Stat 302 Assignment 1 (solution)

Q1.

(a) There are 7!=5040 possible different signals can be made, if there are no restrictions.

(b) There are 2!2!3! arrangements such that the red flags are first in line, then the blue flags, then the green flags. Similarly, for each possible ordering of the colors, there are 2!2!3! possible arrangements. Note that there are 3! possible ordering of the colors, therefore, the desired answer is $3! \times (2!2!3!) = 144$

(c) We can arrange flag R1,B1 and G1 first, there are 3! possible arrangements. Then, we treat R1,B1 and G1 as a whole part (we call it a block of flags). The number of ways to arrange this block of flags together with the other four flags (R2,B2,G2,G3) is (1+4)!=5!. Therefore, the solution is 3!5!=720

(d) We have 2! ways to order flags R2 and B2 and 5! ways to place the other five flags (NOT including R2 and B2). The first step, we put the other five flags that have been ordered in a sequence, denote as $\star\ \star\ \star\ \star\ \star$. The second step, we place R2 and B2 into the above sequence. Since R2 and B2 cannot be put together, there are six possible position for R2 and B2, which are denoted as "Λ" in the expression below,

$$\Lambda \star \Lambda \star \Lambda \star \Lambda \star \Lambda \star \Lambda.$$

The number of ways to place R2 and B2 into the sequence of the other five flags is choosing 2 positions from the 6 possible positions "Λ"s, i.e. $\binom{6}{2}$. Therefore, the solution is $2!5!\binom{6}{2} = 3600$.

Q2.

(a) Since flags of the same color are indistinguishable, the different arrangement of flags of the same color give us the same result. For example, $R1, R2, B1, B2, G1, G2, G3$ is the same as $R2, R1, B1, B2, G1, G2, G3$. So the solution should be

$$\frac{7!}{2!2!3!} = 210.$$

(b) This question is the same as we arrange three colors. So there are 3!=6 signals can be made.

(c) We treat a specific red, blue and green flags as one special big flag, denote as flag S. First, there are 3! signals can be made by flag S. Then we arrange the other four flags together with this special big flag. In other word, we need to arrange flags S, R, B, G, G. Since the color of green flags is indistinguishable, $5!/2!$ different arrangement can be made. Hence, the solution for this question is there are $3!5!/2!$ different signals can be made.

Q3. Jack randomly selects 6 bottles, among these 6 bottles, the number of bottles that are qualified is denote as random variable N. Then,

$$P(N = k)=\binom{8}{k}\ \binom{4}{6-k}/\binom{12}{6},$$

where $k = 2, 3, 4, 5, 6$. Note that, k cannot be 0 or 1, because the total number of unqualified bottles are 4. If we randomly choose 6 bottles, there are at least 2 bottles are qualified.

(a) $P(N = 6)=\binom{8}{6}/\binom{12}{6}=\frac{1}{33} \approx 0.03$.

(b) At least 5 bottles are qualified, means N can be equal to 5 or 6. Therefore, the solution is
$P(N = 5) + P(N = 6)=\binom{8}{5}\ \binom{4}{1}/\binom{12}{6}+\frac{1}{33}=\frac{3}{11} \approx 0.27$

(c) At most 3 bottles are qualified, means N can be equal to 2 or 3. Here, note that N cannot be 0 or 1. So the solution is
$P(N = 2) + P(N = 3)=\binom{8}{2}\ \binom{4}{4}/\binom{12}{6}+\binom{8}{3}\ \binom{4}{3}/\binom{12}{6}=\frac{3}{11} \approx 0.27$

(d) The complement of "at least 1 bottle is unqualified" is "all 6 bottles are qualified". So the solution is
$1 - P(N = 6)=1-\binom{8}{6}/\binom{12}{6}=\frac{32}{33} \approx 0.97$.

Q4. First, we choose 2 months from 12 months, then choose another 6 months from the left 10 months. After these two steps, we get 8 months in which contain 20 people's birthdays. Then, we put 20 people in these 8 months. So the numerator is
$\binom{12}{2}\ \binom{10}{6}\ \binom{20}{4}\ \binom{16}{4}\ \binom{12}{2}\ \binom{10}{2}\ \binom{8}{2}\ \binom{6}{2}\ \binom{4}{2}\ \binom{2}{2}$. And the denominator is 12^{20}, because, for each person, there are 12 choices. So, the solution is

$$\binom{12}{2}\ \binom{10}{6}\ \binom{20}{4}\ \binom{16}{4}\ \binom{12}{2}\ \binom{10}{2}\ \binom{8}{2}\ \binom{6}{2}\ \binom{4}{2}\ \binom{2}{2}/\ 12^{20}$$

Q5. According to the question, we have the following probability table. where

Interview	1st	2nd	3rd	4th	5th	$\cdots$
Success	20%	30%	40%	50%	50%	50%
Failure	80%	70%	60%	50%	50%	50%

"Success" indicates getting an offer and "Failure" indicates being rejected. Denote events

F1 = Failure at the 1st interview;

S1 = Success at the 1st interview;

F2 = Failure at the 2nd interview;

S2 = Success at the 2nd interview;

and so on.

(a) The graduate gets the first offer at the third interview means she fails the first two interviews and wins the third one. That is the event of $F1 \cap F2 \cap S3$. Therefore,

$$\begin{aligned}
& P(F1 \cap F2 \cap S3) \\
= \; & P(S3|F1, F2)P(F1, F2) \\
= \; & P(S3|F1, F2)P(F2|F1)P(F1) \\
= \; & .4 \times .7 \times .8 = .224.
\end{aligned}$$

(b) The counter event of this part of question is that the graduate will have no offer after three interviews. That is

$$\begin{aligned}
& P((F1 \cap F2 \cap F3)^c) = 1 - P(F1 \cap F2 \cap F3) \\
= \; & 1 - P(F3|F1, F2)P(F2|F1)P(F1) \\
= \; & 1 - .6 \times .7 \times .8 \\
= \; & 1 - 0.336 = 0.664.
\end{aligned}$$

You may also list all possibility of having at least one offer during the first three interviews and then calculate the probability.

Q6.

(a) Let A be the event that the item is defective, and B_i, $i = 1, 2, 3, 4$, the event that it is from ith machine. Since the machine is randomly selected, then B_1, B_2, B_3, B_4 have equal chance to occur. That is $P(B_1) = P(B_2) = P(B_3) = P(B_4) = 1/4$.

The question is asking about the probability of B_2 given A occurs. Use Bayes's formula, we have:

$$\begin{aligned}
P(B_2|A) &= \frac{P(A|B_2)P(B_2)}{\sum_{i=1}^{4} P(A|B_i)P(B_i)} \\
&= \frac{.03/4}{.05/4 + .03/4 + .04/4 + .08/4} = 3/20.
\end{aligned}$$

(b) This question is asking about the probability of the count event of $B_2 \cup B_4$ given A occurs. Obviously B_1, B_2, B_3, B_4 are disjoint, then

$$\begin{aligned}
P((B_2 \cup B_4)^c|A) &= 1 - P(B_2 \cup B_4|A) \\
&= 1 - P(B_2|A) - P(B_4|A) \\
&= 1 - \frac{P(A|B_2)P(B_2)}{\sum_{i=1}^4 P(A|B_i)P(B_i)} - \frac{P(A|B_4)P(B_4)}{\sum_{i=1}^4 P(A|B_i)P(B_i)} \\
&= 1 - \frac{.03/4 + .08/4}{.05/4 + .03/4 + .04/4 + .08/4} = 9/20.
\end{aligned}$$

(c) The question is asking about the probability of B_2 given A^c occurs.

$$\begin{aligned}
P(B_2|A^c) &= \frac{P(A^c|B_2)P(B_2)}{\sum_{i=1}^4 P(A^c|B_i)P(B_i)} \\
&= \frac{(1 - P(A|B_2))P(B_2)}{\sum_{i=1}^4 (1 - P(A|B_i))P(B_i)} \\
&= \frac{.97/4}{.95/4 + .97/4 + .96/4 + .92/4} = 97/380.
\end{aligned}$$

Q7.

(a) Apply the definition of independence and Proposition 4.1 to prove the result. Refer to *EXAMPLE 4g* of textbook for details.

(b) Denote events

R1 = Red die shows 1;

G1 = Green die shows 1;

R2 = Red die shows 2;

G2 = Green die shows 2;

and so on.

Based on the notation, we have:

$$\begin{aligned}
A &= \{R3, R4, R5\}, \\
B &= \{G1, G2\}, \\
C &= \{(R1, G6), (R2, G5), (R3, G4), (R4, G3), (R5, G2), (R6, G1)\}, \\
\{A, B\} &= \{(R3, G1), (R3, G2), (R4, G1), (R4, G2), (R5, G1), (R5, G2)\}, \\
\{A, C\} &= \{(R3, G4), (R4, G3), (R5, G2)\}, \\
\{B, C\} &= \{(R6, G1), (R5, G2)\}, \\
\{A, B, C\} &= \{(R5, G2)\}.
\end{aligned}$$

It is not difficult to find that

$$
\begin{aligned}
P(A) &= 1/2, \\
P(B) &= 1/3.
\end{aligned}
$$

And the following table may help you find the probabilities of other events.

$$
\begin{aligned}
P(C) &= 1/6, \\
P(A \cap B) &= 1/6, \\
P(A \cap C) &= 1/12, \\
P(B \cap C) &= 1/18, \\
P(A \cap B \cap C) &= 1/36.
\end{aligned}
$$

	R1	R2	R3	R4	R5	R6
G1	2	3	4	5	6	7
G2	3	4	5	6	7	8
G3	4	5	6	7	8	9
G4	5	6	7	8	9	10
G5	6	7	8	9	10	11
G6	7	8	9	10	11	12

Thus, we can verify that

$$
\begin{aligned}
P(A \cap B) &= P(A) \times P(B), \\
P(A \cap C) &= P(A) \times P(C), \\
P(B \cap C) &= P(B) \times P(C), \\
P(A \cap B \cap C) &= P(A) \times P(B) \times P(C).
\end{aligned}
$$

Therefore, A, B, C are independent.

(c) By applying the part (a), we have

$$
\begin{aligned}
P(A \cup B \cup C) &= 1 - [1 - P(A)][1 - P(B)][1 - P(C)] \\
&= 1 - (1 - 1/2)(1 - 1/3)(1 - 1/6) \\
&= 0.722222.
\end{aligned}
$$

Q8. Denote events

D = Disease;

H = healthy;

+ = positive test;

- = negative test.

According to the context, we have the following probabilities

$$P[D] = 1/1000, P[H] = 1 - P[D], P[+|D] = .98, P[+|H] = .00001.$$

The questions, formulations and calculations are straightforward.

(a)

$$P[+] = P[+|D]P[D] + P[+|H]P[H] = 0.00098999$$

(b)

$$P[D|+] = \frac{P[+|D]P[D]}{P[+|D]P[D] + P[+|H]P[H]} = 0.989909.$$

(c)

$$P[H|-] = \frac{P[-|H]P[H]}{P[-|D]P[D] + P[-|H]P[H]} = 0.99998.$$