Constraint Satisfaction Problems

- Multi-dimensional Selection Problems
- Given a set of variables, each with a set of possible values (a domain), assign a value to each variable that either
  - satisfies some set of constraints: satisfiability problems — "hard constraints"
  - minimizes some cost function, where each assignment of values to variables has some cost: optimization problems — "soft constraints"
- Many problems are a mix of hard and soft constraints.

Relationship to Search

- The path to a goal isn't important, only the solution is.
- Many algorithms exploit the multi-dimensional nature of the problems.
- There are no predefined starting nodes.
- Often these problems are huge, with thousands of variables, so systematically searching the space is infeasible.
- For optimization problems, there are no well-defined goal nodes.

Posing a Constraint Satisfaction Problem

A CSP is characterized by

- A set of variables $V_1, V_2, \ldots, V_n$.
- Each variable V_i has an associated domain $\mathbf{D}_{V_i}$ of possible values.
- For satisfiability problems, there are constraint relations on various subsets of the variables which give legal combinations of values for these variables.
- A solution to the CSP is an n-tuple of values for the variables that satisfies all the constraint relations.

Example: scheduling activities

Variables: A, B, C, D, E that represent the starting times of various activities.

Domains: $\mathbf{D}_A = \{1, 2, 3, 4\}$, $\mathbf{D}_B = \{1, 2, 3, 4\}$, $\mathbf{D}_C = \{1, 2, 3, 4\}$, $\mathbf{D}_D = \{1, 2, 3, 4\}$, $\mathbf{D}_E = \{1, 2, 3, 4\}$

Constraints:

$$(B \neq 3) \wedge (C \neq 2) \wedge (A \neq B) \wedge (B \neq C) \wedge$$
$$(C < D) \wedge (A = D) \wedge (E < A) \wedge (E < B) \wedge$$
$$(E < C) \wedge (E < D) \wedge (B \neq D).$$

Solving CSPs

The finite constraint satisfaction problem is NP-hard. We can

- Try to find algorithms that work well on typical cases even though the worst case may be exponential
- Try to find special cases that have efficient algorithms
- Try to find efficient approximation algorithms
- Develop parallel and distributed algorithms

Generate-and-Test Algorithm

Generate the assignment space $\mathbf{D} = \mathbf{D}_{V_1} \times \mathbf{D}_{V_2} \times \ldots \times \mathbf{D}_{V_n}$.
Test each assignment with the constraints.

Example:

$$\begin{aligned}
\mathbf{D} &= \mathbf{D}_A \times \mathbf{D}_B \times \mathbf{D}_C \times \mathbf{D}_D \times \mathbf{D}_E \\
&= \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} \\
&\quad \times \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} \\
&= \{\langle 1, 1, 1, 1, 1\rangle, \langle 1, 1, 1, 1, 2\rangle, \ldots, \langle 4, 4, 4, 4, 4\rangle\}.
\end{aligned}$$

Generate-and-test is always exponential.

Backtracking Algorithms

Systematically explore **D** by instantiating the variables in some order and evaluating each constraint predicate as soon as all its variables are bound. Any partial assignment that doesn't satisfy the constraint can be pruned.

Example Assignment $A = 1 \land B = 1$ is inconsistent with constraint $A \neq B$ regardless of the value of the other variables.

CSP as Graph Searching

A CSP can be seen as a graph-searching algorithm:

- Totally order the variables, $V_1, \ldots, V_n$.
- A node assigns values to the first j variables.
- The neighbors of node $\{V_1/v_1, \ldots, V_j/v_j\}$ are the consistent nodes $\{V_1/v_1, \ldots, V_j/v_j, V_{j+1}/v_{j+1}\}$ for each $v_{j+1} \in \mathbf{D}_{V_{j+1}}$.
- The start node is the empty assignment $\{\}$.
- A goal node is a total assignment that satisfies the constraints.

Consistency Algorithms

Idea: prune the domains as much as possible before selecting values from them.

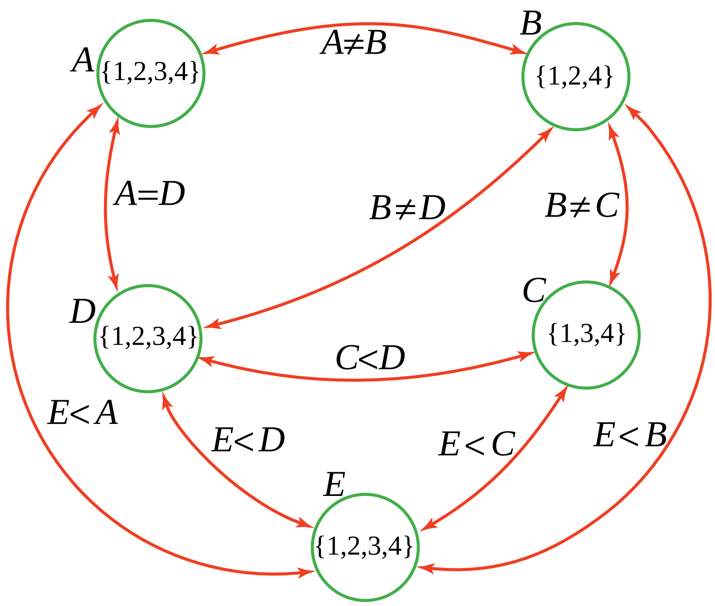
A variable is **domain consistent** if no value of the domain of the node is ruled impossible by any of the constraints.

Example: $\mathbf{D}_B = \{1, 2, 3, 4\}$ isn't domain consistent as $B = 3$ violates the constraint $B \neq 3$.

Arc Consistency

- A **constraint network** has nodes corresponding to variables with their associated domain. Each constraint relation $P(X, Y)$ corresponds to arcs $\langle X, Y\rangle$ and $\langle Y, X\rangle$.
- An arc $\langle X, Y\rangle$ is **arc consistent** if for each value of X in $\mathbf{D}_X$ there is some value for Y in $\mathbf{D}_Y$ such that $P(X, Y)$ is satisfied. A network is arc consistent if all its arcs are arc consistent.
- If an arc $\langle X, Y\rangle$ is *not* arc consistent, all values of X in $\mathbf{D}_X$ for which there is no corresponding value in $\mathbf{D}_Y$ may be deleted from $\mathbf{D}_X$ to make the arc $\langle X, Y\rangle$ consistent.

Example Constraint Network



Arc Consistency Algorithm

The arcs can be considered in turn making each arc consistent.

An arc $\langle X, Y\rangle$ needs to be revisited if the domain of Y is reduced.

Three possible outcomes (when all arcs are arc consistent):

- Each domain is empty $\Longrightarrow$ no solution
- Each domain has a single value $\Longrightarrow$ unique solution
- Otherwise, split a domain & apply arc consistency to each case.

Hill Climbing

Many search spaces are too big for systematic search.

A useful method in practice for some consistency and optimization problems is **hill climbing:**

- Assume a heuristic value for each assignment of values to all variables.
- Maintain a single node corresponding to an assignment of values to all variables.
- Select a neighbor of the current node that improves the heuristic value to be the next current node.

Selecting Neighbors in Hill Climbing

- When the domains are unordered, the neighbors of a node correspond to choosing another value for one of the variables.
- When the domains are ordered, the neighbors of a node are the adjacent values for one of the dimensions.
- If the domains are continuous, you can use **gradient ascent:** change each variable proportional to the gradient of the heuristic function in that direction. The value of variable X_i goes from v_i to $v_i + \eta \frac{\partial h}{\partial X_i}$.
  Gradient descent: go downhill; v_i beomes $v_i - \eta \frac{\partial h}{\partial X_i}$.

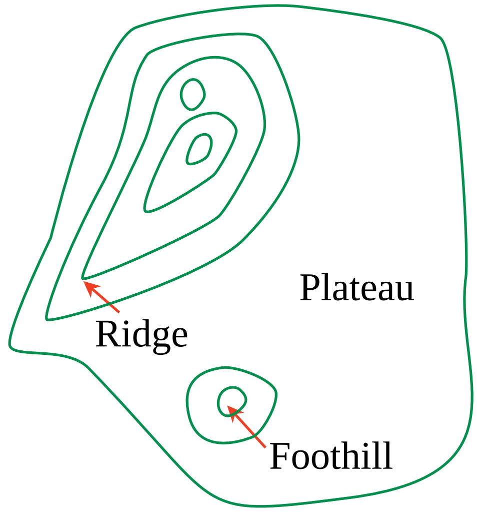
Problems with Hill Climbing

Foothills local maxima that are not global maxima

Plateaus heuristic values are uninformative

Ridge foothill where *n*-step lookahead might help

Ignorance of the peak



Randomized Algorithms

Consider two methods to find a maximum value:

- Hill climbing, starting from some position, keep moving uphill, & report maximum value found
- Pick values at random & report maximum value found

Combinations:

- random-restart hill climbing
- two-phase search: random search, then hill climbing
- maintain multiple nodes, perhaps combine them