Lecture Stat 302
Introduction to Probability - Slides 1

AD

Jan. 2010

Administrative details

- Arnaud Doucet, Office LSK 308c, Department of Statistics & Office ICCS 189, Department of Computer Science.
- Office Hour, Department of Statistics: via appointment.
- Textbook: *A first course in probability*, 8th edition by Sheldon Ross.
- Website: http://people.cs.ubc.ca/~arnaud/stat302.html

Notes, exercises and exams

- Notes posted on the web + Additional material.
- Exercises & solutions from textbook will be posted weekly.
- 3 assignements (20%).
- 1 mid-term (30%).
- 1 final exam (50%).

Introduction to Probability

- Probability theory yields mathematical tools to deal with uncertain events.
- Used everywhere nowadays and its importance is growing.
- Applications include
  - Population genetics: tree-valued stochastic processes,
  - Web search engine: Markov chain theory,
  - Data mining, Machine learning: Stochastic gradient, Markov chain Monte Carlo,
  - Image processing: Markov random fields,
  - Design of wireless communication systems: random matrix theory,
  - Optimization of engineering processes: simulated annealing, genetic algorithms,
  - Computer-aided design of polymers: Markov chain Monte Carlo.
  - Finance (option pricing, volatility models): Monte Carlo, dynamic models,
  - Design of atomic bomb (Los Alamos): Markov chain Monte Carlo.

Plan of the Course - Tentative Schedule

1. Combinatorial analysis; i.e counting (1 week)
2. Axioms of probability (1 week)
3. Conditional probability and inference (1 week)
4. Discrete & continuous random variables (2 weeks)
5. Multivariate random variables (1 week)
6. Properties of expectation, generating function (2 weeks)
7. Limit theorems: SLLN, CLT, inequalities (2 weeks)
8. Additional topics: Poisson and Markov processes (1 week)
9. Simulation and Monte Carlo methods (1 week)

Combinatorial analyses aka Counting

- Many **basic** probability problems are counting problems.
- *Example*: Assume there are 1 man and 2 women in a room. You pick a person randomly. What is the probability P_1 that this is a man? If you pick two persons randomly, what is the probability P_2 that these are a man and woman
- *Answer*: You have the possible outcomes: (M), (W1), (W2) so

$$P_1 = \frac{\#\text{ "successful" events}}{\#\text{ events}} = \frac{\#\text{ boys}}{\#\text{ boys} + \#\text{ girls}} = \frac{1}{3}.$$

  To compute P_2, you can think of all the possible events: (M,W1), (M,W2), (W1,W2) so

$$P_2 = \frac{\#\text{ "successful" events}}{\#\text{ events}} = \frac{2}{3}.$$

- Both problems consists of counting the number of different ways that a certain event can occur.

Basic Principle of Counting

- **Basic Principle of Counting**: Suppose that two experiments are to be performed. Then if experiment 1 can results in any one of n_1 possible outcomes and if, for each outcome of experiment 1, there are n_2 possible outcomes of experiment 2, then there are $n_1 \times n_2$ possible outcomes of the two experiments.
- *Example*: A football tournament consists of 14 teams, each of which has 11 players. If one team and one of its players are to be selected as team and player of the year, how many different choices are possible?
- *Answer*: Selecting the team can be regarded as the outcome of the first experiment and the subsequent choice of one of its player as the outcome of the second experiment, so there are $14 \times 11 = 154$ possibilities.

Generalized Principle of Counting

- **Generalized Principle of Counting**: If r experiments that are to be performed are such that the 1st one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the 2nd experiment; and if, for each of the $n_1 \times n_2$ possible outcomes of the first two experiments, there are n_3 possible outcomes of the 3rd experiment; and if..., then there is a total of $n_1 \cdot n_2 \cdot \cdots \cdot n_r$ possible outcomes of the r experiments.
- *Example*: A university committee consists of 4 undergrads, 5 grads, 7 profs and 2 non-university persons. A subcommitte of 4, consisting of 1 person from each category, is to be chosen. How many different subcommittees are possible?
- *Answer*: The choice of a subcommittee is the combined outcome of the four separate experiments of choosing a single representative from each of the categories. So it follows that, there are

$$4 \times 5 \times 7 \times 2 = 280$$

possible subcommittees.

Generalized Principle of Counting: More Examples

- *Example*: How many different 6-place license plates are possible if the first 3 places are to be occupied by letters and the final 3 by numbers (BC format, http://www.canplates.com/bc.html)?
- *Answer*: We have simply

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,760,000.$$

- *Example*: How many functions defined on n points are possible if each functional value is either 0 or 1?
- *Answer*: We have

$$f = (f(1), f(2), ..., f(n))$$

where $f(i) \in \{0, 1\}$ so there are

$$\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$$

possibilities.

- *Example*: How many different 6-place license plates are possible if the first 3 places are to be occupied by letters, the final 3 by numbers and if repetition among letters were prohibited, repetition among numbers were prohibited, repetition among both letters and numbers were prohibited?
- *Answer*: We have

letters prohib.	$26 \times 25 \times 24 \times 10 \times 10 \times 10 = 15,600,000$
numbers prohib.	$26 \times 26 \times 26 \times 10 \times 9 \times 8 = 12,654,720$
letters & numbers prohib.	$26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11,232,200$

Permutations

- *Example*: Consider the acronym UBC. How many different ordered arrangements of the letters U, B and C are possible?
- *Answer*: We have (B,C,U), (B,U,C), (C,B,U), (C,U,B), (U,B,C) and (U,C,B); i.e. 6 possible arrangement. Each arrangement is known as a *permutation*.
- **General Result**. Suppose you have n objects. The number of permutations of these n objects is given by

$$n\,(n-1)\,(n-2)\cdots 3\cdot 2\cdot 1 = n!$$

  Remember the convention $0! = 1$.
- *Example*: Assume we have an horse race with 12 horses. How many possible rankings are (theoretically) possible?

$$12! = 479,001,600$$

Permutations: Examples

- *Example*: A class in "Introduction to Probability" consists of 40 men and 30 women. An examination is given and the students are ranked according to their performance. Assume that no two students obtain the same score.
  - How many different rankings are possible?
  - If the men are ranked among themselves and the women among themselves, how many different rankings are possible?
- *Solution*:
  - Each ranking corresponds to a particular ordered arrangement of the 40+30=70 people. So there are $70! = 1.1970 \cdot 10^{100}$ possible rankings.
  - Number of ranking for men is $40! = 8.1592 \cdot 10^{47}$ and for women $30! = 2.6525 \cdot 10^{32}$.

Permutations: More Examples

- *Example*: You have 10 textbooks that you want to order on your bookshelf: 3 mathematics books, 3 physics books, 2 chemistry books and 2 biology books. You want to arrange them so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?
- *Solution*: For each ordering of the subject, say M/P/C/B or P/B/C/M, there are

$$3!3!2!2! = 144$$

arrangements. As there are 4! ordering of the subjects, then you have

$$144 \cdot 4! = 3456$$

possible arrangements.

Permutations: More Examples

- *Example*: How many different letter arrangements can be formed from the letters EEPPPR?
- *Solution*: There are 6! possible permutations of letters $E_1E_2P_1P_2P_3R$ but the letters are not labelled so we cannot distinguish E_1 and E_2 and P_1, P_2 and P_3; e.g. $E_1P_1E_2P_2P_3R$ cannot be distinguished from $E_2P_1E_1P_2P_3R$ and $E_2P_2E_1P_1P_3R$. That is if we permuted the E's and the P's among themselves then we still have EPEPPR. We have 2!3! permutations of the labelled letters of the form EPEPPR. Hence there are
$$\frac{6!}{2!3!} = 60$$
possible arrangements of the letters EEPPPR.
- **General Result**. Suppose you have n objects. The number of different permutations of these n objects of which n_1 are alike, n_2 are alike,..., n_r are alike is given by
$$\frac{n!}{n_1!n_2!\cdots n_r!} := \left(\begin{array}{c} n \\ n_1, n_2, \ldots, n_r \end{array} \right).$$

Permutations: More Examples

- *Example*: A speed skating tournament has 4 competitors from South Korea, 3 from Canada, 3 from China, 2 from the USA and 1 from France. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possibles?
- *Answer*: We simply have

$$\frac{(4+3+3+2+1)!}{4!3!3!2!1!} = 3,603,600.$$

Combinations

- We want to determine the number of different groups of r objects that could be formed from a total of n objects.
- *Example*: How many different groups of 3 could be selected from A,B,C,D and E?
- *Answer*: There are 5 ways to select the 1st letter, 4 to select the 2nd and 3 to select the 3rd so $5 \cdot 4 \cdot 3 = 60$ ways to select WHEN the order in which the items are selected is relevant. When it is not relevant, then say the group BCE is the same as BEC, CEB, CBE, EBC, ECB; there are $3! = 6$ permutations. So when the order is irrelevant, we have $60/6 = 10$ different possible groups.
- **General result**: When the order of selection is relevant, there are

$$n\,(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

possible groups. When the order of selection is irrelevant, there are

$$\frac{n!}{(n-r)!r!} := \binom{n}{r} \quad \text{Binomial coefficient}$$

Combinations: Examples

- *Example*: Assume we have an horse race with 12 horses. What is the possible number of combinations of 3 horses when the order matters and when it does not?
- *Answer*: When it matters, we have $\frac{12!}{(12-3)!} = 12 \cdot 11 \cdot 10 = 1320$ and when it does not matter, we have $\frac{12!}{(12-3)!3!} = 220$.
- *Example*: From a group of 5 women and 7 men, how many different committers consisting of 2 women and 3 men can be form? What is 2 of the men are feuding and refuse to be serve on the committee together?
- *Answer*: We have $\begin{pmatrix} 5 \\ 2 \end{pmatrix} = 10$ possible W groups and $\begin{pmatrix} 7 \\ 3 \end{pmatrix} = 35$ possible M groups, so $10 \cdot 35 = 350$ groups. In the 35 groups, we have $5 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ groups where the 2 feuding men can be so there are $10 \cdot 30 = 300$ possible committees.

Combinations: More Examples

- *Example*: Assume we have a set of n antennas of which m are defective. All the defectives and all the functionals are indistiguishable. How many linear orderings are there in which no two defectives are consecutives?
- *Answer*: If two defectives antennas cannot be consecutive, then the space among the $n - m$ functional antennas can contain at most one defective antenna. There are $n - m + 1$ positions possible and select m of them to put the defective antennas, so that is

$$\binom{n-m+1}{m}$$

configurations which is obviously equal to zero if $m > n - m + 1$.