Lecture Stat 302
Introduction to Probability - Slides 22

AD

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Characterizing Joint Distributions/Densities: Covariance

- Consider two r.v. X and Y (either discrete or continuous), then the **covariance** of (X, Y) is defined as

$$\begin{aligned} Cov\,(X, Y) &= E\left(\left(X - E\left(X\right)\right)\left(Y - E\left(Y\right)\right)\right) \\ &= E\left(XY\right) - E\left(X\right)E\left(Y\right) \end{aligned}$$

- The covariance measures the degree to which X and Y vary together. If the covariance is positive, X tends to be larger than its mean when Y is larger than its mean. The covariance of a variable with itself is the variance of that variable.

Independent Variables and Covariance

- If X and Y are two independent r.v. then

$$Cov\left(X,Y\right)=0$$

- **Proof.** We are going to show that $E\left(XY\right)=E\left(X\right)E\left(Y\right)$ if X and Y are independent

$$\begin{aligned}
E\left(XY\right) &= \int\int xy\cdot f\left(x,y\right)dxdy \\
&= \int\int xy\cdot f_X\left(x\right)f_Y\left(y\right)dxdy \text{ (independence)} \\
&= \left[\int x\cdot f_X\left(x\right)dx\right]\left[\int y\cdot f_Y\left(y\right)dy\right] \\
&= E\left(X\right)E\left(Y\right)
\end{aligned}$$

Example: Two Stocks

- Let X and Y denote the values of two stocks at the end of a five-year period. X is uniformly distributed on $(0, 12)$. Given $X = x$, Y is uniformly distributed on the interval $(0, x)$. Determine $Cov(X, Y)$.
- We have for $0 < x < 12$ and $0 < y < x$

$$f(x, y) = f_X(x) f_{Y|X}(y|x) = \frac{1}{12}\frac{1}{x}$$

so

$$\begin{aligned} E(X) &= \int_0^{12} x \cdot \frac{1}{12} dx = 6, \\ E(Y) &= \int_0^{12}\int_0^{x} y \cdot \frac{1}{12}\frac{1}{x} dydx = 3, \\ E(XY) &= \int_0^{12}\int_0^{x} xy \cdot \frac{1}{12}\frac{1}{x} dydx = 24. \end{aligned}$$

Hence we have

$$Cov(X, Y) = 24 - 3 \times 6 = 6.$$

Sum of Random Variables

- Consider two random variables X and Y with variances σ_x^2 and σ_y^2 respectively. Let $Z = X + Y$ then

$$Var(Z) = Var(X) + Var(Y) + 2Cov(X, Y).$$

- **Proof**. We have $Var(Z) = E(Z^2) - E(Z)^2$ where

$$E(Z^2) = E\left((X+Y)^2\right) = E(X^2) + E(Y^2) + 2E(XY)$$

and

$$\begin{aligned} E(Z)^2 &= (E(X) + E(Y))^2 \\ &= E(X^2) + E(Y^2) + 2E(X)E(Y) \end{aligned}$$

and the result follows directly.

Example: Spreading your risk optimally

- You have 2 financial products whose returns can be modelled by the r.v. X and Y such that $E(X) = E(Y) = \mu$, $Var(X) = \sigma_x^2$, $Var(Y) = \sigma_y^2$ and $Cov(X, Y) = \sigma_{xy}$. (These two products are equally priced). You want to buy a proportion λ of product 1 and $(1 - \lambda)$ of product 2 where $\lambda \in [0, 1]$ to spread the risk.
- (a) What is the expectation of the total return $Z = \lambda X + (1 - \lambda) Y$?
- (b) What is the variance of the total return?
- (c) How should you select λ to minimize this variance?
- (d) What is the minimum variance of the return if X and Y are independent?

Example: Minimizing the Variance of Your Return

- (a) The total return is given by $Z = \lambda X + (1-\lambda) Y$ so
$$E(Z) = \lambda E(X) + (1-\lambda) E(Y) = \mu.$$
- (b) We have
$$\begin{aligned} Var(Z) &= \lambda^2 \sigma_x^2 + (1-\lambda)^2 \sigma_y^2 + 2\lambda (1-\lambda) \sigma_{xy} \\ &= \lambda^2 \left(\sigma_x^2 + \sigma_y^2 - 2\sigma_{xy}\right) + 2\lambda \left(\sigma_{xy} - \sigma_y^2\right) + \sigma_y^2 \end{aligned}$$
- (c) We just differentiate $Var(Z)$ w.r.t. λ and obtain
$$\lambda_{\text{opt}} = \frac{\sigma_y^2 - \sigma_{xy}}{\sigma_x^2 + \sigma_y^2 - 2\sigma_{xy}}$$
- (d) For X, Y independent, we have $\sigma_{xy} = 0$ so $\lambda_{\text{opt}} = \sigma_y^2 / \left(\sigma_x^2 + \sigma_y^2\right)$
$$Var(Z) = \frac{\sigma_y^4}{\left(\sigma_x^2+\sigma_y^2\right)^2}\sigma_x^2 + \frac{\sigma_x^4}{\left(\sigma_x^2+\sigma_y^2\right)^2}\sigma_y^2$$
$$= \frac{\sigma_x^2\sigma_y^2\left(\sigma_x^2+\sigma_y^2\right)}{\left(\sigma_x^2+\sigma_y^2\right)^2} = \frac{\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_y^2}$$

Example: Surgical Claim

- Let X denote the size of a surgical claim and let Y denote the size of the associated hospital claim. An actuary is using a model in which $E(X) = 5$, $E(X^2) = 27.4$, $E(Y) = 7$, $E(Y^2) = 51.4$ and $Var(X+Y) = 8$. Let $C_1 = X + Y$ denote the size of the combined claims before the application of a 20% surcharge on the hospital portion of the claim, and let C_2 the size of the combined claims after the application of that surcharge. Calculate $Cov(C_1, C_2)$.
- We have $C_1 = X + Y$ and $C_2 = X + 1.2Y$ so

$$
\begin{aligned}
Cov(C_1, C_2) &= E[(X+Y)(X+1.2Y)] - E[(X+Y)]\,E[(X+1.2Y)] \\
&= E(X^2) + 1.2E(Y^2) + 2.2E(XY) - E(X)^2 - 1.2E(Y)^2 - 2.2E(X)E(Y) \\
&= Var(X) + 1.2Var(Y) + 2.2Cov(X, Y)
\end{aligned}
$$

and $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X, Y)$ so
$Cov(X, Y) = \frac{1}{2}\left\{E(X^2) - E(X)^2 + E(Y^2) - E(Y)^2\right\} = 1.6$ and
$Cov(C_1, C_2) = 8.8.$

- The correlation of (X, Y) is defined as

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

- The correlation is a measure of "dependence" between X and Y. It is a uniteless measure which takes values in $[-1, 1]$. Proof can be established using Cauchy-Schwartz inequality $\left(\left(\int \alpha(u)\beta(u)\,du\right)^2 \leq \left(\int \alpha^2(u)\,du\right)\left(\int \beta^2(u)\,du\right)\right)$.
- If X and Y are two independent r.v. then $\rho(X, Y) = 0$ as $Cov(X, Y) = 0$.

Example: Two Stocks

- We want to compute the correlation

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

- We have computed $Cov(X, Y) = 6$ so we need to compute $Var(X)$ and $Var(Y)$. We have

$$E(X^2) = \int_0^{12} x^2 \cdot \frac{1}{12} dx = \frac{1}{12}\left[\frac{x^3}{3}\right]_0^{12} = \frac{144}{3} = 48,$$

$$E(Y^2) = \int_0^{12} \frac{1}{12}\frac{1}{x}\left(\int_0^x y^2 \cdot dy\right) dx = \frac{1}{36}\int_0^{12} x^2 dx = \frac{144}{9} = 16.$$

Hence

$$\rho(X, Y) = \frac{6}{\sqrt{48 - 6^2}\sqrt{16 - 3^2}} = 0.3631$$

Uncorrelated Variables Are Not Necessarily Independent

- Independence does imply uncorrelation but the reverse is NOT true.
- Counter example for discrete r.v.: let X be such that

$$P(X=-1) = P(X=0) = P(X=1) = \frac{1}{3}$$

and $Y = X^2$ then X and Y are dependent but X and Y are uncorrelated as

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= \underbrace{E(X^3)}_{=0} - \underbrace{E(X)}_{=0} E(X^2) = 0 \end{aligned}$$

- Counter example for continuous r.v.: let X be a standard normal and $Y = X^2$ then clearly X and Y are dependent but X and Y are uncorrelated as

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= \underbrace{E(X^3)}_{=0} - \underbrace{E(X)}_{=0} E(X^2) = 0 \end{aligned}$$