Lecture Stat 302
Introduction to Probability - Slides 18

AD

March 2010

Jointly Distributed Random Variables

- If both X and Y are continuous r.v., then their joint p.d.f. is a non-negative function $f(x, y)$ such that for any set C

$$P\{(X, Y) \in C\} = \int\int_{(x,y)\in C} f(x, y)\, dxdy.$$

- In particular, we have the following multivariate c.d.f.

$$F(a, b) = P(X \leq a, Y \leq b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x, y)\, dxdy.$$

and, when we differentiate, we obtain

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y).$$

- For X and Y are discrete r.v., then their joint p.m.f. is

$$P(X = x, Y = y) = p(x, y)$$

and

$$P\{(X, Y) \in C\} = \sum_{(x,y)\in C} p(x, y).$$

Independent Random Variables

- The r.v. X and Y are said to be independent if, for any sets A and B we have

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$$

- It can be shown that X and Y are independent if and only if

$$F(x, y) = F_X(x)\, F_Y(y)$$

  and

$$f(x, y) = f_X(x)\, f_Y(y)$$

  that is the joint c.d.f. (resp. the joint p.d.f.) is the product of the marginal c.d.f.s (resp. the marginal p.d.f.s)

Example

- The joint density of two r.v. X and Y is given by
$$f(x,y)=\begin{cases} x\ e^{-(x+y)} & \text{for } x>0 \text{ and } y>0 \\ 0 & \text{otherwise.}\end{cases}$$
Are the r.v. X and Y independent?
- We need to check whether $f(x,y)=f_X(x)\,f_Y(y)$ or not. We have for $x>0$
$$f_X(x)=\int_0^\infty f(x,y)\,dy = xe^{-x}\int_0^\infty e^{-y}dy = xe^{-x}$$
and for $y>0$
$$\begin{aligned} f_Y(y) &= \int_0^\infty f(x,y)\,dx = e^{-y}\int_0^\infty xe^{-x}dx \\ &= e^{-y}\left\{\left[-xe^{-x}\right]_0^\infty + \int_0^\infty e^{-x}dx\right\} \\ &= e^{-y}\end{aligned}$$
so X and Y are independent rv.

Example: Insurance Policies

- An insurance company sells two types of auto insurance policies: Basic and Deluxe. The time until the next Basic policy claim is an epxonential random variable with mean two days. The time until the next Deluxe policy claim is an independent exponential random variable with mean three days. What is the probability that the next claim will be a Deluxe policy claim?
- In this case, we have for $0 < t_1 < \infty$, $0 < t_2 < \infty$

$$f(t_1, t_2) = \left(\frac{1}{2}\exp(-t_1/2)\right)\left(\frac{1}{3}\exp(-t_2/3)\right)$$

and we want to find

$$P(T_2 < T_1) = \int_0^\infty \int_0^{t_1} f(t_1, t_2)\, dt_1 dt_2.$$

Example: Insurance Policies

- We have

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\begin{aligned}
P\left(T_2 < T_1\right) &= \int_0^\infty \left[\int_0^{t_1} f\left(t_1, t_2\right) dt_2\right] dt_1 \\
&= \int_0^\infty \exp\left(-t_1/2\right)\left[-\frac{1}{2}\exp\left(-t_2/3\right)\right]_0^{t_1} dt_1 \\
&= \int_0^\infty \left(\frac{1}{2}\exp\left(-t_1/2\right) - \frac{1}{2}\exp\left(-t_1/2\right)\exp\left(-t_1/3\right)\right) \\
&= \int_0^\infty \left(\frac{1}{2}\exp\left(-t_1/2\right) - \frac{1}{2}\exp\left(-5t_1/6\right)\right) dt_1 \\
&= \left[-\exp\left(-t_1/2\right) + \frac{3}{5}\exp\left(-5t_1/6\right)\right]_0^\infty = 1 - \frac{3}{5} = \frac{2}{5}.
\end{aligned}
$$

Example: Operational Cost

- A device containing two key components fails when, and only when, both components fail. The lifetimes, T_1 and T_2, of these components are independent with common density function $f(t) = \exp(-t)$ for $t \geq 0$. The cost X of operating the device until failure is $2T_1 + T_2$. What is the density function of X?
- We have $f_X(x) > 0$ only if $x \geq 0$. For $x \geq 0$, we have

$$\begin{aligned}
P(X \leq x) &= P(2T_1 + T_2 \leq x) \\
&= \int_0^x \exp(-t_2) \left[\int_0^{\frac{1}{2}(x-t_2)} \exp(-t_1)\, dt_1 \right] dt_2 \\
&= \int_0^x \exp(-t_2) \left[-\exp(-t_1) \right]_0^{\frac{1}{2}(x-t_2)} dt_2 \\
&= \int_0^x \exp(-t_2) \left[1 - \exp\left(-\frac{1}{2}(x - t_2) \right) \right] dt_2 \\
&= \left[-\exp(-t_2) + 2\exp\left(-\frac{x}{2}\right) \exp\left(-\frac{t_2}{2}\right) \right]_0^x
\end{aligned}$$

Example: Operational Cost

- We have

$$P(X \leq x) = \left[-\exp(-t_2) + 2\exp\left(-\frac{x}{2}\right)\exp\left(-\frac{t_2}{2}\right) \right]_0^x$$
$$= 1 - 2\exp\left(-\frac{x}{2}\right) + \exp(-x).$$

- Hence, the density is given by

$$f_X(x) = \frac{dP(X \leq x)}{dx}$$
$$= \exp\left(-\frac{x}{2}\right) - \exp(-x)$$

for $x \geq 0$.

Sum of Independent Random Variables

- Consider two integer-valued independent r.v. X and Y of respective p.m.f. $p_X(x)$ and $p_Y(y)$.
- Consider $Z = X + Y$, we want to compute the p.m.f. of Z denoted $p_Z(z)$.
- Assume $Y = y$ then $Z = z$ if and only if $X = z - y$ and

$$P(X = z - y \cap Y = y) = p_X(z - y)\, p_Y(y)$$

so, as Y can take integer values and the events $(X = z - y) \cap (Y = y)$ and $(X = z - y') \cap (Y = y')$ are mutually exclusive for $y \neq y'$, we have

$$p_Z(z) = \sum_{y=-\infty}^{\infty} p_X(z - y)\, p_Y(y).$$

Sum of Poisson Random Variables

- Consider two Poisson independent r.v. X and Y of respective p.m.f. $p_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}$ for $x = 0, 1, 2, ...$ and $p_Y(y) = e^{-\lambda'}\frac{\lambda'^y}{y!}$.
- Consider $Z = X + Y$, then we can prove that Z is Poisson of parameter $\lambda + \lambda'$.
- **Proof**: For $z \geq 0$ we have $Y = y \leq z$ that $X = z - y$

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\begin{aligned}
p_Z(z) &= e^{-\left(\lambda+\lambda'\right)} \sum_{y=0}^{z} \frac{\lambda^{(z-y)}}{(z-y)!} \frac{(\lambda')^y}{y!} \\
&= \frac{e^{-\left(\lambda+\lambda'\right)}}{z!} \sum_{y=0}^{z} \binom{z}{y} (\lambda')^y \lambda^{(z-y)} \\
&= \frac{e^{-\left(\lambda+\lambda'\right)}}{z!} \left(\lambda+\lambda'\right)^z \quad \text{(by binomial formula)}
\end{aligned}
$$

Sum of Independent Random Variables

- In numerous scenarios, we have to sum independent continuous r.v.; signal + noise, sums of different random effects etc.
- Assume that X, Y are continuous r.v. of respective pdf $f_X(x)$ and $f_Y(y)$ then $Z = X + Y$ admits the pdf

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\, dy \\ &= \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx \end{aligned}$$

- The pdf $f_Z(z)$ is the so-called "convolution" of $f_X(x)$ and $f_Y(y)$.

Proof

- We have

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\begin{aligned}
F_Z(z) &= P(Z \leq z) = P(X+Y \leq z) \\
&= \int\int_{(x+y\leq z)} f_X(x)\ f_Y(y)\ dxdy \\
&= \int\int_{(y\in\mathbb{R};x\leq z-y)} f_X(x)\ f_Y(y)\ dxdy \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-y} f_X(x)\,dx \right)\ f_Y(y)\,dy \\
&= \int_{-\infty}^{\infty} F_X(z-y)\ f_Y(y)\,dy
\end{aligned}
$$

- Now we obtain

$$
\begin{aligned}
f_Z(z) &= \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} \frac{dF_X(z-y)}{dz}\ f_Y(y)\,dy \\
&= \int_{-\infty}^{\infty} f_X(z-y)\ f_Y(y)\,dy\ \ \text{(chain rule)}
\end{aligned}
$$

Sum of Uniform Random Variables

- In this case, we have $f_X(x) = 1_{[0,1]}(x)$ and $f_Y(y) = 1_{[0,1]}(y)$ so $f_Z(z)$ is non-null in $[0,2]$
- We have

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(z-y)\ f_Y(y)\,dy \\ &= \int_0^1 f_X(z-y)\ dy = \int_0^1 1_{[0,1]}(z-y)\,dy \end{aligned}$$

where

$$\int_0^1 1_{[0,1]}(z-y)\,dy = \int_{z-1}^{z} 1_{[0,1]}(u)\,du = \begin{cases} z & \text{if } 0 \le z \le 1 \\ 2-z & \text{if } 1 \le z \le 2 \end{cases}$$