Preferences

- Actions result in outcomes
- Agents have preferences over outcomes
- A rational agent will do the action that has the best outcome for them
- Sometimes agents don’t know the outcomes of the actions, but they still need to compare actions
- Agents have to act (doing nothing is (often) an action).

Preferences Over Outcomes

If o_1 and o_2 are outcomes

- $o_1 \succeq o_2$ means o_1 is at least as desirable as o_2.
- $o_1 \sim o_2$ means $o_1 \succeq o_2$ and $o_2 \succeq o_1$.
- $o_1 \succ o_2$ means $o_1 \succeq o_2$ and $o_2 \not\succeq o_1$

Lotteries

- An agent may not know the outcomes of their actions, but only have a probability distribution of the outcomes.
- A lottery is a probability distribution over outcomes. It is written

$$[p_1 : o_1, p_2 : o_2, \ldots, p_k : o_k]$$

where the o_i are outcomes and $p_i > 0$ such that

$$\sum_i p_i = 1$$

The lottery specifies that outcome o_i occurs with probability p_i.

- When we talk about outcomes, we will include lotteries.

Properties of Preferences

- Agents have to act, so they must have preferences:

$$\forall o_1 \forall o_2 \; o_1 \succeq o_2 \text{ or } o_2 \succeq o_1$$

- Preferences must be transitive:

$$\text{if } o_1 \succeq o_2 \text{ and } o_2 \succeq o_3 \text{ then } o_1 \succeq o_3$$

otherwise $o_1 \succeq o_2$ and $o_2 \succeq o_3$ and $o_3 \succ o_1$. If they are prepared to pay to get from o_1 to o_3 $\longrightarrow$ money pump.

Properties of Preferences (cont.)

Monotonicity. An agent prefers a larger chance of getting a better outcome than a smaller chance:

- If $o_1 \succ o_2$ and $p > q$ then

$$[p : o_1, 1 - p : o_2] \succ [q : o_1, 1 - q : o_2]$$

Consequence of axioms

- Suppose $o_1 \succ o_2$ and $o_2 \succ o_3$. Consider whether the agent would prefer
  - o_2
  - the lottery $[p : o_1, 1 - p : o_3]$

  for different values of $p \in [0, 1]$.

- You can plot which one is preferred as a function of p:

Properties of Preferences (cont.)

Continuity Suppose $o_1 \succ o_2$ and $o_2 \succ o_3$, then there exists a $p \in [0, 1]$ such that

$$o_2 \sim [p : o_1, 1 - p : o_3]$$

Properties of Preferences (cont.)

Decomposability (no fun in gambling). An agent is indifferent between lotteries that have same probabilities and outcomes. This includes lotteries over lotteries. For example:

$$[p : o_1, 1 - p : [q : o_2, 1 - q : o_3]]$$
$$\sim \quad [p : o_1, (1 - p)q : o_2, (1 - p)(1 - q) : o_3]$$

Substitutivity if $o_1 \sim o_2$ then the agent is indifferent between lotteries that only differ by o_1 and o_2:

$$[p : o_1, 1 - p : o_3] \sim [p : o_2, 1 - p : o_3]$$

What we would like

- We would like a measure of preference that can be combined with probabilities. So that

$$\begin{aligned} & value([p : o_1, 1-p : o_2]) \\ = \; & p \times value(o_1) + (1-p) value(o_2) \end{aligned}$$

- Money does not act like this.

  What you you prefer

  $$\$1,000,000 \text{ or } [0.5 : \$0, 0.5 : \$2,000,000]?$$

- It may seem that preferences are too complex and muti-faceted to be represented by single numbers.

Theorem

If preferences follows the preceding properties, then preferences can be measured by a function

$$utility : outcomes \rightarrow [0, 1]$$

such that

- $o_1 \succeq o_2$ if and only if $utility(o_1) \geq utility(o2)$.
- Utilities are linear with probabilities:

$$\begin{aligned} & utility([p_1 : o_1, p_2 : o_2, \ldots, p_k : o_k]) \\ = & \sum_{i=1}^{k} p_i \times utility(o_i) \end{aligned}$$

Proof

- if all outcomes are equally preferred, set $utility(o_i) = 0$ for all outcomes o_i.
- Otherwise, suppose the best outcome is *best* and the worst outcome is *worse*.
- For any outcome o_i, define $utility(o_i)$ to be the number u_i such that

$$o_i \sim [u_i : best, 1 - u_i : worst]$$

This exists by the Continuity property.

Proof (cont.)

- Suppose $o_1 \succeq o_2$ and $\mathit{utility}(o_i) = u_i$, then by Substitutivity,

$$
\begin{aligned}
& [u_1 : \mathit{best}, 1 - u_1 : \mathit{worst}] \\
\succeq \quad & [u_2 : \mathit{best}, 1 - u_2 : \mathit{worst}]
\end{aligned}
$$

Which, by monotonicity implies $u_1 \geq u_2$.

Proof (cont.)

- Suppose $p = utility([p_1 : o_1, p_2 : o_2, \ldots, p_k : o_k])$.
- Suppose $utility(o_i) = u_i$. We know:

$$o_i \sim [u_i : best, 1 - u_i : worst]$$

- By substitutivity, we can replace each o_i by $[u_i : best, 1 - u_i : worst]$, so

$$\begin{aligned} p = utility(\quad [\quad & p_1 : [u_1 : best, 1 - u_1 : worst] \\ & \ldots \\ & p_k : [u_k : best, 1 - u_k : worst]]) \end{aligned}$$

- By decomposability, this is equivalent to:

$$p = utility(\quad [\quad p_1u_1 + \ldots + p_ku_k$$

$$: best,$$

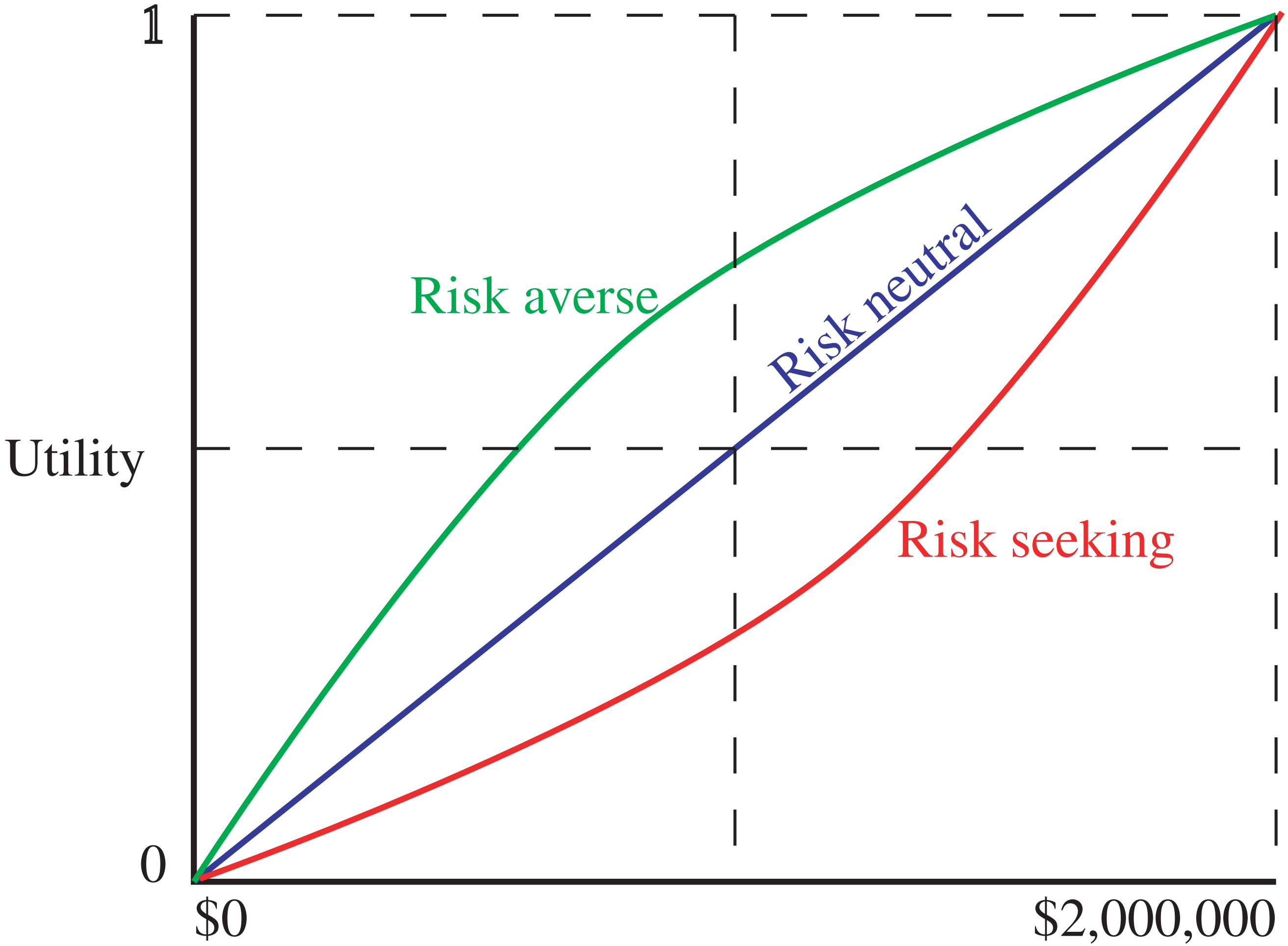
$$p_1(1-u_1) + \ldots + p_k(1-u_k)$$

$$: worst]])$$

- Thus, by definition of utility,

$$p = p_1 \times u_1 + \ldots + p_k \times u_k$$

Utility as a function of money



Possible utility as a function of money

Someone who really wants a toy worth $50, but who would also like one worth $30: