

- The mean/expected value of X is given by

$$E(X) = np$$

- **Proof**. We have

$$\begin{aligned} E(X) &= \textstyle\sum_{k=0}^{n} k \ P(X=k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \\ &= \textstyle\sum_{k=1}^{n} k \ \frac{n \ (n-1)!}{k \ (k-1)! \ (n-k)!} p \ p^{k-1} (1-p)^{n-k} \\ &= np \ \textstyle\sum_{k=1}^{n} \ \frac{(n-1)!}{(k-1)! \ (n-k)!} \ p^{k-1} (1-p)^{n-k} \\ &= np \ \textstyle\sum_{l=0}^{m} \ \frac{m!}{l! \ (m-l)!} \ p^{l} (1-p)^{m-l} \quad (m \leftarrow n-1, \ l \leftarrow k-1) \\ &= np \end{aligned}$$

- A much more straightforward proof follows from $X = X_1 + X_2 + \cdots + X_n$ where X_i is a Bernoulli random variable with proba. p so

$$\begin{aligned} E(X) &= E(X_1 + \cdots + X_n) \\ &= E(X_1) + \cdots + E(X_n) = np. \end{aligned}$$