Stat 302 Midterm (201)

Q1. (a)

$$\begin{aligned}
P(\text{same face}) &= P(\text{both heads} \cup \text{both tails}) \\
&= P(\text{both heads}) + P(\text{both tails}) \quad \text{(disjoint events)} \\
&= P(\text{first head}) \times P(\text{second head}) \\
&\quad + P(\text{first tail}) \times P(\text{second tail}) \quad \text{(independent coins)} \\
&= p_1 p_2 + (1 - p_1)(1 - p_2) \\
&= 2p_1 p_2 - p_1 - p_2 + 1.
\end{aligned}$$

(b)

$$\begin{aligned}
P(\text{two heads}|\text{same face}) &= \frac{P(\text{both heads} \cap \text{same face})}{P(\text{same face})} \\
&= \frac{P(\text{both heads})}{P(\text{same face})} \\
&= \frac{p_1 p_2}{2p_1 p_2 - p_1 - p_2 + 1}.
\end{aligned}$$

Q2. (a) Denote X as the number of heads, and we know that X follows a binomial distribution $Binomial(10, 0.5)$.

$$P(X = 5) = \binom{10}{5} 0.5^5 0.5^5 = 0.246.$$

(b)

$$\begin{aligned}
& P(3 \text{ heads in former 5 tosses} \cap 5 \text{ heads}) \\
= \ & P(3 \text{ heads in former 5 tosses} \cap 2 \text{ tails in latter 5 tosses}) \\
= \ & P(3 \text{ heads in former 5 tosses}) \times P(2 \text{ tails in latter 5 tosses}) \\
= \ & \binom{5}{3} 0.5^3 0.5^2 \times \binom{5}{2} 0.5^2 0.5^3 \\
= \ & 0.099.
\end{aligned}$$

(c)

$$\begin{aligned}
& P(3\text{ heads in former 5 tosses}|5\text{ heads}) \\
= & \frac{P(3\text{ heads in former 5 tosses} \cap 2\text{ tails in latter 5 tosses})}{P(5\text{ heads})} \\
= & \frac{0.098}{0.246} \\
= & 0.397.
\end{aligned}$$

Q3. Define: TD = transformer damage; LD = line damage. We have $P(TD) = 0.04, P(LD) = 0.6, P(TD \cap LD) = 0.01$.

(a) $P(LD|TD) = \frac{P(TD \cap LD)}{P(TD)} = \frac{0.01}{0.04} = 0.25$.

(b) $P(TD|LD^c) = \frac{P(TD \cap LD^c)}{P(LD^c)} = \frac{0.04-0.01}{1-0.6} = 0.075$.

(c) $P(TD \cup LD) = P(TD) + P(LD) - P(TD \cap LD) = 0.04 + 0.6 - 0.01 = 0.63$.

(d) $P(TD^c \cap LD^c) = 1 - P(TD \cup LD) = 1 - 0.63 = 0.37$.

Q4. (a) Suppose $P(X = 0) = \beta$, then we have

$$\begin{aligned}
P(X = 1) &= \alpha P(X = 0) = \alpha\beta, \\
P(X = 2) &= \alpha P(X = 1) = \alpha^2\beta, \\
&\cdots \\
P(X = n) &= \alpha P(X = n-1) = \alpha^n\beta, \\
&\cdots
\end{aligned}$$

We know that

$$\begin{aligned}
1 &= \sum_{i=0}^{\infty} P(X = i) \\
&= \sum_{i=0}^{\infty} \alpha^i\beta = \beta\sum_{i=0}^{\infty} \alpha^i = \frac{\beta}{1-\alpha}.
\end{aligned}$$

Therefore, $\beta = 1 - \alpha$.

(b) Based on part (a), we have $P(X = i) = (1-\alpha)\alpha^i$.

$$\begin{aligned} E(X) &= \sum_{i=0}^{\infty} iP(X=i) \\ &= \sum_{i=0}^{\infty} i(1-\alpha)P(X=i) \\ &= (1-\alpha)\sum_{i=0}^{\infty} i\alpha^i = \frac{\alpha}{1-\alpha}. \end{aligned}$$

(c)

$$\begin{aligned} E(Y) &= E(\frac{(1-\alpha)^3}{\alpha}X + 1) \\ &= \frac{(1-\alpha)^3}{\alpha}E(X) + 1 \\ &= (1-\alpha)^2 + 1. \end{aligned}$$