Lecture Stat 302
Introduction to Probability - Slides 16

AD

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Change of Variables

- Let X be a r.v. of pdf $f_X(x)$ and consider the r.v. $Y = g(X)$.
- A legitimate question is to ask what is the pdf $f_Y(y)$ of Y.
- This has numerous applications: converting measurements, computing returns on investments etc.

Example: Conversion Celsius to Farenheit

- Consider X the temperature at a given time instant in Celsius. It is assumed that X has a pdf $f_X(x)$ and associated distribution function $F_X(x)$. Assume you want to convert this temperature in Fahrenheit, hence you introduce the r.v.

$$Y = \frac{9}{5}X + 32.$$

What is the distribution $F_Y(y)$ and its associated density $f_Y(y)$?

- Consider the general case where $Y = aX + b$ then for $a > 0$

$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) = \Pr(aX + b \leq y) = \Pr\left(X \leq \frac{y-b}{a}\right) \\ &= F_X\left(\frac{y-b}{a}\right) \Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right). \end{aligned}$$

- For $a < 0$, $F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right)$ so $f_Y(y) = \frac{-1}{a} f_X\left(\frac{y-b}{a}\right)$.

Example: Polar to Cartesian coordinates

- Assume you evolve on a circle of radius $R = 1$. The cartesian coordinates associated to an angle θ are

$$X = \cos\theta, \;\; Y = \sin\theta.$$

  Assuming that the angle θ is distributed uniformly on $\left[0, \frac{\pi}{2}\right]$, what is the pdf of Y?

- We have $P\left(Y \leq y\right) = 0$ for $y \leq 0$ and $P\left(Y \leq y\right) = 1$ for $y \geq 1$. For $0 \leq y \leq 1$

$$P\left(Y \leq y\right) = P\left(\sin\theta \leq y\right) = P\left(\theta \leq \sin^{-1} y\right) = \frac{2}{\pi}\sin^{-1} y.$$

- By differentiating $P\left(Y \leq y\right)$, we obtain

$$f_Y\left(y\right) = \frac{2}{\pi\sqrt{1 - y^2}} \text{ for } 0 \leq y \leq 1$$

Example: Squared random variable

- Consider a r.v. X of pdf $f_X(x)$ and associated distribution function $F_X(x)$. You are not interested in X per se but in $Y = X^2$. What is the distribution $F_Y(y)$ and its associated density $f_Y(y)$?
- For $y \leq 0$, we have $F_Y(y) = 0$ and $f_Y(y) = 0$. For $y \geq 0$, we have

$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) = \Pr\left(X^2 \leq y\right) \\ &= \Pr(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

- Using the chain rule, we obtain

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}).$$

Change of variables

- Consider the case where $g(x)$ is a strictly monotonic differentiable function of x. Then the r.v. $Y = g(X)$ has a pdf given by

$$f_Y(y) = \begin{cases} f_X\left(g^{-1}(y)\right)\left|\frac{dg^{-1}(y)}{dy}\right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

- Proof follows for increasing $g(x)$ from

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(g(X) \leq y) \\ &= P\left(X \leq g^{-1}(y)\right) \text{ if } y = g(x) \text{ for some } x \\ &= F_X\left(g^{-1}(y)\right) \end{aligned}$$

- By the chain rule,we obtain

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dg^{-1}(y)}{dy} f_X\left(g^{-1}(y)\right)$$

Example: Investment

- An investment account earns an annual interested rate R that follows a uniform distribution on the interval $(0.04, 0.08)$. The value of a 10,000$ initial investment in this account after one year is given by $V = 10,000 \exp(R)$. Determine the cdf and pdf of V.
- We have $F_V(v) = 0$ for $v \leq 10,000 \times \exp(0.04)$ and for $v \geq 10,000 \times \exp(0.04)$

$$\begin{aligned} F_V(v) &= P(V \leq v) = P(R \leq \log v - \log 10,000) \\ &= \frac{1}{0.04} \int_{0.04}^{\log v - \log 10,000} dr = 25 \log(v) - 25 \log(10,000) - 1 \\ &= 25 \left[\log \frac{v}{10,000} - 0.04 \right] \end{aligned}$$

so $f_V(v) = \frac{dF(v)}{dv} = \frac{25}{v}$ or directly for
$g(r) = 10,000 \exp(r) = v \Leftrightarrow r = g^{-1}(v) = \log \frac{v}{10,000}$

$$f_V(v) = \left| \frac{dg^{-1}(v)}{dv} \right| f_R\left(g^{-1}(v)\right) = \frac{1}{v} \frac{1}{0.04} = \frac{25}{v}$$

Example: Cauchy distribution

- Consider a real r.v. X of pdf

$$f_X(x) = \frac{1}{\pi}\frac{1}{(1+x^2)}$$

What is the pdf of the r.v. $Y = g(X) = 1/X$?

- We have $X = g^{-1}(Y) = 1/Y$

$$\begin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)\right)\left|\frac{dg^{-1}(y)}{dy}\right| \\ &= f_X(1/y)\left|-\frac{1}{y^2}\right| = \frac{1}{\pi}\frac{1}{(1+1/y^2)} \times \frac{1}{y^2} \\ &= \frac{1}{\pi}\frac{1}{(1+y^2)} = f_X(y). \end{aligned}$$