STAT302 Assignment 3 (Solution)

1. (a) The possible values of X and Y are 0,1,2,3. Consider

$$P(X=0,Y=0)=P(\text{all three drawn balls are blue})=\frac{C_{5,3}}{C_{12,3}}=0.0455$$

$$P(X=0,Y=1)=P(\text{one white ball and 2 blue balls})=\frac{C_{4,1}\times C_{5,2}}{C_{12,3}}=0.1818$$

……

Similar calc ulations giv es the joint probability function of X and Y display ed in the following table.

X	***Y***				**Total**
	0	1	2	3	
0	0.0455	0.1818	0.1364	0.0182	0.3819
1	0.1364	0.2727	0.0818	0	0.4909
2	0.0682	0.0545	0	0	0.1227
3	0.0045	0	0	0	0.0045
Total	0.2546	0.5090	0.2182	0.0182	1

(b) $P(X = Y) = f(0,0) + f(1,1) + f(2,2) + f(3,3)$
$= 0.0455 + 0.2727 + 0 + 0 = 0.3182$

$P(X > Y) = f(1,0) + f(2,0) + f(3,0) + f(2,1) + f(3,1) + f(3,2)$
$= 0.1364 + 0.0682 + 0.0045 + 0.0545 + 0 + 0 = 0.2636$

$$P(X + Y > 2) = f(0,3) + f(1,2) + f(1,3) + f(2,1) + f(2,2) + f(2,3) + f(3,0) + f(3,1) + f(3,2) + f(3,3)$$
$$= 0.0182 + 0.0818 + 0 + 0.0545 + 0 + 0 + 0.0045 + 0 + 0 + 0$$
$$= 0.159$$

$$P(XY \le 3) = 1 - P(XY > 3)$$
$$= 1 - (f(2,2) + f(2,3) + f(3,2) + f(3,3))$$
$$= 1 - (0 + 0 + 0 + 0) = 1$$

(c) From the margins of above table, the marginal probability function of X and Y are given by

$$f_X(x) = \begin{cases} 0.3819 & \text{if } x = 0 \\ 0.4909 & \text{if } x = 1 \\ 0.1227 & \text{if } x = 2 \\ 0.0045 & \text{if } x = 3 \end{cases}, \qquad f_Y(y) = \begin{cases} 0.2546 & \text{if } y = 0 \\ 0.5090 & \text{if } y = 1 \\ 0.2182 & \text{if } y = 2 \\ 0.0182 & \text{if } y = 3 \end{cases}.$$

(d) X and Y are not independent as

$$f_X(0) f_Y(0) = 0.3819 \times 0.2546 = 0.0972 \ne f(0,0) = 0.0455.$$

(e) $$f(x \mid y = 1) = \frac{f(x,1)}{f_Y(1)} = \begin{cases} 0.1818/0.5090 = 0.3572 & \text{if } x = 1 \\ 0.2727/0.5090 = 0.5358 & \text{if } x = 2 \\ 0.0545/0.5090 = 0.1071 & \text{if } x = 3 \\ 0 & \text{if } x = 4 \end{cases}$$

$$f(x \mid y = 2) = \frac{f(x,2)}{f_Y(2)} = \begin{cases} 0.1364/0.2182 = 0.6251 & \text{if } x = 1 \\ 0.0818/0.2182 = 0.3749 & \text{if } x = 2 \\ 0 & \text{if } x = 3 \\ 0 & \text{if } x = 4 \end{cases}$$

2. (a) Since the range of x depends on the value of y, X and Y are not independent.

(b) $$f_Y(y)=\int_{-y}^{y}\frac{1}{8}\left(y^2-x^2\right)e^{-y}dx=\frac{1}{8}e^{-y}\left[xy^2-\frac{x^3}{3}\right]_{-y}^{y}=\frac{1}{6}y^3e^{-y},\qquad 0\le y<\infty$$

$$f_X(x)=\int_{|x|}^{\infty}\frac{1}{8}\left(y^2-x^2\right)e^{-y}dy=\frac{1}{4}e^{-|x|}\left(1+|x|\right),\qquad -\infty<x<\infty$$

(c) $$f(x\mid y)=\frac{f(x,y)}{f_Y(y)}=\frac{1}{8}\left(y^2-x^2\right)e^{-y}\Big/\frac{1}{6}y^3e^{-y}$$

$$=\frac{3\left(y^2-x^2\right)}{4y^3},\qquad -y\le x\le y$$

(d) $$F(x\mid y)=\int_{-y}^{x}f(t\mid y)dt$$

$$=\int_{-y}^{x}\frac{3\left(y^2-t^2\right)}{4y^3}dt$$

$$=\left[\frac{3y^2t-t^3}{4y^3}\right]_{-y}^{x}$$

$$=\frac{3y^2x-x^3}{4y^3}-\frac{-3y^3+y^3}{4y^3}$$

$$=\frac{2y^3+3y^2x-x^3}{4y^3},\qquad -y\le x\le y$$

$$P(X>1\mid Y=2)=1-F(1\mid 2)=1-\frac{2\times2^3+3\times2^2-1}{4\times2^3}=0.1563$$

$$P(X>1\mid Y=3)=1-F(1\mid 3)=1-\frac{2\times3^3+3\times3^2-1}{4\times3^3}=0.2593$$

3.

(a) $f_X(x) = \int_0^{2-x} \frac{1}{2}\, dy = 1 - \frac{x}{2}, \qquad 0 < x < 2$

$f_Y(y) = \int_0^{2-y} \frac{1}{2}\, dx = 1 - \frac{y}{2}, \qquad 0 < y < 2$

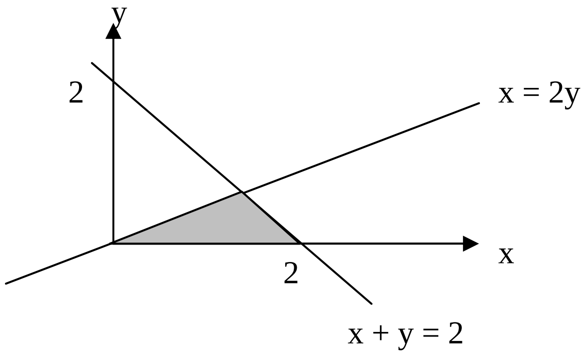
(b) They are not independent as the ranges of *x* and *y* depend on each other.

(c) $P(X > 2Y) = \int_0^{2/3} \int_{2y}^{2-y} \frac{1}{2}\, dx dy$

$= \int_0^{2/3} \frac{2-3y}{2}\, dy$

$= \left[y - \frac{3y^2}{4} \right]_0^{2/3}$

$= \frac{1}{3}$



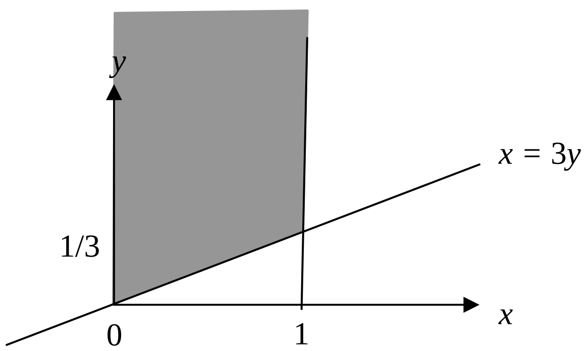
(d) Conditional pdf of Y given $X = x$ is given by

$$f(y \mid x) = \frac{f(x, y)}{f_X(x)} = \frac{1}{2} \Big/ \left(1 - \frac{x}{2}\right) = \frac{1}{2 - x}, \qquad 0 < y < 2 - x$$

4. Since X and Y are independent, their joint pdf is given by

$$f(x, y) = f_X(x) f_Y(y) = \begin{cases} e^{-y} & \text{for } 0 < x < 1, y > 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) $P(X < 3Y) = \int_0^1 \int_{x/3}^{\infty} e^{-y}\, dy dx = \int_0^1 e^{-x/3}\, dx = \left[-3e^{-x/3}\right]_0^1 = 3\left(1 - e^{-1/3}\right)$



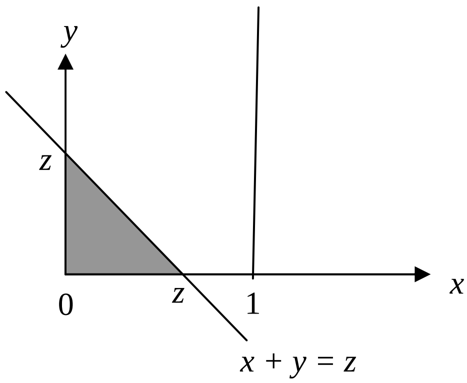
(b) $Z = X + Y$

The distribution function of Z is given by

$$F_Z(z) = \Pr(Z \le z) = \Pr(X + Y \le z)$$

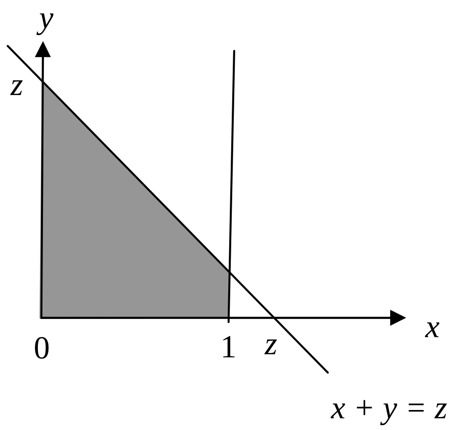
When $0 \le z \le 1$,

$$F_Z(z) = \int_0^z \int_0^{z-y} e^{-y} dxdy = \int_0^z (z-y)e^{-y} dy = \left[(y+1-z)e^{-y}\right]_0^z = e^{-z} - (1-z)$$



When $1 < z < \infty$,

$$F_Z(z) = \int_0^1 \int_0^{z-x} e^{-y} dydx = \int_0^1 \left(1 - e^{-(z-x)}\right) dx = \left[x - e^{-(z-x)}\right]_0^1 = 1 - (e-1)e^{-z}$$



Hence
$$F_Z(z) = \begin{cases} 0 & z < 0 \\ e^{-z} - (1 - z) & 0 \le z \le 1 \\ 1 - (e-1)e^{-z} & 1 < z < \infty \end{cases}.$$

The pdf is given by
$$f_Z(z) = \begin{cases} 1 - e^{-z} & 0 \le z \le 1 \\ (e-1)e^{-z} & 1 < z < \infty \\ 0 & \text{otherwise} \end{cases}.$$

5. (a) $\int_0^1\int_0^2 f(x,y)dxdy = \int_0^1\int_0^2 cy^2 dxdy = \int_0^1 2cy^2 dy = 2c\left[\frac{y^3}{3}\right]_0^1 = \frac{2c}{3}$

$c = \frac{3}{2}$

(b) $P(X+Y>2) = P(X>2-Y)$

$$= \int_0^1\int_{2-y}^2 f(x,y)dxdy$$
$$= \int_0^1\int_{2-y}^2 \frac{3}{2}y^2 dxdy$$
$$= \int_0^1 \frac{3}{2}y^2(2-(2-y))dy$$
$$= \int_0^1 \frac{3}{2}y^3 dy$$
$$= \frac{3}{2}\left[\frac{y^4}{4}\right]_0^1 = \frac{3}{8}$$

(c) $P\left(Y<\frac{1}{2}\right) = \int_0^{1/2}\int_0^2 f(x,y)dxdy = \int_0^{1/2}\int_0^2\left(\frac{3}{2}y^2\right)dxdy$

$$= \int_0^{1/2} 3y^2 dy = \left[y^3\right]_0^{1/2} = \frac{1}{8}$$

(d) $P(X<1) = \int_0^1\int_0^1 f(x,y)dxdy = \int_0^1\int_0^1\left(\frac{3}{2}y^2\right)dxdy$

$$= \int_0^1\left(\frac{3}{2}y^2\right)dy = \left[\frac{y^3}{2}\right]_0^1 = \frac{1}{2}$$

(e) $P(X=3Y)=0$ ($x=3y$ is a plane which has zero volume)

Q6. (a) The marginal p.d.f. of Y is

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2-2\rho xy+y^2+\rho^2y^2-\rho^2y^2}{2(1-\rho^2)}\right\} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{[(x-\rho y)^2+(1-\rho^2)y^2]}{2(1-\rho^2)}\right\} dx \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right\} dx. \end{aligned}$$

Clearly, $\exp\{-\frac{(x-\rho y)^2}{2(1-\rho^2)}\}/\sqrt{2\pi(1-\rho^2)}$ is p.d.f. of a normal distribution, and thus its integral equals to 1. Thus,

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right).$$

Based on this p.d.f., we know that Y is a standard normal random variable. Similarly, we can prove that X is also a standard normal random variable.

(b, c)

$$\begin{aligned} f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} &= \frac{\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\}}{\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\}} \\ &= \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)} + \frac{y^2}{2}\right\} dx \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right\}. \end{aligned}$$

According to the p.d.f., we know that $X|Y$ follows a normal distribution with mean ρy and variance $1-\rho^2$.