A New Method to Order Functions by Asymptotic Growth Rates

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ABSTRACT

A new method is described to determine the complexity classes of functions by comparison. This method is a similar to that of L'Hospital. It applies the logarithmic function to the given functions to and then find the limits.

The new rule can compare the complexity classes of most functions, giving more information than L'Hospital's Rule, and in some cases, solving some that are difficult to solve, by purely using L'Hospital's Rule.

There are some functions, however, which one has to revert to applying L'Hospital's Rule.

Keywords: Algorithms, Complexity, Big-Oh, Asymptotic Growth, L'Hospital's Rule.

1. Introduction

In this paper, we discuss the use of a new approach in dealing with complexity rankings of functions in an Algorithm Analysis Course. We introduce a new method and attempt to analyze some obvious, and some not so obvious functions, in terms of their growth.

The method gives a better analytical basis, rather than the sometimes popular and intuitional (among students), plug 'n play (plugging large numbers in the functions and hoping for the best) method. Another method is to either use Maple or Matlab, to plot graphs or write a program that would tell you the comparative orders of complexity. The method introduced in this paper is still preferable, in that, one -- it would take a very brief time to use it to evaluate the relative complexity orders of two functions, thus quickly assessing whether one algorithm is better than another. Secondly, by continual use of it, one becomes adept to stating whether one function grows faster than another.

2. Definitions

This section gives first an informal reminder of the symbols used in complexity, and then some formal definitions of these terms. These symbols (O, Ω, o, θ, ω) also called Landau symbols, have precise mathematical definitions. They have many uses, such as in the analysis of algorithms. These symbols are used to evaluate and to concisely describe the performance of an algorithm in time and space.

Since the properties related to these symbols hold for asymptotic notations, one can draw an analogy between the asymptotic comparison of two functions *f* and *g* and the comparison of two real numbers *a* and *b*. We will use this analogy, in the table below to give a brief informal reminder of the symbols names and their use:

Table 2.1 Landau Symbols

Symbol	Name	Usage	Analogy	
O	Big Oh	Upper Bound	$f = O(g)$	$a \le b$
Ω	Big Omega	Lower Bound	$f = \Omega(g)$	$a \ge b$
θ	Theta	Same Order	$f = \theta(g)$	$a = b$
o	Little Oh	Strict Upper Bound	$f = o(g)$	$a < b$
ω	Little Omega	Strict Lower Bound	$f = \omega(g)$	$a > b$

An illustration of **O** and **Ω** are given in the Figures 2.1 and 2.2 below:

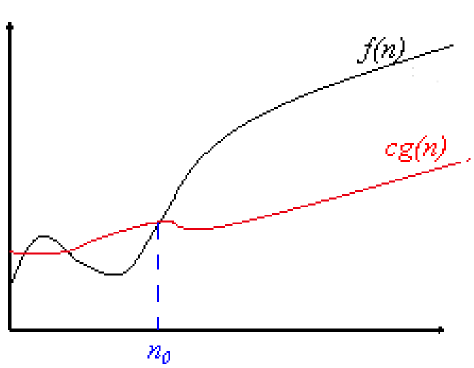

Figure 2.1 $f(n) = \Omega(g(n))$

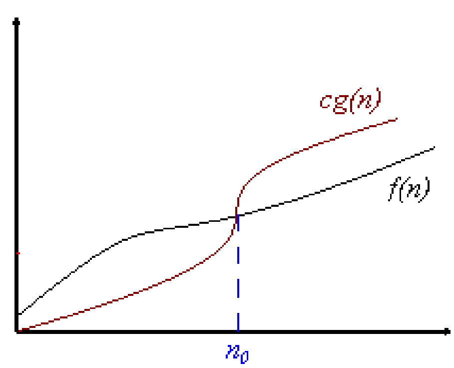

Figure 2.2 $f(n) = O(g(n))$

We now formally define Big "Oh", Theta and Big "Omega". The reader can skip this part, as it is not mandatory to understand the rest of the paper. We define f and g to be functions from the set of integers to the set of positive real numbers.

Definition 2.1 [Big "Oh"] [Upper Bound]

***f*(*n*) = O(*g*(*n*))** iff there exist positive constants c and n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

Example: $3n + 2 = O(n)$ as $3n + 2 \geq 4n$ for $n \geq 2$.

Definition 2.2 [Big "Omega"] [Lower Bound]

***f*(*n*) = Ω(*g*(*n*))** iff there exist positive constants c and n_0 such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$.

Example: $3n + 2 = \Omega(n)$ as $3n + 2 \geq 3n$ for $n \geq 1$

Definition 2.3 [Theta] [Tight Bound]

***f*(*n*) = θ(*g*(*n*))** iff there exist positive constants c_1, c_2 and n_0 such that $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0$.

Example: $3n + 2 = \theta(n)$ as $3n \leq 3n + 2 \leq 4n$ for $n \geq 2$.

2.1 Calculation of Big Oh Relations

Let $\lim_{n \to \infty} \frac{f(n)}{g(n)} = L.$

Then if $L = 0$, $f(n) = O(g(n))$. In particular, $f(n) = o(g(n))$.

If $L = \infty$, $f(n) = \Omega(g(n))$ and in particular, $f(n) = \omega(g(n))$.

If $0 < L < \infty$, $f(n) = \theta(g(n))$. (2.1)

3. L′Hospital (l′Hôpital)′s Rule

$\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is not always easy to calculate. For example take $n^2/3^n$. Since both n^2 and 3^n go to ∞ as n goes to ∞ and there is no apparent factor common to both, the calculation of the limit is not immediate. One tool we may be able to use in such cases is L'Hospital's Rule, which is given as a theorem below.

Theorem 3.1 [L'Hospital's Rule]

Let $\lim_{n\to\infty} f(n) = \infty$ and $\lim_{n\to\infty} g(n) = \infty$, and let $f(n)$ and $g(n)$ both have their first derivatives, $f'(n)$ and $g'(n)$, respectively, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$.

Example 3.1: Let $f(n) = n^2$ and $g(n) = 3^n$. Then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^2}{3^n} = \lim_{n\to\infty} \frac{(n^2)'}{(3^n)'} = \lim_{n\to\infty} \frac{2n}{3^n \ln 3} = \lim_{n\to\infty} \frac{(2n)'}{(3^n \ln 3)'} = \lim_{n\to\infty} \frac{2}{3^n \ln^2 3} = 0$$

Thus $f(n) = O(g(n))$ by Equation (2.1). $f(n)$ is in fact $o(g(n))$, since 3^n is a loose upper bound of n^2.

Note that this rule can be applied repeatedly as long as the conditions are satisfied.

Example 3.2: Now consider the functions $f(n) = n^n$ and $g(n) = 3^n$.

It is easy to see that the first is a loose upper-bound of the second, i.e. $f(n) = \omega(g(n))$.

However, if we attempt to apply L′Hôpital′s Rule, the first few steps would be as follows. We first calculate the derivatives of the functions $f(n)$ and $g(n)$.

$f'(n)$ = ?. Let $y = n^n$ then, by Logarithmic Differentiation,

$\ln y = \ln n^n$ thus $\ln y = n \ln n$.

$$\therefore \frac{1}{y} y' = n\frac{1}{n} + \ln n = 1 + \ln n \quad \Rightarrow \quad y' = y(1+\ln n) = n^n(1+\ln n)$$

This means $f'(n) = n^n(1+\ln n)$ and $g'(n) = 3^n \ln 3$

Thus $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = \lim\limits_{n\to\infty} \frac{n^n}{3^n} = \lim\limits_{n\to\infty} \frac{(n^n)'}{(3^n)'} = \lim\limits_{n\to\infty} \frac{n^n(1+\ln n)}{3^n \ln 3} = \ldots$ ☹

At this point it seems to be a good time to stop doing it analytically, since the expression appears to get only worse.

Below are other pairs of functions which one may find it difficult to compare by using L′Hôpital′s Rule directly, without using MatLab, Maple, or trying to plot.

1. $2^{\lg n}$ and $n^{\lg\lg n}$

2. 2^{2^n} and $2^{2^{n+1}}$

3. $2^{\sqrt{2\lg n}}$ and $\lg n!$

The last one is compounded by the fact that the derivative of *n*! cannot be found directly. One may use Stirling's Approximation. This, however, does not make the determination of complexity ordering using L′Hôpital′s Rule any easier.

Aside from the use of Mathematical and Graphing packages, analytical determination becomes a nightmare.

In the following section, we look at a rule that is a modification of L′Hôpital′s Rule, which facilitates the analytical determination of the complexity orders of most functions.

4. Charlie′s Rule

The main difference between L′Hôpital′s Rule and Charlie's Rule is that in Charlie's Rule, we take the logs of the functions and then differentiate, if necessary. This makes sense in that if the log of a positive increasing monotone grows faster than another, i.e. the limit of their quotient, as *n* approaches infinity, is greater than 1, then the former necessarily grows faster than the later. This is evident due to the facts that, were their exponents of the same base (i.e. say for example for n^2 and n^3 – the base is *n*), then in taking the logs, what we are actually comparing are the exponents. [This indeed is a rudimentary proof.]

Before we state the Theorem, let us demonstrate an example or two. Indeed we can look at Examples 3.1 and 3.2 which have been done (and in the second case, attempted to be done) using L'Hôpital's Rule.

Example 4.1: (Similar to Example 3.1) Let $f(n) = n^2$ and $g(n) = 3^n$. Then

By L'Hôpital's Rule:

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^2}{3^n} = \lim_{n\to\infty}\frac{(n^2)'}{(3^n)'} = \lim_{n\to\infty}\frac{2n}{3^n \ln 3} = \lim_{n\to\infty}\frac{(2n)'}{(3^n \ln 3)'} = \lim_{n\to\infty}\frac{2}{3^n \ln^2 3} = 0$$

By Charlie's Rule:

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^2}{3^n} \xleftrightarrow{CR} \lim_{n\to\infty}\frac{\lg n^2}{\lg 3^n} = \underset{\substack{\uparrow\\ \mathbf{1}}}{\lim_{n\to\infty}}\frac{2\lg n}{n \lg 3} = \frac{2}{\lg 3}\underset{\substack{\uparrow\\ \mathbf{2}}}{\lim_{n\to\infty}}\frac{(1/n)}{(1)} = 0$$

The notation $\xleftrightarrow{CR}$ is just merely to show that we are applying Charlie's Rule, and thus the left hand side may not necessarily be equal to the right hand side. However, if the RHS tends to $L > 1$, then the left hand limit tends to infinity, and the converse, RHS tends to $0 \le L < 1$, then the left hand limit tends to zero. Also in step **1** (with the arrow) with prior knowledge that the log function (lg n) is sublinear (the n in the denominator), we can already tell that $f(n) = o(g(n))$. In step **2** we applied L'Hôpital's Rule.

Example 4.2: (Similar to Example 3.2)

Now let us reconsider the functions $f(n) = n^n$ and $g(n) = 3^n$.

Recall that in Example 3.2 we finally got to the statement:

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^n}{3^n} = \lim_{n\to\infty}\frac{(n^n)'}{(3^n)'} = \lim_{n\to\infty}\frac{n^n(1+\ln n)}{3^n \ln 3} = \ldots \quad ☹$$

which was not exactly desirable, using L'Hôpital's Rule.

Using Charlie's Rule we get:

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^n}{3^n} \leftrightarrow \lim_{n\to\infty}\frac{\lg n^n}{\lg 3^n} = \lim_{n\to\infty}\frac{n\lg n}{n\lg 3} = \lim_{n\to\infty}\frac{\lg n}{\lg 3} = \infty$$

Thus: $f(n) = \omega(g(n))$ ($f(n)$ grows much faster than $g(n)$).

We now state the new rule as follows:

Theorem 4.1 [Charlie's Rule]

Let $f(n)$ and $g(n)$ be positive monotones over the interval $[a, \infty)$, for some $a>0$.

Also let $\lim_{n\to\infty} f(n) = \infty$ and $\lim_{n\to\infty} g(n) = \infty$.

(In other words we have an indeterminate form $\frac{\infty}{\infty}$).

Also let $\lim_{n\to\infty} \frac{\lg f(n)}{\lg g(n)} = L$, then $\begin{cases} \text{if} \quad L<1 \quad \text{then} \quad f(n) = o(g(n)) \\ \text{if} \quad L>1 \quad \text{then} \quad f(n) = \omega(g(n)) \end{cases}$

Note that for this rule, the limit (L) does not have to be ∞. If it is greater than 1, then the function in the numerator grows faster. If it is between 0 and 1, including 0, then the numerator grows slower. For example n^2 and n^3:

$$\lim_{n\to\infty} \frac{n^2}{n^3} \leftrightarrow \lim_{n\to\infty} \frac{2\lg n}{3\lg n} = \frac{2}{3} < 1$$

Thus $n^2 = O(n^3)$, or in particular $n^2 = o(n^3)$ (n^2 grows slower than n^3).

4.1 What more do we get with Charlie's Method

Indeed there is new information obtained by using Charlie's Method, which may be not as evident when using L'Hôpital's Rule. One is that,where as in L'Hôpital's Rule we are just concerned with the limit of the Quotient attaining infinity, a constant or zero, and it just tells us whether the function grow faster, slower, or is of the same order, in Charlie's method we may get a number between 0 and 1, or 1, or a number between 1 and infinity.

We can therefore tell, depending on the quotient obtained by the logarithmic limit, whether a function belongs to the same class. For example, in the last example discussed in Section 4, (n^2 and n^3), we can tell that they belong to the same family of functions (polynomials), since their logarithmic limit is 2/3. In Example 4.1, however, n^2 and 3^n, do not belong to the same class since the limit of their logarithmic quotient is 0. Indeed n^2 belongs to the Polynomial class, whereas, 3^n belongs to the Exponential Class.

Secondly, we can now easily solve new problems analytically that couldn't be easily solved before using L'Hôpital's Rule (without other aids). The example below illustrates just one of the many examples.

Example 4.3: Consider the functions $f(n) = 2^{\sqrt{2\lg n}}$ and $g(n) = \lg n!$.

Then $\lg f(n) = \lg 2^{\sqrt{2\lg n}} = \sqrt{2\lg n}\lg 2 = \sqrt{2\lg n}$ and $\lg g(n) = \lg\lg n!$

[Recall that lg n = $\log_2 n$]

However, since $\lg n! = \theta(n\lg n)$, then

$$\lg g(n) = \lg\lg n! = \theta(\lg(n\lg n)) = \theta(\lg n + \lg\lg n)) = \theta(\lg n).$$

Therefore $$\lim_{n\to\infty}\frac{\lg f(n)}{\lg g(n)} = \lim_{n\to\infty}\frac{2\sqrt{\lg n}}{\lg n} = \lim_{n\to\infty}\frac{2}{\sqrt{\lg n}} = 0$$

Hence $2^{\sqrt{2\lg n}} = o(\lg n!)$, or $2^{\sqrt{2\lg n}}$ grows much slower than $\lg n!$.

5. Discussions and Conclusions

5.1 Problems that cannot be solved with Charlie's Method

There are problems that cannot be solved using Charlie's Method. These are the ones whose logarithmic quotients give a result of 1 (i.e. L =1 in Theorem 4.1).

Example 4.4: Let $f(n) = n$ and $g(n) = n\lg n$. Then in applying Charlie's Rule, we get:

$$\lim_{n\to\infty}\frac{\lg f(n)}{\lg g(n)} = \lim_{n\to\infty}\frac{\lg n}{\lg(n\lg n)} = \lim_{n\to\infty}\frac{\lg n}{\lg n + \lg\lg n} = 1$$

The only information obtained in this case is that they belong to the same class of algorithms (polynomials in this case), but we can't tell whether one grows faster or slower by using Charlie's Method, as depicted in Theorem 4.1. These problems are, however trivial to solve in most cases, and do not even need L'Hôpital's Rule. However, in cases that are slightly more complex, one can apply L'Hôpital's Rule.

5.1 Students View of the New Method

The students in the Algorithms: Analysis and Design Courses (CIS3490) at the University of Guelph and (CS352) at the University of Prince Edward Island, have been taught this method. One of their tasks was to analyze a group of functions in increasing asymptotic order. The example we took was from the CLR (now CLRS) book [2], problem 3-3 in Chapter 3 (page 58).

The problem asks to rank 30 functions (we only include the first two lines of them) by order of growth; i.e. to find an arrangement $g_1, g_2, \cdots, g_n$ of functions satisfying $g_1 = \Omega(g_2) = \Omega(g_3) = \cdots = \Omega(g_n)$. Some of the functions given are:

$$\lg(\lg^* n) \quad 2^{\lg^* n} \quad \left(\sqrt{2}\right)^{\lg^* n} \quad n^2 \quad n! \quad (\lg n)!$$

$$\left(\frac{3}{2}\right)^n \quad n^3 \quad \lg^2 n \quad 2^{\sqrt{2\lg n}} \quad \lg(n!) \quad n2^n$$

For many students coming to Computer Science nowadays, and for many of us, this is an intimidating ordeal. However, there was a great response from the students, once they applied this new method. They found it faster, less agonizing, and once they had their heads around logs, they found they didn't have to remember much to be able to do the problems with ease, and thus apply it to other parts of the course to compare the different algorithms.

6. References

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