Lecture Stat 302
Introduction to Probability - Slides 3

AD

Jan. 2010

Sample Space

- **Definition**. The *sample space* S of an experiment (whose outcome is uncertain) is the set of all possible outcomes of the experiment.
- *Example* (child): Determining the sex of a newborn child in which case $S = \{boy, girl\}$.
- *Example* (horse race): Assume you have an horse race with 12 horses. If the experiment is the order of finish in a race, then

$$S = \{\text{all 12! permutations of } (1, 2, 3, ..., 11, 12)\}.$$

- *Example* (coins): If the experiment consists of flipping two coins, then the sample space is

$$S = \{(H, H), (H, T), (T, H), (T, T)\}.$$

- *Example* (lifetime): If the experiment consists of measuring the lifetime (in years) of your pet then the sample space consists of all nonnegative real numbers: $S = \{x; 0 \leq x < \infty\}$.

Events

- Any *subset* E of the sample space S is known as an *event*; i.e. an event is a set consisting of possible outcomes of the experiment.
- If the outcome of the experiment is in E, then we say that E has occurred.
- *Example* (child): The event $E = \{boy\}$ is the event that the child is a boy.
- *Example* (horse race): The event $E = \{\text{all outcomes in } S \text{ starting with a } 7\}$ is the event that the race was won by horse 7.
- *Example* (coins): The event $E = \{(H, T), (T, T)\}$ is the event that a tail appears on the second coin.
- *Example* (lifetime): The event $E = \{x : 3 \leq x \leq 5.5\}$ is the event that your pet will live more than 3 years but won't live more than 5 years and 6 months.

Union of Events

- Given events E and F, $E \cup F$ is the set of all outcomes *either* in E or F or in *both* E and F. $E \cup F$ occurs if *either* E or F occurs. $E \cup F$ is the **union** of events E and F
- *Example* (coins): If we have $E = \{(H, T)\}$ and $F = \{(T, H)\}$ then $E \cup F = \{(H, T), (T, H)\}$ is the event that one coin is head and the other is tail.
- *Example* (horse race): If we have $E = \{\text{all outcomes in } S \text{ starting with a } 7\}$ and $F = \{\text{all outcomes in } S \text{ finishing with a } 3\}$ then $E \cup F$ is the event that the race was won by horse 7 or/and the last horse was horse 3.
- *Example* (lifetime): If $E = \{x : 0 \leq x \leq 5\}$ and $F = \{x : 10 \leq x < \infty\}$ then $E \cup F$ is the event that your pet will die before 5 or will die after 10.
- If $\{E_i\}_{i \geq 1}$ are events then the union is denoted $\cup_{i=1}^{\infty} E_i$: it is the event which consists of all the outcomes in $\{E_i\}_{i \geq 1}$.

Intersection of Events

- Given events E and F, $E \cap F$ is the set of all outcomes which are *both* in E and F. $E \cap F$ is also denoted EF.
- *Example* (coins): If we have $E = \{(H, H), (H, T), (T, H\}$ (event that one H at least occurs) and $F = \{(H, T), (T, H), (T, T)\}$ (even that one T at least occurs) then $E \cap F = \{(H, T), (T, H)\}$ is the event that one H and one T occur.
- *Example* (horse race): If we have $E = \{\text{all outcomes in } S \text{ starting with a } 7\}$ and $F = \{\text{all outcomes in } S \text{ starting with a } 8\}$ then $E \cap F$ does not contain any outcome and is denoted by $\varnothing$.
- *Example* (lifetime): If we have $E = \{x : 0 \leq x \leq 5\}$ and $F = \{x : 3 \leq x < 7\}$ then $E \cap F = \{x : 3 \leq x \leq 5\}$ is the event that your pet will die between 3 and 5.
- If $\{E_i\}_{i \geq 1}$ are events then the intersection is denoted $\cap_{i=1}^{\infty} E_i$: it is the event which consists of the outcomes which are in all of the events $\{E_i\}_{i \geq 1}$.

Notation and Properties

- For any event E, E^c denote the *complement* set of all outcomes in S which are not in E. Hence we have $E \cup E^c = S$ and $E \cap E^c = \varnothing$.
- For any two events E and F, we write $E \subset F$ is all the outcomes of E are in F.
- "Algebra"
  - *Commutative laws*

$$E \cup F = F \cup E \text{ and } E \cap F = F \cap E.$$

  - *Associative laws*

$$\begin{aligned} (E \cup F) \cup G &= E \cup (F \cup G), \\ (E \cap F) \cap G &= E \cap (F \cap G). \end{aligned}$$

  - *Distributive laws*

$$\begin{aligned} (E \cup F) \cap G &= (E \cap G) \cup (F \cap G), \\ (E \cap F) \cup G &= (E \cup G) \cap (F \cup G). \end{aligned}$$

DeMorgan's Laws

- We have

$$\begin{aligned} \left(\cup_{i=1}^{n} E_i\right)^c &= \cap_{i=1}^{n} E_i^c \\ \left(\cap_{i=1}^{n} E_i\right)^c &= \cup_{i=1}^{n} E_i^c \end{aligned}$$

- *Proof*: Suppose x is an outcome of $\left(\cup_{i=1}^{n} E_i\right)^c$, then x is not in $\left(\cup_{i=1}^{n} E_i\right)$. Thus it is not in any of the event E_i, $i = 1, ..., n$. Hence it is in E_i^c for all $i = 1, ..., n$. Hence this proves $\left(\cup_{i=1}^{n} E_i\right)^c \subset \cap_{i=1}^{n} E_i^c$. Suppose x is an outcome of $\cap_{i=1}^{n} E_i^c$, hence it is in E_i^c for all $i = 1, ..., n$. Hence it is in none of the event E_i, $i = 1, ..., n$. Thus it is $\left(\cup_{i=1}^{n} E_i\right)^c$ and we have proven that $\cap_{i=1}^{n} E_i^c \subset \left(\cup_{i=1}^{n} E_i\right)^c$. The result $\left(\cup_{i=1}^{n} E_i\right)^c = \cap_{i=1}^{n} E_i^c$ follows.
- To prove $\left(\cap_{i=1}^{n} E_i\right)^c = \cup_{i=1}^{n} E_i^c$, we remark that

$$\underbrace{\left(\cup_{i=1}^{n} E_i^c\right)^c = \cap_{i=1}^{n} \left(E_i^c\right)^c}_{\text{previous result applied to events } \left\{E_i^c\right\}} \quad = \cap_{i=1}^{n} E_i \text{ as } \left(E_i^c\right)^c = E_i.$$

Thus by taking the complement on both sides, we obtain the result.

Axioms of Probability

- Consider an experiment with sample space S. For each event E, we assume that a number $P(E)$, the *probability* of the event E, is defined and satisfies the following 3 axioms.
- **Axiom 1**
$$0 \leq P(E) \leq 1$$
- **Axiom 2**
$$P(S) = 1$$
- **Axiom 3**. For any sequence of mutually exclusive events $\{E_i\}_{i\geq 1}$, i.e. $E_i \cap E_j = \varnothing$ when $i \neq j$, then
$$P\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$
- Direct consequences include $P(\varnothing) = 0$ and for mutually exclusive events $\{E_i\}_{i\geq 1}$
$$P\left(\cup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i).$$