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Particle Motions in Absorbing Medium with Hard and Soft Obstacles

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ABSTRACT

We consider a particle evolving according to a Markov motion in an absorbing medium. We analyze the long term behavior of the time at which the particle is killed and the distribution of the particle conditional upon survival. Under given regularity conditions, these quantities are characterized by the limiting distribution and the Lyapunov exponent of a nonlinear Feynman-Kac operator.

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We propose to approximate numerically this distribution and this exponent based on various interacting particle system interpretations of the Feynman-Kac operator. We study the properties of the resulting estimates.

INTRODUCTION

We consider an homogeneous and discrete time Markov motion in an absorbing medium. The particle X_n evolves according to a Markov transition kernel $M(x, dy)$ in a measurable space $(E, \mathscr{E})$. At each time n, it is killed with a probability $1 - G(X_n)$ where G is a given potential function for E into [0, 1]. When the particle hits the set $G^{-1}(0)$, it is instantly killed. In this sense $G^{-1}(0)$ can be regarded as a set of hard obstacles. In the opposite situation, the motion is not affected by the killing transition in the set G^{-1}.[1] Regions where $0 < G(x) < 1$ can be regarded as soft obstacles. The particle may explore these regions but its life time decreases during these visits.

Let T be the random time at which the particle X_n is killed. In this article we analyze the long time behavior of the quantities

$$\mathbb{P}(T > n) \quad \text{and} \quad \mathbb{P}(X_n \in \cdot \mid T > n).$$

We also propose an alternative model in which the hard obstacles are turned into repulsive ones. We connect these two physical models with two interacting particle systems approximating models. In the special case $G(x) = 1_S(x)$, $S \in \mathscr{E}$, these particle interpretations are the discrete time versions of a recent Fleming Viot particle model introduced in Ref.[1] for the spectral analysis of the Laplacian with Dirichlet boundary conditions. We design a general interacting process approach which simplifies and extends the asymptotic results presented in the above referenced paper. Uniform convergence results and unbiased particle estimates for the spectral radius of the sub-Markov killing operator are presented. This analysis has been motivated by the recent work of one of the authors with Del Moral and Miclo[2]. The present article is an extension of the techniques presented in Ref.[2] to treat the hard obstacle situation. The semi-group and martingale techniques presented here lay solid theoretical foundations for the asymptotic analysis of the long time behavior of a class of interacting particle models arising in physics. In contrast to Ref.[1], these techniques are not specific to Gaussian explorations of the medium. Our approach can also be extended without further work to study the path-space distribution

$$\mathbb{P}((X_0, \ldots, X_n) \in \cdot \mid T > n). \tag{1}$$

We refer the reader to the Ref.[3] where the authors provide a natural path-space extension and show that the occupation measures of the genealogical tree associated to a sequence of absorbed and interacting particle models converge to the desired distribution[1]. Incidentally our analysis also applies to study some asymptotic properties of the ‘‘go with the winners’’ algorithm presented in Ref.[4].

At the heart of this article is a functional Feynman-Kac representation of the absorbing time distribution. Namely let μ_n be the distribution flow on E defined for each bounded measurable function f by the formulae

$$\mu_n(f) = \lambda_n(f)/\lambda_n(1) \quad \text{with } \lambda_n(f) = \mathbb{E}\left(f(X_n)\prod_{p=0}^{n} G(X_p)\right).$$

When the absorbing medium is regular enough, we will check that

$$\lambda_n(1) = \mathbb{P}(T > n) > 0 \quad \text{and} \quad \mu_n = \mathrm{Law}(X_n \mid T > n).$$

The power of this representation comes from the following formula

$$\mathbb{P}(T > n) = \prod_{p=0}^{n} \eta_p(G) \quad \text{with } \eta_0 = \mathrm{Law}(X_0) \quad \text{and} \quad \eta_p = \mu_{p-1}M.$$

The study of the exponential tail of the distribution of the absorbing time depends on the long time behavior of the flow η_n. We will present a natural condition on the pair (G, M) under which the flows μ_n converges to a unique distribution μ as $n \to \infty$ and in some sense we will have

$$\mathbb{P}(T > n) \underset{n\to\infty}{\sim} e^{n\Lambda(G)} \quad \text{with } \Lambda(G) = \log \mu M(G) \leq 0.$$

The quantity $\Lambda(G)$ also represents the logarithmic Lyapunov exponent of the semigroup $f \to M(Gf)$ on the Banach space $B_b(S)$ of all bounded test functions on $S = G^{-1}((0,1])$. We will extend the techniques presented in Ref.[2] to non necessarily positive potential and we will provide exponential decays to the equilibrium for the nonlinear distribution flow μ_n with respect to the total variation norm. Basically we will prove that

$$\|\mu_n - \mu\|_{\mathrm{TV}} \leq c e^{-n\alpha}$$

for some finite constant $c < \infty$ and $\alpha > 0$ depending on the pair (G, M). The second part of this article is concerned with the effective numerical approximation of the flow μ_n and the limiting distribution μ. We provide essentially two distinct interacting particle interpretations dependent on two distinct nonlinear Markovian interpretations of the measure-valued dynamical system associated to the flow μ_n. More precisely, we exhibit two collections of Markov transition K_ν indexed by the set of probability measures ν on E and such that

$$\mu_{n+1} = \mu_n K_{\mu_n}.$$

Let $\overline{\mathbb{P}}$ be the McKean measure on the product space $\Omega = E^{\mathbb{N}}$ defined by its marginals on E^{n+1}

$$\overline{\mathbb{P}}_n(d(x_0, \ldots, x_n)) = \mu_0(dx_0) K_{\mu_0}(x_0, dx_1) \cdots K_{\mu_{n-1}}(x_{n-1}, dx_n).$$

By definition of $\overline{\mathbb{P}}$, the canonical process $(\Omega, \overline{\mathbb{P}}, Z)$ is a time inhomogeneous Markov chain on E with transitions

$$\overline{\mathbb{P}}(Z_{n+1} \in dx \mid Z_n) = K_{\mu_n}(Z_n, dx) \quad \text{with } \mu_n = \overline{\mathbb{P}}\text{-law}(Z_n).$$

The discrete generation interacting particle system associated to this interpretation consists in a Markov chain $\xi_n = (\xi_n^1, \ldots, \xi_n^N)$ in the product space E^N with elementary transitions

$$\mathbb{P}(\xi_{n+1} \in d(x^1, \ldots, x^N) \mid \xi_n) = \prod_{p=1}^{N} K_{\mu_n^N}(\xi_n^p, dx_p) \quad \text{with } \mu_n^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi_n^i}.$$

In the first interpretation the set of hard obstacles will be turned into repulsive obstacles. The particle system will again behave as a set of particles evolving in an absorbing medium. Between absorption times the particles evolve independently one from each other according to a Markov transition on S and related to the pair (G, M). When a particle visiting a soft obstacle is killed then instantly another suitably chosen particle is splitted into two offsprings. The choice of this new particle depends on its ability to stay alive in the next stage. In this context we will give several uniform estimates w.r.t. the time parameter. For instance we will prove that for each fixed $N \geq 1$, $p \geq 1$ and for any test function $f \in B_b(E)$, where $B_b(E)$ is the set of bounded measurable functions on E, one has

$$\sup_{n \geq d \log N} \sqrt{N} \mathbb{E}(|\mu_n^N(f) - \mu(f)|^p)^{1/p} < \infty$$

for a constant $d < \infty$. In the second particle interpretation the system evolves in absorbing medium with hard and soft obstacles. The time evolution of the particles is essentially the same as before but a particle visiting the hard obstacle set is instantly killed and in the same time another suitably selected particle splits into two offsprings.

The essential difference with the previous particle model is that the whole configuration $\xi_n = (\xi_n^1, \ldots, \xi_n^N)$ may hit the set of hard obstacles $E - S$. Let τ^N be the first time this event happens

$$\tau^N = \inf\{n \geq 0 : \eta_n^N(G) = 0\}.$$

In this context we will prove for instance the following asymptotic result

$$\lim_{N \to \infty} (|\mu_{n(N)}^N(f) 1_{\tau^N \geq n(N)} - \mu(f)|^p)^{1/p} = 0.$$

for some sequence of integers $n(N) \to \infty$ as $N \to \infty$. We end this introductory section with some comments on the genealogy of the particles. If we interpret the above particle models as a birth and death model then arises the important notion of the ancestral line

$$(\xi_{0,n}^i, \xi_{1,n}^i, \ldots, \xi_{n,n}^i) \in E^{n+1}$$

of each individual ξ_n^i. In these notations $\xi_{p,n}^i$ represents the ancestor at level p of the particle ξ_n^i. The path-particle model coincides with the corresponding interacting particle interpretations of the Feynman-Kac distribution flow defined as in Ref.[3] by replacing X_n by the path $(X_0, \ldots, X_n)$ and the potential function $G(X_n)$ by $G_n(X_0, \ldots, X_n) = G(X_n)$. For path evaluation problems, this enlargement technique of the state space can be used to conclude that for any $f_n \in B_b(E^{n+1})$ and $p \geq 1$

$$\left| \mathbb{E}\left(\frac{1}{N} \sum_{i=1}^N f_n((\xi_{p,n}^i)_{0 \leq p \leq n}) 1_{\tau^N \geq n} \right) - \mathbb{E}(f(X_0, \ldots, X_n) \mid T > n) \right| \leq \frac{c(n)}{N}$$

and

$$\mathbb{E}\left(\left|\frac{1}{N}\sum_{i=1}^{N} f_n((\xi^i_{p,n})_{0\le p\le n})1_{\tau^N\ge n} - \mathbb{E}(f(X_0,\ldots,X_n)\,|\,T>n)\right|^p\right)^{1/p} \le \frac{c(n)}{\sqrt{N}}$$

for some finite constant $c(n)=O(n)$.

HARD, SOFT AND REPULSIVE OBSTACLES

In this section, we review some probabilistic models of particle evolution in an absorbing medium. The motion of the particle is represented by a Markov transition $M(x, dy)$ on some measurable space $(E, \mathcal{E})$ representing the medium. After each elementary move the particle X_n at time n remains in its location with a probability $G(X_n)$, otherwise it is killed and goes into an auxiliary cemetery of coffin state c. The potential function G is an $\mathcal{E}$-measurable function on E taking values in $[0, 1]$ and we recall that $S=G^{-1}([0,1])$. The Markov evolution X^c_n in the absorbing medium $E_c=E\cup\{c\}$ is therefore defined as a two-step transition

$$X^c_n \xrightarrow{\text{absorption}} \widehat{X}^c_n \xrightarrow{\text{exploration}} X^c_{n+1}.$$

If we extend (G, M) to E_c by setting

$$M(c, dy) = \delta_c(dy) \quad \text{and} \quad G(c) = 0,$$

then the absorption/exploration transition can be written more formally as follows

$$\mathbb{P}(\widehat{X}^c_n \in dy\,|\,\widehat{X}^c_n = x) = G(x)\delta_x(dy) + (1-G(x))\delta_c(dy),$$
$$\mathbb{P}(X^c_{n+1} \in dy\,|\,\widehat{X}^c_n = x) = M(x, dy).$$

We let $\mathbb{E}^c_{\eta 0}$ and $\mathbb{P}^c_{\eta 0}$ the expectation and the probability measure of the killed Markov evolution $(X^c_n, \widehat{X}^c_n)$ on the state space E_c with initial distribution η_0 on E. We also denote by $\mathbb{E}_{\eta_0}$ and $\mathbb{P}_{\eta_0}$ the expectation and probability measure associated to the homogeneous Markov chain X_n with Markov transition M and initial distribution η_0. These distributions are defined in a traditional way on the canonical spaces associated to the canonical realizations of the chains X^c_n or X_n. For indicator potential function $G(x)=1_S(x)$, the chain X^c_n evolving in S is not affected by the killing transition but it is instantly killed as soon as it enters in the set of hard obstacles $E - S$. We define T as the first time the particle is killed

$$T = \inf\{n \geq 0, \widehat{X}_n^c = c\}.$$

Note that

$$\begin{aligned}
&\mathbb{P}^c_{\eta_0}(T > n)\\
&\quad = \mathbb{P}^c_{\eta_0}(\widehat{X}_0^c \in E, \ldots, \widehat{X}_n^c \in E)\\
&\quad = \int_{E^{n+1}} \eta_0(dx_0)G(x_0)M(x_0, dx_1)\cdots G(x_{n-1})M(x_{n-1}, dx_n)G(x_n)\\
&\quad = \mathbb{E}_{\eta_0}\left(\prod_{p=0}^{n} G(X_p)\right).
\end{aligned}$$

Suppose the state space E is a rooted tree with an unknown finite maximum depth D and the particle traverses the paths from the root to the set of leaves. The particle at a vertex x of depth $d \leq D$ chooses its path according to a given probability distribution $M(x, dy)$

$$M(x, dy) = \sum_{k=1}^{K} p(x, y_k)\delta_{y_k}(dy).$$

If we set $S = E - L$ and $G(x) = 1_S(x)$ then the killed Markov model X_n^c consists in exploring the tree according to M and when the particle X_n^c is at a leaf then it is killed and we set $X_n^c = c$. By construction, we have in this context

$$\mathbb{P}^c_{\eta_0}(T > D + 1) = 0.$$

One way to prevent the particle to be killed in a given finite horizon is to introduce the following accessibility condition:

Assumption (A). We have $\eta_0(S) > 0$ and for each $n \geq 1$ and $x \in S$, $M(x, S) > 0$.

This condition is not met in the previous situation but it holds true in most physical evolution models in absorbing medium. Loosely speaking this condition states that a particle evolving in the absorbing medium has always a chance to survive. Thus we have $\mathbb{P}_{\eta_0}(T > n) > 0$ for any $n \geq 0$. In this sense, Assumption (A) ensures that the medium does not contain some trapping regions where the particle cannot escape. More interestingly under (A) we can write

$$M(x, dy)G(y) = G'(x)M'(x, dy)$$

with the pair potential/Markov kernel

$$G'(x) = M(G)(x) = \int M(x, dy)G(y),$$
$$M'(x, dy) = \frac{1}{M(G)(x)} M(x, dy)G(y).$$

For indicator function $G(x) = 1_S(x)$, we have for instance

$$G'(x) = M(x, S) \quad \text{and} \quad M'(x, dy) = \frac{M(x, dy)1_S(x)}{M(x, S)}.$$

In the same way since we have $\eta_0(S) > 0$, we can also write

$$\eta_0(dx)G(x) = \eta_0(G)\eta'_0(dx) \quad \text{with } \eta'_0(dx) = \frac{G(x)}{\eta_0(G)}\eta_0(dx).$$

The killed particle model

$$X'^c_n \xrightarrow{\text{absorption}} \widehat{X}'^c_n \xrightarrow{\text{exploration}} X'^c_{n+1}$$

associated to the pair (G', M') is again defined by a absorption/exploration transition but the novelty here is that the potential function G' is strictly positive. In this alternative particle model, the random exploration is restricted to S and cannot visit hard obstacles. Furthermore if we set

$$T' = \inf\{n \geq 0, \widehat{X}'^c_n = c\}$$

then we readily find the formula

$$\mathbb{P}^c_{\eta_0}(T > n) = \eta_0(G) \times \mathbb{P}'^c_{\eta'_0}(T' \geq n). \tag{2}$$

It is also important to notice that M' is a Markov kernel from S into itself. This shows that the absorbed Markov particle evolution is restricted to the measurable state space $(S, \mathscr{S})$ where $\mathscr{S}$ is the trace on S of the σ-field $\mathscr{E}$.

FEYNMAN-KAC REPRESENTATION

Let Q' be the bounded operator on the Banach space of bounded measurable function $f \in B_b(S)$ defined for any $x \in S$ by

$$Q'(f)(x) = \int M(x, dy) G(y) f(y) = M(Gf)(x) = G'(x) M'(f)(x).$$

We denote by $Q'^{(n)}$ the semi-group (on $B_b(S)$) associated to the operator Q' and defined by $Q'^{(n)} = Q'^{(n-1)} Q'$ with the convention $Q'^{(0)} = Id$. The logarithmic Lyapunov exponent or spectral radius of Q' on $B_b(S)$ is the quantity defined by

$$\Lambda(G) = \log \mathrm{Lyap}(Q') = \lim_{N \to \infty} \frac{1}{n} \sup_{x \in S} \log Q'^{(n)}(1)(x) \tag{3}$$

In a symbolic form, under regularity assumptions detailed further, we have in the logarithmic scale

$$\mathbb{P}^c_x(T' \geq n) \quad \text{and} \quad \mathbb{P}'^c_x(T' > n) \sim e^{\Lambda(G) n}.$$

Our next objective is to connect $\Lambda(G)$ with the long time behavior of the distribution flow models

$$\mu_n(dx) = \mathbb{P}^c_{\eta_0}(\widehat{X}^c_n \in dx \mid T > n),$$
$$\eta_n(dx) = \mathbb{P}^c_{\eta_0}(X^c_n \in dx \mid T \geq n).$$

Notice that for any $f \in B_b(E)$, we have the Feynman-Kac functional representation formulae

$$\mu_n(f) = \lambda_n(f)/\lambda_n(1) \quad \text{with } \lambda_n(f) = \mathbb{E}_{\eta_0}\left(f(X_n) \prod_{p=0}^{n} G(X_p)\right),$$

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1) \quad \text{with } \gamma_n(f) = \mathbb{E}_{\eta_0}\left(f(X_n) \prod_{p=0}^{n-1} G(X_p)\right).$$

Let X'_n be the time homogeneous Markov chain with initial distribution η'_o and Markov transition M'. Using the superscript $(\ )'$ to denote the positive measures μ'_n, λ'_n, η'_n, γ'_n, defined as above by replacing (η_0, X_n, G) by (η'_n, X'_n, G') we find that

$$\lambda_n(dx) = \eta_0(G)\gamma'_n(dx) \quad \text{and} \quad \mu_n = \eta'_n.$$

Note that $\eta'_n(S) = 1$ and η'_n can be regarded as a distribution flow model taking values in $\mathscr{P}(S)$. From these observations, we obtain the two following interpretations

$$\mathbb{P}^c_{\eta_0}(T > n) = \lambda_n(1) = \prod_{p=0}^{n} \eta_p(G) \tag{4}$$

and

$$\mathbb{P}'^c_{\eta'_0}(T \geq n) = \gamma'_n(1) = \prod_{p=0}^{n-1} \eta'_p(G')\left(= \prod_{p=0}^{n-1} \mu_p(G')\right).$$

When $\eta_0 = \eta'_0 = \delta_x$, for some $x \in S$, we use the superscript $(\cdot)^{(x)}$ to define the corresponding distribution. In these notations, we find that for any $x \in S$ and $n \geq 1$

$$Q'^{(n)}(1)(x) = \mathbb{E}_x\left(\prod_{p=1}^{n} G(X_p)\right) = G^{-1}(x)\lambda_n^{(x)}(1) = \gamma_n'^{(x)}(1) \tag{5}$$

and by (4) we obtain

$$\Lambda_n^{(x)}(G) = \frac{1}{n}\log Q'^{(n)}(1)(x) = \frac{1}{n}\sum_{p=1}^{n} \log \eta_p^{(x)}(G) = \frac{1}{n}\sum_{p=0}^{n-1} \log \eta_p'^{(x)}(G').$$

The above display indicates that the exponent $\Lambda(G)$ is related to the long time behavior of the distribution flows $\eta_n^{(x)} = \mu_{n-1}^{(x)} M$ and $\eta_n'^{(x)} = \mu_n^{(x)}$. Let us introduce the following mixing assumption.

Assumption. $(G', Q')_m$: there exists an integer parameter $m \geq 1$ and a pair $(r', \delta') \in [0, 1]$ such that for each $x, y \in S$

$$G'(x) \geq r' G'(y) \quad \text{and} \quad Q'^{(m)}(x, \cdot) \geq \delta' Q'^{(m)}(y, \cdot).$$

Theorem 1. *Under assumption* $(G', Q')_m$, *there exists a unique distribution* $\mu \in \mathscr{P}(S)$ *such that* $\Lambda(G) = \log \mu M(G)$ *and for any* $f \in B_b(S)$

$$\mu M(Gf) = e^{\Lambda(G)} \mu(f).$$

In addition, we have the uniform estimates

$$\sup_{x \in S} \|\mu_n^{(x)} - \mu\| \leq \frac{2}{\delta'} (1 - \delta'^2)^{[\frac{n}{m}]} \tag{6}$$

as well as

$$\sup_{x \in S} |\Lambda_n^{(x)}(G) - \Lambda(G)| \leq \frac{2m}{r'\delta'^3} \times \frac{1}{n}$$

and

$$\sup_{x \in S} |\log \mu_n^{(x)} M(G) - \Lambda(G)| \leq \frac{4}{\delta'} (1 - \delta'^2)^{[\frac{n}{m}]}.$$

This theorem and its proof parallel earlier results of one of the two authors and Miclo[2] for the soft obstacle situations. The strategy we have used here to extend the result to the hard obstacle case is to analyze the long term behavior of the distribution flow η'_n in the set of probability measures on S. The set $\mathscr{P}(S)$ can be regarded as the subset of probability measures μ over E such that $\mu(E - S) = 0$. On the other hand M' is a Markov kernel for the range S of G into itself. From these observations, it is intuitively clear that the analysis of the long time behavior of $\mu_n = \eta'_n \in \mathscr{P}(S)$ can be studied using the same line of arguments as those presented in Ref.[2] in the soft obstacle case by replacing the set $\mathscr{P}(E)$ by the set $\mathscr{P}(S)$. The semi-group approach proposed is based on contraction properties of a collection of nonhomogeneous Markov transitions related to the pair (G, M). The main difficulty in extending this technique is to check that these nonhomogeneous kernels are Markov transitions on S. For the convenience of the reader we give a complete proof with some precise estimates. Before getting into its proof, we give next an immediate corollary of this theorem. We denote by $\mathscr{P}_G(E)$ the set of probability measures η on E such that $\eta(G) > 0$. Under our assumptions we notice that for any $\eta_0 \in \mathscr{P}_G(E)$ the Feynman-Kac flow $\eta_n \in \mathscr{P}_G(E)$ is well

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defined and we have for any $f \in \mathscr{B}_b(E)$

$$\eta_{n+1}(f) = \Phi(\eta_n)(f) = \eta_n(GM(f))/\eta_n(G).$$

Corollary 1. *Under the assumptions of Theorem* 1, *the distribution* $\eta = \mu M \in \mathscr{P}_G(E)$ *is the unique fixed point of the nonlinear mapping* $\Phi : \mathscr{P}_G(E) \to \mathscr{P}_G(E)$ *and we have for each* $n \geq 1$

$$\sup_{x \in S} \|\eta_n^{(x)} - \eta\| \leq \frac{2}{\delta'}(1 - \delta'^2)^{[\frac{n-1}{m}]} \tag{7}$$

and

$$\sup_{x \in S} |\log \eta_n^{(x)}(G) - \Lambda(G)| \leq \frac{4}{\delta'}(1 - \delta'^2)^{[\frac{n}{m}-1]}.$$

To check this corollary, we first notice that

$$\eta(G) = \mu M(G) = \mu(G') > 0$$

from which we conclude that $\Phi(\eta) \in \mathscr{P}_G(E)$ is well defined and for any $f \in B_b(E)$

$$\Phi(\eta)(f) = \frac{\eta(GM(f))}{\eta(G)} = \frac{\mu M(GM(f))}{\mu M(G)} = \mu M(f) = \eta(f).$$

The estimate (7) is easily checked by noting that

$$\|\eta_n^{(x)} - \eta\| = \|\mu_{n-1}^{(x)} M - \mu M\| \leq \|\mu_{n-1}^{(x)} - \mu\|.$$

Proof. By definition of the distribution flow $\eta'_n \in \mathscr{P}(S)$, we have for any $f \in B_b(S)$ and $p \leq n$

$$\eta'_{p+n}(f) = \Phi'_n(\eta_p)(f) = \eta'_p Q'^{(n)}(f)/\eta'_p Q'^{(n)}(1).$$

Next we observe that for any $\eta \in \mathscr{P}(S)$

$$\Phi'_n(\eta)(f) = \eta(g_n K_n(f))/\eta(g_n)$$

with the function $g_n \in B_b(S)$ and the Markov kernel K_n from S into S defined by

$$g_n = Q'^{(n)}(1) \quad \text{and} \quad K_n(f) = \frac{Q'^{(n)}(f)}{Q'^{(n)}(1)}.$$

Under our assumption, we have for any $(x, y) \in S^2$ and $n \geq m$

$$\frac{g_n(x)}{g_n(y)} = \frac{Q'^{(n)}(1)(x)}{Q'^{(n)}(1)(y)} = \frac{Q'^{(m)}(Q'^{(n-m)}(1))(x)}{Q'^{(m)}(Q'^{(n-m)}(1))(y)} \geq \delta'. \tag{8}$$

It is also convenient to observe that

$$K_n(f) = \frac{Q'^{(n)}(f)}{Q'^{(n)}(1)} = \frac{Q'^{(m)}(Q'^{(n-m)}(f))}{Q'^{(m)}(Q'^{(n-m)}(1))} = \frac{Q'^{(m)}(g_{n-m}K_{n-m}(f))}{Q'^{(m)}(g_{n-m})}.$$

Therefore we obtain the decomposition formula

$$K_n = R_m^{(n)} K_{n-m}$$

with the Markov kernel $R_m^{(n)}$ from S into S defined for each $f \in B_b(S)$ by

$$R_m^{(n)}(f) = \frac{Q'^{(m)}(g_{n-m}f)}{Q'^{(m)}(g_{n-m})}.$$

We conclude that

$$\begin{aligned} K_n &= R_m^{(n)} R_m^{(n-m)} K_{n-2m} \\ &= R_m^{(n)} R_m^{(n-m)} \cdots R_m^{(n-([\frac{n}{m}]-1)m)} K_{n-[\frac{n}{m}]m}. \end{aligned} \tag{9}$$

For any $\mu \in \mathscr{P}(S)$ the distribution $\Phi'_n(\mu) \in \mathscr{P}(S)$ and we have

$$\Phi'_n(\mu) = \Psi'_n(\mu) K_n$$

with the Boltzmann–Gibbs transformation $\Psi'_n : \mathscr{P}(S) \to \mathscr{P}(S)$ defined for each $f \in B_b(S)$ as

$$\Psi'_n(\mu)(f) = \mu(g_n f)/\mu(g_n).$$

Therefore we conclude that for any pair $(\mu_1, \mu_2) \in \mathscr{P}(S)^2$

$$\|\Phi'_n(\mu_1) - \Phi'_n(\mu_2)\| \leq \beta_S(K_n) \|\Psi'_n(\mu_1) - \Psi'_n(\mu_2)\|$$

with the Dobrushin's contraction coefficient defined by the formulae

$$\beta_S(K_n) = \sup_{x,y\in S} \|K_n(x,\cdot) - K_n(y,\cdot)\|, \tag{10}$$

$$= \sup_{\mu_1,\mu_2\in\mathscr{P}(S)} \frac{\|\mu_1 K_n - \mu_2 K_n\|}{\|\mu_1 - \mu_2\|}. \tag{11}$$

Using (9) and (11), we have the estimate

$$\beta_S(K_n) \leq \prod_{p=0}^{[\frac{n}{m}]-1} \beta_S\big(R_m^{(n-pm)}\big). \tag{12}$$

Under our assumption, we note that for each pair $(x, y) \in S^2$, $n \geq m$ and $f \in B_b(S)$, $f \geq 0$

$$R_m^{(n)}(f)(x) \geq \delta'^2 R_m^{(n)}(f)(y).$$

This together with (10) implies that $\beta_S(R_m^{(n)}) \leq (1 - \delta'^2)$ and by (12) we conclude that

$$\beta_S(K_n) \leq (1 - \delta'^2)^{[\frac{n}{m}]}.$$

It is now not difficult to check that

$$\begin{aligned} \|\Psi'_n(\mu_1) - \Psi'_n(\mu_2)\| &\leq 2 \sup_{(x,y)\in S^2} \frac{g_n(x)}{g_n(y)} \|\mu_1 - \mu_2\| \\ &\leq \frac{2}{\delta'} \|\mu_1 - \mu_2\| \ (\text{by}(8)), \end{aligned}$$

from which we conclude that

$$\|\Phi'_n(\mu_1) - \Phi'_n(\mu_2)\| \leq \frac{2}{\delta'} (1 - \delta'^2)^{[\frac{n}{m}]} \|\mu_1 - \mu_2\|. \tag{13}$$

By the Banach fixed point theorem we conclude the existence of a unique distribution $\mu \in \mathscr{P}(S)$ with for each $n \geq 0$

$$\mu = \Phi'_n(\mu).$$

From this fixed point equation we find that for each $f \in B_b(S)$

$$\mu(f)\mu(M(G)) = \mu M(Gf) = \mu(Q' f).$$

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If we take $f = Q'^{(n)}(1)$, we get the recursive formula

$$\mu(Q'^{(n+1)}(1)) = \mu(Q'^{(n)}(1))\mu(Q'(1))$$
$$= \mu(Q'(1))^{n+1} = \mu(M(G))^{n+1}$$

from which we conclude that

$$\log \mu(M(G)) = \frac{1}{n} \log \mu(Q'^{(n)}(1)) = \Lambda_n^{(\mu)}(G).$$

Using the inequality

$$|\log x - \log y| \leq \frac{|x-y|}{|x \wedge y|}$$

we readily find the estimate

$$|\Lambda_n^{(x)}(G) - \Lambda_n^{(\mu)}(G)| = \left| \frac{1}{n} \sum_{p=0}^{n-1} [\log \mu_p^{(x)} M(G) - \log \mu M(G)] \right|$$
$$\leq \sup_{(x,y) \in S^2} \frac{M(G)(x)}{M(G)(y)} \times \frac{1}{n} \sum_{p=0}^{n-1} \|\mu_p^{(x)} - \mu\|.$$

This together with (13) implies that

$$|\Lambda_n^{(x)}(G) - \log \mu M(G)| \leq \frac{1}{r'} \frac{1}{n} \sum_{p=0}^{n-1} \frac{2}{\delta'} (1 - \delta'^2)^{\left[\frac{n}{m}\right]} \leq \frac{c(r', \delta')}{n}$$

for some finite constant $c(r', \delta') \leq \frac{2m}{r'\delta'^3} \frac{1}{n}$. This last assertion is a simple consequence of

$$\sum_{p=0}^{n-1} a^{\left[\frac{p}{m}\right]} = \sum_{p=0}^{m-1} a^{\left[\frac{p}{m}\right]} + \sum_{p=m}^{n-1} a^{\left[\frac{p}{m}\right]}$$
$$= \sum_{p=0}^{m-1} a^{\left[\frac{p}{m}\right]} + \sum_{p=m}^{2m} a^{\left[\frac{p}{m}\right]} + \cdots + \sum_{p=\left[\frac{n-1}{m}\right]m}^{n-1} a^{\left[\frac{p}{m}\right]}$$
$$= m\left(1 + a + \cdots + a^{\left[\frac{n-1}{m}\right]m}\right) \leq \frac{m}{1-a}.$$

This ends the proof of the theorem. $\square$

Next we propose a simple sufficient condition for $(G', Q')_m$ in terms of the pair (G', M').

Assumption. $(G', M')_m$: there exists an integer $m \geq 1$ and a pair $(r', \varepsilon') \in [0, 1]$ such that for each $x, y \in S$

$$G'(x) \geq r' G'(y) \quad \text{and} \quad M'^{(m)}(x, \cdot) \geq \varepsilon' M'^{(m)}(y, \cdot).$$

In the case of indicator functions $G(x) = 1_S(x)$, this condition is related to the mixing properties of the restricted Markov evolution on S.

Proposition 1. *When assumption $(G', M')_m$ holds for some integer $m \geq 1$ and some pair (ε', r') then assumption $(G', Q')_m$ holds with the same $m \geq 1$ and $\delta' \geq r'^m \varepsilon'$.*

Proof. For any nonnegative function $f \in B_b(S)$ and $x, y \in S$ we prove easily the following chain of inequalities

$$\begin{aligned} Q'^{(m)}(f)(x) &= G'(x) M'(Q^{(m-1)}(f))(x) \\ &\geq \left(\inf_{z \in S} G'(z)\right) M'(Q^{(m-1)}(f))(x) \\ &\geq \left(\inf_{z \in S} G'(z)\right)^m M'^{(m)}(f)(x) \\ &\geq \left(\inf_{z \in S} G'(z)\right)^m \times \varepsilon M'^{(m)}(f)(y). \end{aligned}$$

In the same way, we find that

$$Q'^{(m)}(f)(x) \leq \left(\sup_{z \in S} G'(z)\right)^m M'^{(m)}(f)(y)$$

from which we conclude that

$$Q'^{(m)}(f)(x) \geq \left(\inf_{(u,v) \in S^2} \frac{G'(u)}{G'(v)}\right)^m \times \varepsilon Q'^{(m)}(f)(y) \geq r'^m \varepsilon' Q'^{(m)}(f)(y).$$

This ends the proof of the proposition. □

Condition $(G', M')_m$ is not a finite state space condition. We refer the reader to Refs.[5,6] for a collection of examples. In the next example, we show that it holds for a restricted Gaussian transition and simple random walks on compact sets.

Example 1. Suppose M is the Gaussian transition on $E = \mathbb{R}$

$$M(x, dy) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-a(x))^2} dy$$

where a is a measurable drift function and let S be a Borel subset of $\mathbb{R}$ such that

$$\|a\|_S = \sup\{|a(x)|; x \in S\} < \infty \quad \text{and} \quad |S| = \sup\{|x|; x \in S\} < \infty.$$

One has for any $x,\ y \in S$

$$\begin{aligned}\frac{dM(x, .)}{dM(x', .)}(y) &= \exp\{(a(x) - a(x'))(y - [a(x) + a(x')/2])\}\\ &\in [e^{-c(a)}, e^{c(a)}]\end{aligned}$$

with $c(a) \leq 2\|a\|_S(|S| + \|a\|_S)$. This clearly implies that $M(x, S) \geq e^{-c(a)} M(y,\ S)$. For the indicator potential functiofn $G(x) = 1_S(x)$ we conclude that $(G', M')_m$ holds true with $r' = e^{-c(a)}$ and $\varepsilon' = e^{-2c(a)}$.

Example 2. Suppose M is the Markov transition on $E = \mathbb{Z}$ defined by

$$M(x, dy) = p(-1)\delta_{x-1}(dy) + p(0)\delta_x(dy) + p(1)\delta_{x+1}(dy)$$

with $p(i) > 0$ and $\sum_{i=-1}^{1} p(i) = 1$. If we take $S = [0,\ m]$ and $G(x) = 1_S(x)$ for some $m \geq 1$ we find that for any $(x, y) \in [0, m]$, $M'^{(m)}(x,\ y) > 0$ and $M(x,\ S) > 0$. More precisely we have the rather crude estimate

$$M'^{(m)}(x, y) \geq (p(-1) \wedge p(0) \wedge p(1))^m$$

and $M(x, S) = G'(x) \geq p(-1) \wedge p(0) \wedge p(1)$. This readily yields that $(G',\ M')_m$ is met with

$$r' = p(-1) \wedge p(0) \wedge p(1) \quad \text{and} \quad \varepsilon' = (p(-1) \wedge p(0) \wedge p(1))^m.$$

PARTICLE INTERPRETATIONS

In the second part of this article, we design interacting particle systems to approximate the eigen-measure μ and the exponent $\Lambda(G)$. Two strategies can be used depending on the two interpretations of

$\Lambda_n^{(x)}(G)$, namely

$$\Lambda_n^{(x)}(G) = \frac{1}{n}\sum_{p=1}^{n}\eta_p^{(x)}(G) \quad \text{and} \quad \Lambda_n^{(x)}(G) = \frac{1}{n}\sum_{p=0}^{n-1}\eta_p'^{(x)}(G').$$

In the first interpretation we start by observing that under condition **(A)** for any $\eta_0 \in \mathscr{P}_G(E)$ the distribution flow η_p takes values in $\mathscr{P}_G(E)$. To see this claim, we simply notice that for each $n \geq 1$ we have

$$\eta_{n-1}(G) = \gamma_n(1) = \mathbb{E}_{\eta_0}\left(\prod_{p=0}^{n-1} G(X_p)\right) > 0.$$

In addition it satisfies a nonlinear equation

$$\eta_n = \Phi(\eta_{n-1}). \tag{14}$$

The mapping $\Phi : \mathscr{P}(E) \to \mathscr{P}(E)$ is defined for any $\eta \in \mathscr{P}(E)$, $\eta(G) > 0$, by the equation

$$\Phi(\eta) = \eta K_\eta.$$

The collection of Markov transitions $K_\eta(x, dy)$ from E to E are defined by

$$K_\eta(x, dz) = S_\eta M(x, dz) = \int S_\eta(x, dy) M(y, dz)$$

with

$$S_\eta(x, dy) = G(x)\delta_x(dy) + (1 - G(x))\frac{G(y)}{\eta(G)}\eta(dy).$$

The evolution equations of the distribution flow η'_n are defined the same way by replacing (G, M) by (G', M'). We do not describe these structures, we simply use the superscript $(\cdot)'$ to denote the corresponding objects Φ, S'_η, K'_η. The nonlinear measure valued interpretations

$$\eta_n = \Phi(\eta_{n-1}) = \eta_{n-1} K_{\eta_{n-1}} \quad \text{and} \quad \eta'_n = \Phi'(\eta'_{n-1}) = \eta'_{n-1} K'_{\eta'_{n-1}}$$

are related to four different types of interacting particle approximating algorithms.

- The two particle models associated to the distribution flow η'_n can be studied in a simpler way. Indeed, they correspond to particle evolution in an absorbing medium with soft and repulsive obstacles. The asymptotic analysis of these particle models can be studied in essentially the same way as in Ref.[7]. However a more careful analysis is required as the potential function G is non necessarily strictly positive. We will present a detailed analysis of these algorithms including some precise uniform estimates with respect to the time parameter in subsection.

- The two particle interpretations associated to the evolution equation of $\eta_n \in \mathscr{P}(E)$ consists in evolving a set of particles in an absorbing medium with hard obstacles. At a given time it may happen that the whole particle configuration hits the set of hard obstacles. At that time the algorithm stopped and we need to resort to another realization. In subsection, we will see that this event happens with an exponentially small probability. We will propose a novel strategy to control uniformly in time the probability of this event. Roughly speaking the idea is to increase slightly in a logarithmic scale the size of the population so as to control the probability of extinction of the model.

From an algorithmical point of view, the two particles methods associated to particle evolution in an absorbing medium with soft and repulsive obstacles are the most interesting ones as there is no need to increase the size of the population over time. However, it is not always possible to implement these methods as it might not be possible to compute $G'(x) = M(G)(x)$ in closed-form.

Soft and Repulsive Obstacles

The first particle interpretation associated to the flow η'_n consists in a Markov chain

$$\xi'_n = (\xi'^1_n, \ldots, \xi'^N_n) \in S^N$$

with initial distribution $\eta'^{\otimes N}_0 \in \mathscr{P}(S)^{\otimes N}$ and Markov transitions

$$\mathbb{P}(\xi'_n \in d(x^1, \ldots, x^N) | \xi'_{n-1}) = \prod_{p=1}^{N} \Phi'\left(\frac{1}{N}\sum_{i=1}^{N} \delta_{\xi'^i_{n-1}}\right)(dx^p).$$

Noting that

$$\Phi'\left(\frac{1}{N}\sum_{i=1}^{N} \delta_{\xi'^i_{n-1}}\right) = \sum_{i=1}^{N} \frac{G'\left(\xi'^i_{n-1}\right)}{\sum_{j=1}^{N} G'\left(\xi'^j_{n-1}\right)} M'(\xi'^i_{n-1}, \cdot).$$

We conclude that each elementary transition $\xi'_{n-1} \to \xi'_n$ is decomposed into two separate generic selection/mutation transitions

$$\xi'_{n-1} \xrightarrow{\text{selection}} \widehat{\xi}'_{n-1} \xrightarrow{\text{mutation}} \xi'_n.$$

The selection stage consists in sampling N conditionally independent random variables $\widehat{\xi}'^i_{n-1}$ with common law

$$\sum_{i=1}^{N} \frac{G'(\xi'^i_{n-1})}{\sum_{j=1}^{N} G'(\xi'^j_{n-1})} \delta_{\xi'^i_{n-1}} \quad \text{with } G'(\xi'^i_{n-1}) = M(G)(\xi'^i_{n-1}).$$

During the mutation stage each selected particle $\widehat{\xi}'^i_{n-1}$ evolves independently of the others according to the mutation transition

$$M'\left(\widehat{\xi}'^i_{n-1}, dx\right) = \frac{M\left(\widehat{\xi}'^i_{n-1}, dx\right) G(x)}{M(G)\left(\widehat{\xi}'^i_{n-1}\right)}.$$

It is instructive to examine the situation where $G = 1_S$. In this case, we have

$$G'(x) = M(x, S) \quad \text{and} \quad M'(x, dy) = \frac{M(x, dy) 1_S(y)}{M(x, S)}.$$

The selection stage gives more opportunities to a particle ξ'^i_{n-1} to reproduce when the probability $M(\xi'^i_{n-1}, S)$ to stay in S at the next step is large. The mutation transition consists in evolving the selected particle $\widehat{\xi}'^i_{n-1}$ according to the Markov transition M restricted to the set S. An alternative particle scheme consists in defining a Markov chain

$$\xi'_n = (\xi'^1_n, \ldots, \xi'^N_n) \in E^N$$

with initial distribution $\eta'^{\otimes N}_0$ and Markov transitions

$$\mathbb{P}(\xi'_n \in d(x^1, \ldots, x^N) \mid \xi'_{n-1}) = \prod_{p=1}^{N} S_{\frac{1}{N}\sum_{i=1}^{N} \delta_{\xi'^i_{n-1}}} M(\xi'^i_{n-1}, dx^p).$$

Similar arguments as above show that this algorithm is again defined in terms of two separate genetic selection/mutation transitions but the selection transition consists in selecting randomly the particle $\widehat{\xi}'^i_{n-1}$ with the distribution

$$S_{\frac{1}{N}\sum_{i=1}^{N}\delta_{\xi'^i_{n-1}}}(\xi'^i_{n-1}, dx) = G'(\xi'^i_{n-1})\delta_{\xi'^i_{n-1}}(dx) + (1 - G'(\xi'^i_{n-1}))\sum_{j=1}^{N}\frac{G'(\xi'^j_{n-1})}{\sum_{k=1}^{N} G'(\xi'^k_{n-1})}\delta_{\xi'^j_{n-1}}(dx)$$

This particle algorithm contains less randomness than the previous one. For instance, in the lattice model examined in Example 2, we have

$$G'(x) = M(x, S) = 1 \quad \text{for each } x \in (0, m) = S - \{0, m\},$$
$$G'(0) = M(0, S) = p(0) + p(1) \quad \text{and} \quad G'(m) = p(-1) + p(0).$$

In this context, the particles evolving in $S - \{0, m\}$ are not affected by the selection mechanism while the particles in location $\{0, m\}$ tend to change and move to another location. These two particle approximating models can be studied using the same line of arguments as the one performed in Ref.[6]. More precisely, if we set

$$\eta'^{(N)}_n = \frac{1}{N}\sum_{i=1}^{N}\delta_{\xi'^i_n}$$

then under the regularity mixing condition $(G', M')_m$ we have for any $f \in B_b(S)$, $\|f\| \leq 1$, $p \geq 1$, the uniform estimate

$$\sup_{n\geq 0} \mathbb{E}\left(\left|\eta'^{(N)}_n(f) - \eta'_n(f)\right|^p\right)^{1/p} \leq c(m)\frac{b_p}{\sqrt{N}} \tag{15}$$

for some universal constant b_p only depending on the parameter $p \geq 1$ and

$$c(m) \leq \frac{m}{\varepsilon'^4 r'^{(3m-1)}}.$$

Since we have $\eta'^{(N)}_n$, $\eta'_n(f) \in \mathscr{P}(S)$ the above estimate clearly holds for each $f \in B_b(E)$. Recalling that η'_n coincides with the distribution μ_n, if we combine the above estimate with Theorem 1 and Proposition 1 we get the following result.

Theorem 2. *Suppose condition $(G', M')_m$ is satisfied for some $m \geq 1$ and some pair $(r', \varepsilon') \in [0, 1]$ then the limiting distribution μ of the flow $\mu_n = \eta'_n$ is the eigen-measure associated to the Lyapunov exponent $e^{\Lambda(G)}$ and for*

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any $f \in B_b(E)$, $\|f\| \leq 1$, $p \geq 1$ *we have the uniform estimate*

$$\sup_{n \geq d(m)\log N} \mathbb{E}\left(\left|\eta_n'^{(N)}(f) - \mu(f)\right|^p\right)^{1/p} \leq c(m)\frac{b(p)}{\sqrt{N}}$$

for some universal constant $b(p)$,

$$c(m) \leq \frac{m}{\varepsilon'^4 r'^{(3m-1)}} \vee \frac{2}{\varepsilon' r'^m (1 - (\varepsilon' r'^m)^2)}$$

and

$$d(m) \leq \frac{m}{2\log\left(1/\left(1 - (\varepsilon' r'^m)^2\right)\right)}.$$

Proof. It is sufficient to notice that

$$\begin{aligned}
&\mathbb{E}\left(\left|\eta_n'^{(N)}(f) - \mu(f)\right|^p\right)^{1/p} \\
&\quad \leq c(m)\frac{b_p}{\sqrt{N}} + \frac{2}{\varepsilon' r'^m}\left(1 - (\varepsilon' r'^m)^2\right)^{[\frac{n}{m}]} \\
&\quad \leq c(m)\frac{b_p}{\sqrt{N}} + \frac{2}{\varepsilon' r'^m}\frac{1}{\left(1 - (\varepsilon' r'^m)^2\right)}\left(1 - (\varepsilon' r'^m)^2\right)^{\frac{n}{m}} \\
&\quad \leq c'(m)\left(\frac{b_p}{\sqrt{N}} + (1 - a_m)^{\frac{n}{m}}\right)
\end{aligned}$$

with $a_m = r'^m \varepsilon'$ and $c'(m) = c'(m) \vee \frac{2}{a_m(1-a_m^2)}$. For any pair (n, N) we have

$$\frac{1}{\sqrt{N}} \geq (1 - a_m)^{n/m} \Longleftrightarrow \frac{1}{2}\log N \leq \frac{n}{m}\log\left(\frac{1}{1 - a_m}\right).$$

This implies that for any n with $n \geq m/\left[2\log\left(\frac{1}{1-a_m}\right)\right]$ we have

$$\mathbb{E}\left(\left|\eta_n'^{(N)}(f) - \mu(f)\right|^p\right)^{1/p} \leq c'(m)\frac{b_p + 1}{\sqrt{N}}.$$

This ends the proof of the theorem. □

From the above theorem and by Theorem 1, we can construct two particle approximations of the logarithmic Lyapunov exponent $\Lambda(G)$ introduced in (3). We define

$$\Lambda_n'^{(N,x)}(G) = \frac{1}{n}\sum_{p=0}^{n-1} \log \eta_p'^{(N,x)}(G')$$

where $\eta_n'^{(N,x)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_n'^{(i,x)}}$ is the particle density profile associated to the particle model starting at $\eta_0^{\otimes N} = \delta_x^{\otimes N}$. By the definition of the genetic particle models, it is well known (cf. for instance Ref.[3] that

$$e^{n\Lambda_n^{(N,x)}(G)} = \prod_{p=0}^{n-1} \eta_p'^{(N,x)}(G') = \gamma_n'^{(N,x)}(1)$$

is an unbiased estimate of $e^{n\Lambda_n^{(x)}(G)}$, that is we have

$$\mathbb{E}\left(e^{n\Lambda_n^{(N,x)}(G)}\right) = e^{n\Lambda_n^{(x)}(G)} = \mathbb{E}\left(\gamma_n'^{(N,x)}(1)\right) = \gamma_n'^{(x)}(1).$$

Corollary 2. *Under the assumption $(G', M')_m$, we have for any $p \geq 1$ and $x \in S$*

$$\sup_{n \geq d(m)\log N} \mathbb{E}\left(\left|\eta_n'^{(N,x)}(MG) - \exp(\Lambda(G))\right|^p\right)^{1/p} \leq c(m)\frac{b(p)}{\sqrt{N}}$$

with the same constants $b(p)$, $c(m)$, $d(m)$ as in Theorem 2. In addition we have the uniform estimate

$$\sup_{x\in S}\sup_{n\geq\sqrt{N}} \mathbb{E}\left(\left|\Lambda_n'^{(N,x)}(G) - \Lambda(G)\right|^p\right)^{1/p} \leq \frac{b(p)\cdot c}{\sqrt{N}}$$

for some finite constants $b(p)$ and c which only depend respectively on p and (m, ε', r').

Proof. If we take $f = M(G) = G'$ in Theorem 2 we readily find the first estimate. To prove the second one, we observe that

$$\mathbb{E}\left(\left|\Lambda_n'^{(N,x)}(G) - \Lambda_n^{(x)}(G)\right|^p\right)^{1/p} \leq \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{E}(|\log \eta_k'^{(N,x)}(G') - \log \eta_k'^{(x)}(G')|^p)^{1/p}.$$

Recalling that for any $u,\ v>0$ we have $|\log u - \log v| \le |u-v|/(u\wedge v)$ we find that

$$\mathbb{E}\left(\left|\Lambda_n'^{(N,x)}(G) - \Lambda_n^{(x)}(G)\right|^p\right)^{1/p} \le \frac{1}{\inf_S G'}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{E}(|\eta_k'^{(N,x)}(G') - \eta_k'^{(x)}(G')|^p)^{1/p}.$$

Under our assumptions we have for any $x,\ y\in S$

$$r'G'(y) \le G'(x) \le 1.$$

Therefore we have $(\inf_S G') \ge r'(\sup_S G')$ and by (15) we prove the uniform estimate

$$\mathbb{E}\left(\left|\Lambda_n'^{(N,x)}(G) - \Lambda_n^{(x)}(G)\right|^p\right)^{1/p} \le c(m)\frac{b(p)}{\sqrt{N}} \quad \text{with } c(m) \le \frac{m}{\varepsilon'^4 r'^{(3m)}}.$$

By Theorem 1 and Proposition 1, we conclude that

$$\begin{aligned}\mathbb{E}&\left(\left|\Lambda_n'^{(N,x)}(G) - \Lambda(G)\right|^p\right)^{1/p}\\ &\le \frac{b(p).c(m)}{\sqrt{N}} + \frac{2m}{\varepsilon'^3 r'^{(3m+1)}}\frac{1}{n}\\ &\le (b(p)\vee 1)\left(c(m)\vee\frac{2m}{\varepsilon'^3 r'^{(3m+1)}}\right)\left(\frac{1}{\sqrt{N}}+\frac{1}{n}\right).\end{aligned}$$

and the end of the proof of the corollary is now clear. □

Soft and Hard Obstacles

The particle interpretations of the flow η_n are conducted along the same construction as the ones for the flow η_n' by replacing (G', M') by (G, M). To distinguish the particle models associated to η_n to the particle models associated to η_n' we simply suppress the superscript $(\cdot)'$. The essential difference with the previous particle models is that the whole configuration $\xi_n = (\xi_n^1, \ldots, \xi_n^N)$ may hit the set of hard obstacles $E - S$.

Let τ^N be this hitting time

$$\tau^N = \inf\{n \geq 0 : \eta_n^N(G) = 0\}$$

As usual we add a cemetery point to the state space $E_c^N = E^N \cup \{c\}$ and we set

$$\forall n \geq \tau^N \quad \widehat{\xi}_n = \xi_{n+1} = c.$$

We also extend the functions $f \in B_b(E)$ on E_c by setting $f(c) = 0$. Without any assumption on the pair (G, M) but assuming that $\gamma_n(1) > 0$, we proved in Ref.[8] that

$$\mathbb{P}(\tau^N \leq n) \leq c(n) e^{-N/c(n)}$$

for some finite constant $c(n) < N$ only depending on the time parameter. Consider now the rooted tree example discussed earlier. In this example, we know the algorithm stops almost surely at the maximal unknown depth D, that is we have that

$$\mathbb{P}(\tau^N \leq D) = 1.$$

Using the above exponential estimate, we readily conclude that

$$\mathbb{P}(\tau^N = D) = 1 - \mathbb{P}(\tau^N < D) \geq 1 - c(D) e^{-N/c(D)}.$$

This shows that the particle model dies at time D on the deepest depth with an exponential rate. Unfortunately the constant $c(D)$ obtained in Ref.[8] is too large to study with more precision if the algorithm finds with a given probability the deepest depth in polynomial time or not. Next we strength condition **(A)** and we suppose that the Markov kernel M satisfies the uniform accessibility condition.

Assumption (B). There exists a constant $\alpha(S) > 0$ such that

$$\sup_{x \in S} M(x, E - S) \leq e^{-\alpha(S)}.$$

This condition clearly holds true in the two examples given in Section.

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Lemma 1. *When assumption (B) is satisfied for some* $\alpha(S) > 0$ *then for any* $n \geq 0$ *and* $\eta_0 \in \mathscr{P}(E)$, $\eta_0(S) = 1$, *we have*

$$\mathbb{P}(\tau^N \leq n) \leq n e^{-N\alpha(S)}.$$

Proof. By a simple induction argument we first observe that

$$\begin{aligned}
\mathbb{P}(\tau^N > n) &= \mathbb{P}(\tau^N > n-1 \text{ and } \eta_n^N(G) > 0) \\
&= \mathbb{E}(\mathbb{P}(\exists\, 1 \leq i \leq N \ \xi_n^i \in S \,|\, \xi_0, \ldots, \xi_{n-1}) 1_{\tau^N > n-1}) \\
&= \mathbb{P}(\tau^N > n-1) \\
&\quad - \mathbb{E}(\mathbb{P}(\forall\ 1 \leq i \leq N \ \xi_n^i \in E - S \,|\, \xi_0, \ldots, \xi_{n-1}) 1_{\tau^N > n-1}).
\end{aligned}$$

By definition of the particle models, we have in the first interpretation

$$\begin{aligned}
&\mathbb{P}(\forall\ 1 \leq i \leq N \ \xi_n^i \in E - S \,|\, \xi_0, \ldots, \xi_{n-1}) 1_{\tau^N > n-1} \\
&\quad = \left(\frac{1}{N} \sum_{i=1}^N S_{\frac{1}{N}\sum_{j=1}^N \delta_{\xi_{n-1}^j}} M(\xi_{n-1}^i, E - S) \right)^N 1_{\tau^N > n-1}
\end{aligned}$$

or in the second one

$$= \prod_{i=1}^N \left(S_{\frac{1}{N}\sum_{j=1}^N \delta_{\xi_{n-1}^j}} M \right)(\xi_{n-1}^i, E - S) 1_{\tau^N > n-1}.$$

Since for any $\eta \in \mathscr{P}(E)$ with $\eta(S) > 0$ we have $S_\eta(x, S) = 1$ for any $x \in E$, we readily get the estimate

$$\forall x \in E \ S_\eta M(x, E - S) \leq e^{-\alpha(S)}$$

from which we conclude that

$$\begin{aligned}
\mathbb{P}(\tau^N > n) &\geq \mathbb{P}(\tau^N > n-1) - \mathbb{P}(\tau^N > n-1) e^{-N\alpha(S)} \\
&\geq \mathbb{P}(\tau^N > n-1) - e^{-N\alpha(S)} \geq 1 - n e^{-N\alpha(S)}.
\end{aligned}$$

This ends the proof of the lemma. $\square$

The particle approximating measures of γ_n and η_n are defined for any $f \in B_b(E_c)$ by

$$\gamma_n^N(f) = \eta_n^N(f)\gamma_n^N(1) \quad \text{with } \gamma_n^N(1) = \prod_{p=0}^{n-1} \eta_p^N(G).$$

Note that for $f = 1$ we have

$$\gamma_n^N(1)1_{\tau^N \geq n} = \prod_{p=0}^{n-1} \eta_p^N(G)1_{\tau^N > p}, \quad \text{and} \quad 1_{\tau^N \geq n} = \prod_{p=0}^{n-1} 1_{\tau^N > p}$$

and $\eta_n^N(f)1_{\tau^N \geq n} = \frac{\gamma_n^N(f)}{\gamma_n^N(1)} 1_{\tau^N \geq n}$ with the convention $0/0 = 0$. We denote by Q the bounded integral operator on $\mathscr{B}_b(E)$ defined for any $f \in \mathscr{B}_b(E)$ and $x \in E$ by

$$Q(f)(x) = G(x)M(f)(x)$$

As usual we denote by $Q^{(n)}$ the corresponding semigroup defined by the inductive formula $Q^{(n)} = Q^{(n-1)}Q$ and we use the convention $Q^{(0)} = Id$.

Proposition 2. *If assumption (B) holds, then for each $n \geq 0$ and $f \in B_b(E)$ the $\mathbb{R}$-valued process*

$$p \leq n \to \Gamma_{p,n}^{(N)}(f) = \gamma_p^N(Q^{(n-p)}f)1_{\tau^N \geq p} - \gamma_p(Q^{(n-p)}f)$$

is a martingale with respect to the filtration $F_n^N = \sigma(\xi_0, \ldots, \xi_n)$ generated by the particle model. In the first particle interpretation its angle bracket is given by

$$\left\langle \Gamma_{0,n}^{(N)}(f) \right\rangle_p = \frac{1}{N} \sum_{q=0}^{p} \gamma_q^N(1)^2 1_{\tau^N \geq q} \Phi(\eta_{q-1}^N)$$
$$\times ((Q^{(n-p)}f - \Phi(\eta_{q-1}^N)(Q^{(n-p)}f))^2)$$

and in the second particle interpretation

$$\left\langle \Gamma_{0,n}^{(N)}(f) \right\rangle_p = \frac{1}{N} \sum_{q=0}^{p} \gamma_q^N(1)^2 1_{\tau^N \geq q} \eta_{q-1}^N$$
$$\times \left(K_{\eta_{q-1}^N}\left(\left[Q^{(n-p)}f - \Phi(\eta_{q-1}^N)\left(Q^{(n-p)}f\right)\right]^2\right)\right)$$

with the convention for $q = 0$, $\eta_{-1}^N = \Phi(\eta_{-1}^N) = K\eta_{-1}^N = \eta_0$.

Proof. We use the decomposition for each $\varphi \in \mathscr{B}_b(E)$

$$\gamma_p^N(\varphi)1_{\tau^N \geq p} - \gamma_p(\varphi) = \sum_{q=0}^{p} [\gamma_q^N(Q^{(p-q)}\varphi)1_{\tau^N \geq q} - \gamma_{q-1}^N(Q^{(p-q+1)}\varphi)1_{\tau^N \geq q-1}] \tag{16}$$

with the convention for $q = 0$, $\gamma_{-1}^N(Q^{(p+1)}\varphi)1_{\tau^N \geq -1} = \gamma_p(\varphi)$. Observe that

$$\gamma_q^N(Q^{(p-q)}\varphi)1_{\tau^N \geq q} = \gamma_q^N(1)1_{\tau^N \geq q}\eta_q^N(Q^{(p-q)}\varphi) \tag{17}$$

and

$$\begin{aligned}
\gamma_{q-1}^N(Q^{(p-q+1)}\varphi)1_{\tau^N \geq q-1} &= \gamma_{q-1}^N(GM(Q^{(p-q)}\varphi))1_{\tau^N \geq q-1} \\
&= \gamma_{q-1}^N(1)1_{\tau^N \geq q-1}\eta_{q-1}^N(GM(Q^{(p-q)}\varphi)).
\end{aligned}$$

Since $1 = 1_{\eta_{q-1}^N(G)=0} + 1_{\eta_{q-1}^N(G)>0}$ and

$$\eta_{q-1}^N(GM(Q^{(p-q)}\varphi))1_{\eta_{q-1}^N(G)=0} = 0$$

we conclude that

$$\begin{aligned}
\gamma_{q-1}^N(Q^{(p-q+1)}\varphi)1_{\tau^N \geq q-1} &= \gamma_{q-1}^N(1)1_{\tau^N \geq q}\eta_{q-1}^N(GM(Q^{(p-q)}\varphi)) \\
&= \gamma_{q-1}^N(1)1_{\tau^N \geq q}\Phi(\eta_{q-1}^N)(Q^{(p-q)}\varphi).
\end{aligned}$$

If we set $\varphi = Q^{(n-p)}(f)$, for some $f \in \mathscr{B}_b(E)$ we find that

$$\begin{aligned}
&\gamma_p^N(Q^{(n-p)}f)1_{\tau^N \geq p} - \gamma_p(Q^{(n-p)}f) \\
&\quad = \sum_{q=0}^{p} \gamma_q^N(1)1_{\tau^N \geq q}[\eta_q^N(Q^{(n-p)}f) - \Phi(\eta_{q-1}^N)(Q^{(n-p)}f)] \\
&\qquad \text{(1st interpretation)} \\
&\quad = \sum_{q=0}^{p} \gamma_q^N(1)1_{\tau^N \geq q}[\eta_q^N(Q^{(n-p)}f) - \eta_{q-1}^N K_{\eta_{q-1}^N}(Q^{(n-p)}f)] \\
&\qquad \text{(2nd interpretation).}
\end{aligned}$$

The end of the proof is now clear. □

Corollary 3. *For each $f \in B_b(E_c)$, $\|f\| \leq 1$ and $n \geq 0$, $p \geq 1$ we have*

$$\mathbb{E}(\gamma_n^N(f)1_{\tau^N \geq n}) = \gamma_n(f) \tag{18}$$

and

$$\mathbb{E}\left(\left|\gamma_n^N(f)1_{\tau^N \geq n} - \gamma_n(f)\right|^p\right)^{1/p} \leq \frac{b_p c(n)}{\sqrt{N}} \tag{19}$$

for some universal constant b_p and a finite constant $c(n) \leq n+1$.

Proof. The first assertion is a simple consequence of Proposition 2. We also prove the second estimate using Marcinkiwicz-Zygmund's inequality (cf. for instance Ref.[9]).

To conclude this discussion on the un-normalized approximating measures, we notice that if we define the particle log-Lyapunov exponent estimates as

$$\Lambda_n^{(N,x)}(G) = \frac{1}{n}\log(\gamma_n^{(N,x)}(1)1_{\tau^N \geq n}) = \frac{1}{n}\sum_{p=0}^{n-1}\log(\eta_p^{(N,x)}(G)1_{\tau^N > p})$$

then (18) reads

$$\mathbb{E}(e^{n\Lambda_n^{(N,x)}(G)}) = e^{n\Lambda_n^{(N,x)}(G)}.$$

By Markov's inequality, it also follows from (19) that for any $p \geq 1$ and $\varepsilon \in (0, 1)$

$$\mathbb{P}(\gamma_n^N(1)1_{\tau^N \geq n} \geq \varepsilon\gamma_n(1)) \geq 1 - b_p\left(\frac{\tilde{c}(n)}{1-\varepsilon}\right)^p \frac{1}{N^{p/2}}$$

for some universal constant b_p and $\tilde{c}(n) \leq (n+1)/\gamma_n(1)$. □

Theorem 3. *If assumption (B) holds for some $\alpha = \alpha(S) > 0$ then for any $n \geq 0$ and $f \in B_b(E)$ with $\|f\| \leq 1$, we have for any $N \geq N(\alpha)$*

$$|\mathbb{E}(\eta_n^N(f)1_{\tau^N \geq n}) - \eta_n(f)| \leq \frac{c(n)}{N}$$

for some finite constant $c(n) \leq c \cdot n/\gamma(1)$, $c < \infty$ and a collection of integer parameters $N(\alpha) \geq 1$ depending on α. In addition for any $p \geq 1$ we have for $N \geq N(\alpha)$

$$\mathbb{E}\left(\left|\eta_n^N(f)1_{\tau^N \geq n} - \eta_n(f)\right|^p\right)^{1/p} \leq \frac{b_p \cdot c(n)}{\sqrt{N}}$$

for some universal constants $b_p < \infty$ *which only depend on the parameter* $p \geq 1$ *and* $c(n) \leq n/\gamma(1)$.

Proof. We use the decomposition

$$\begin{aligned}(\eta_n^N(f) - \eta_n(f))1_{\tau^N \geq n} &= \left(\frac{\gamma_n^N(f)}{\gamma_n^N(1)} - \frac{\gamma_n(f)}{\gamma_n(1)}\right)1_{\tau^N \geq n} \\ &= \frac{\gamma_n(1)}{\gamma_n{}^N(1)}\gamma_n^N\left(\frac{1}{\gamma_n(1)}(f - \eta_n(f))\right)1_{\tau^N \geq n}. \qquad (20)\end{aligned}$$

If we set

$$f_n = \frac{1}{\gamma_n(1)}(f - \eta_n(f))$$

then, since $\gamma_n(f_n) = 0$, (20) also reads

$$(\eta_n^N(f) - \eta_n(f))1_{\tau^N \geq n} = \frac{\gamma_n(1)}{\gamma_n^N(1)}(\gamma_n^N(f_n)1_{\tau^N \geq n} - \gamma_n(f_n))1_{\tau^N \geq n}.$$

By Proposition 2, we have

$$\mathbb{E}(\gamma_n^N(f_n)1_{\tau^N \geq n}) = \gamma_n(f_n)$$

This implies that

$$\begin{aligned}&\mathbb{E}((\eta_n^N(f) - \eta_n(f))1_{\tau^N \geq n}) \\ &\quad = \mathbb{E}\left(\left(\frac{\gamma_n(1)}{\gamma_n^N(1)} - 1\right)(\gamma_n^N(f_n)1_{\tau^N \geq n} - \gamma_n(f_n))1_{\tau^N \geq n}\right) \\ &\quad = \mathbb{E}\left(\frac{\gamma_n(1)}{\gamma_n^N(1)}\left(1 - \frac{\gamma_n^N(1)}{\gamma_n(1)}\right)(\gamma_n^N(f_n)1_{\tau^N \geq n} - \gamma_n(f_n))1_{\tau^N \geq n}\right) \\ &\quad = \mathbb{E}\left(\frac{\gamma_n(1)}{\gamma_n^N(1)}\left(1 - \frac{\gamma_n^N(1)}{\gamma_n(1)}\right)1_{\tau^N \geq n}(\gamma_n^N(f_n)1_{\tau^N \geq n} - \gamma_n(f_n))\right).\end{aligned}$$

If we set $h_n = \frac{1}{\gamma_n(1)} - 1$ we get the formula

$$\begin{aligned}\mathbb{E}((\eta_n^N(f_n) - \eta_n(f_n))1_{\tau^N \geq n}) &= -\mathbb{E}(\frac{\gamma_n(1)}{\gamma_n^N(1)}(\gamma_n^N(h_n)1_{\tau^N \geq n} - \gamma_n(h_n)) \\ &\qquad \times (\gamma_n^N(f_n)1_{\tau^N \geq n} - \gamma_n(f_n))1_{\tau^N \geq n}). \qquad (21)\end{aligned}$$

Let $\Omega^N_{\varepsilon,n}$ be the set of events

$$\begin{aligned}\Omega^N_{\varepsilon,n} &= \{\gamma^N_n(1)1_{\tau^N\geq n} \geq \gamma_n(1)\varepsilon\}\\ &= \{\gamma^N_n(1) \geq \gamma_n(1)\varepsilon \text{ and } \tau^N \geq n\}\\ &= \left\{\frac{\gamma_n(1)}{\gamma^N_n(1)} \leq \frac{1}{\varepsilon} \text{ and } \tau^N \geq n\right\} \subset \{\tau^N \geq n\}.\end{aligned}$$

We recall that for any $p \geq 1$ we have

$$\mathbb{P}(\Omega^N_{\varepsilon,n}) \geq 1 - b_p\left(\frac{\tilde{c}(n)}{1-\varepsilon}\right)^p \frac{1}{N^{p/2}}$$

with $\tilde{c}(n) \leq n/\gamma_n(1)$ and a universal constant $b_p < \infty$. If we combine this estimate with (21) we find that for any $f \in \mathscr{B}_b(E)$, with $\|f\| \leq 1$,

$$\begin{aligned}&|\mathbb{E}((\eta^N_n(f_n) - \eta_n(f_n))1_{\tau^N\geq n})|\\ &\quad\leq \left|\mathbb{E}\left((\eta^N_n(f_n) - \eta_n(f_n))1_{\Omega^N_{\varepsilon,n}}\right)\right| + 2\mathbb{P}((\Omega^N_{\varepsilon,n})^c)\\ &\quad\leq \frac{1}{\varepsilon}\mathbb{E}(|\gamma^N_n(h_n)1_{\tau^N\geq n} - \gamma_n(h_n)||\gamma^N_n(f_n)1_{\tau^N\geq n} - \gamma_n(f_n)|)\\ &\qquad + 2b_p\left(\frac{\widetilde{c}(n)}{1-\varepsilon}\right)^p \frac{1}{N^{p/2}}\end{aligned}$$

By Corollary 3 and Cauchy–Schwartz inequality this implies that

$$|\mathbb{E}((\eta^N_n(f_n) - \eta_n(f_n))1_{\tau^N\geq n})| \leq \frac{1}{\varepsilon}a\frac{c(n)}{N} + 2b_p\left(\frac{\tilde{c}(n)}{1-\varepsilon}\right)^p\frac{1}{N^{p/2}}$$

for some universal constant a, $\tilde{c}(n) \leq n/\gamma_n(1)$, $c(n) \leq n$ and a universal constant $b(p)$ which only depends on the parameter p. Finally by Lemma 1 and setting $\varepsilon = 1/2$, we conclude that

$$|\mathbb{E}(\eta^N_n(f_n)1_{\tau^N\geq n} - \eta_n(f_n))| \leq a\left(\frac{c(n)}{N} + b_p\tilde{c}(n)^p\frac{1}{N^{p/2}} + ne^{-N\alpha(S)}\right)$$

for some universal constant $a < \infty$ and $\tilde{c}(n) \leq n/\gamma_n(1)$, $c(n) \leq n$. For N sufficiently large and taking $p = 2$ we have the crude estimate

$$|\mathbb{E}(\eta^N_n(f_n)1_{\tau^N\geq n} - \eta_n(f_n))| \leq \frac{c(n)}{N}$$

for some finite constant $c(n) \leq c_0(n+1)/\gamma_n(1)$ Using similar arguments, we prove the second assertion and complete easily the proof of the theorem. □

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Combining Corollary 1 and Theorem 3, we obtain a particle approximation of the log-Lyapunov exponent $\Lambda(G) = \log \eta(G)$ and the corresponding eigen-measure $\eta \in \mathscr{P}(E)$

$$\eta(GM(f)) = e^{\Lambda(G)} \eta(f).$$

Corollary 4. *If Assumption* (B) *holds then there exists a non-decreasing sequence of time parameters* $n(N) \to \infty$ *as* $N \to \infty$ *such that for any* $f \in B_b(E)$ *and* $p \geq 1$ *then*

$$\lim_{N\to\infty} \sup_{x \in S} \mathbb{E}\left(\left(\eta_{n(N)}^{(N,x)}(f) 1_{\tau^N \geq n(N)} - \eta(f)\right)^p\right)^{1/p} = 0$$

as well as

$$\lim_{N\to\infty} \sup_{x \in S} \mathbb{E}\left(\left(\log \eta_{n(N)}^{(N,x)}(G) 1_{\tau^N \geq n(N)} - \Lambda(G)\right)^p\right)^{1/p} = 0,$$

and

$$\lim_{N\to\infty} \sup_{x \in S} \mathbb{E}\left(\left(\Lambda_{n(N)}^{(N,x)}(G) - \Lambda(G)\right)^p\right)^{1/p} = 0$$

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