CPSC 403/542
Assignment 4 - Solutions

1. (a) Differentiating

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$$

with respect to $\mathbf{c}$ we have by the chain rule

$$\mathbf{y}'_{\mathbf{c}} = \mathbf{f}_{\mathbf{y}} \mathbf{y}_{\mathbf{c}}, \quad where \ \mathbf{y}_{\mathbf{c}} = \frac{\partial \mathbf{y}}{\partial \mathbf{c}}.$$

Differentiating $\mathbf{y}(0) = \mathbf{c}$ with respect to $\mathbf{c}$ gives $\mathbf{y}_{\mathbf{c}}(0) = I$. Denoting $Y(t) = \mathbf{y}_{\mathbf{c}}(t)$ we get

$$\begin{aligned} Y' &= A(t)Y \\ Y(0) &= I \end{aligned}$$

where $A(t) = \mathbf{f}_{\mathbf{y}}(t, \mathbf{y}(t; \mathbf{c}))$.

(b) Let

$$\mathbf{z} = \hat{\mathbf{y}} - \mathbf{y}.$$

We know that $\mathbf{z}(t) = O(\epsilon)$ (from the fundamental theorem of IVPs). A Taylor expansion of $\mathbf{f}(\hat{\mathbf{y}})$ about $\mathbf{f}(\mathbf{y})$ then yields

$$\mathbf{z}' = \mathbf{f}(t, \hat{\mathbf{y}}) - \mathbf{f}(t, \mathbf{y}) = A(t)\mathbf{z} + O(\epsilon^2)$$

The $O(\epsilon^2)$ term can be considered (for ϵ small enough) as an inhomogeneity for the linear problem for $\mathbf{z}$ which has the initial conditions

$$\mathbf{z}(0) = \epsilon \mathbf{d}.$$

So,

$$\mathbf{z}(t) = \epsilon Y(t)\mathbf{d} + O(\epsilon^2).$$

For the jth initial value alone we look at the vector $\mathbf{d}^T = (0, \ldots, 0, 1, 0, \ldots, 0)$, i.e. the jth unit vector. Thus, $Y(t)\mathbf{d}$ is the jth column of $Y(t)$. If the modes are separated then this would be the jth mode.

(c) For the boundary value problem we are looking at

$$B_0 \hat{\mathbf{y}}(0) + B_b \hat{\mathbf{y}}(b) = \mathbf{b} + \epsilon \mathbf{d}$$

For $\mathbf{z} = \hat{\mathbf{y}} - \mathbf{y}$ the same differential equation holds as in the IVP case, but with boundary conditions

$$B_0 \mathbf{z}(0) + B_b \mathbf{z}(b) = \epsilon \mathbf{d}$$

So, define the fundamental solution $\Phi(t)$ by

$$\begin{aligned} \Phi' &= A(t)\Phi \\ B_0\Phi(0) + B_b\Phi(b) &= I \end{aligned}$$

(which we assume to exist and be bounded). Clearly, $\Phi = \mathbf{y_b}$, indicating that this is the sensitivity matrix function with respect to the boundary data, and we also obtain

$$\hat{\mathbf{y}}(t) = \mathbf{y}(t) + \epsilon\Phi(t)\mathbf{d} + O(\epsilon^2)$$

2. (a) Using the trapezoidal scheme on a given mesh we have

$$\mathbf{y}(t_n) - \mathbf{y}_n = \mathbf{c}_1 h_n^2 + \mathbf{c}_2 h_n^4 + \mathbf{c}_3 h_n^6 + \dots$$

So, if we halve each step and repeat the computation we get

$$\mathbf{y}(t_n) - \tilde{\mathbf{y}}_{2n} = \mathbf{c}_1 h_n^2/4 + \mathbf{c}_2 h_n^4/16 + \mathbf{c}_3 h_n^6/64 + \dots$$

We can now divide each step to 3 or 4 equal pieces and repeat. To be concrete, subdivide to 4 and call the result $\hat{\mathbf{y}}_j$:

$$\mathbf{y}(t_n) - \hat{\mathbf{y}}_{4n} = \mathbf{c}_1 h_n^2/16 + \mathbf{c}_2 h_n^4/2^8 + \mathbf{c}_3 h_n^6/2^{12} + \dots$$

These three expressions can be used to eliminate $\mathbf{c}_1$ and $\mathbf{c}_2$, yielding

$$\mathbf{y}(t_n) - \frac{64\hat{\mathbf{y}}_{4n} - 20\tilde{\mathbf{y}}_{2n} + \mathbf{y}_n}{45} = O(h_n^6) + O(h^7).$$

(b) For the problem of Examples 8.1 and 8.3 we get the results in the tables below. These results are qualitatively similar to those obtained for

N	λ	error	rate	λ	error	rate	λ	error	rate
10	1	.33e-8		50	.50e-1		500	.19	
20		.51e-10	6.0		.94e-2	2.4		.12	.66
40		.80e-12	6.0		.59e-3	4.0		.14	-.22
80		.20e-13	5.3		.17e-4	5.1		.83e-1	.75

Table 0.1: Maximum errors for Example 8.1 using extrapolation: uniform meshes.

collocation at 3 Gaussian points.

Comment: Note that the nonuniform meshes used here and in the text are very ad hoc. Better meshes can certainly be constructed.

3. This is a 'do it any way you can' question, more difficult than usual.

N	λ	error	rate	λ	error	rate
10	50	.34e-2		500	.35e-2	
20		.22e-3	3.9		.22e-3	4.0
40		.62e-5	5.1		.17e-4	3.7
80		.10e-6	6.0		.24e-4	0.5

Table 0.2: Maximum errors for Example 8.1 using extrapolation: nonuniform meshes.

(a) It is useful to read Exercise 7.4 for this particular question. The problem on a semi-infinite interval can be approximated in the following 3 ways:

- Solve

$$v'' + \frac{4}{t}v' + (tv-1)v = 0, \qquad 0 < t < L$$
$$v'(0) = 0, \quad v(L) = 0.$$

  This is the simplest approach, and it is suggested in Exercise 7.4. Here L is a large enough interval size, e.g. $L = 10$.

- The solution v decays exponentially. Thus, for t large, $\frac{4}{t}v'$ and tv^2 are negligibly small, and we get $v'' \approx v$, so $v(t) \approx e^{-t}$. This can be enforced by replacing the boundary condition $v(L) = 0$ by

$$v'(L) + v(L) = 0\,.$$

  Note also that the above argument justifies assuming that $L = 10$ is "large enough": because e^{-10} is close enough to 0.

- It is also possible to apply a nonlinear transformation of the independent variable:

$$\tau = 1/(t+1)$$

  or

$$\tau = t/(t+1)\,.$$

  Then the problem in τ is defined on $[0,1]$. But singularities arise and have to be dealt with. This idea still works for this simple example. But more generally, it seems less robust than the previous options.

I chose the first of these reformulations. Note that if the trapezoidal scheme is used then we end up evaluating $\frac{0}{0}$ at $t = 0$. No difficulty arises if Gauss collocation is used. But for the trapezoidal scheme the trouble is not serious either: By L'Hopitalle's rule we have

$$\frac{4v'}{t} \rightarrow 4v''(0)$$

and this is used to modify the discretized differential equation at $t = 0$.

Some trial and error for the choice of initial solution followed, to avoid the trivial solution. With the initial guess

$$\begin{aligned} v(t) &= 1 + 2t(2-t), \qquad t \le 2 \\ &= e^{2-t}, \qquad t > 2 \end{aligned}$$

collocation at 3 Gaussian points and 80 uniform mesh points yields the results in the following figure.

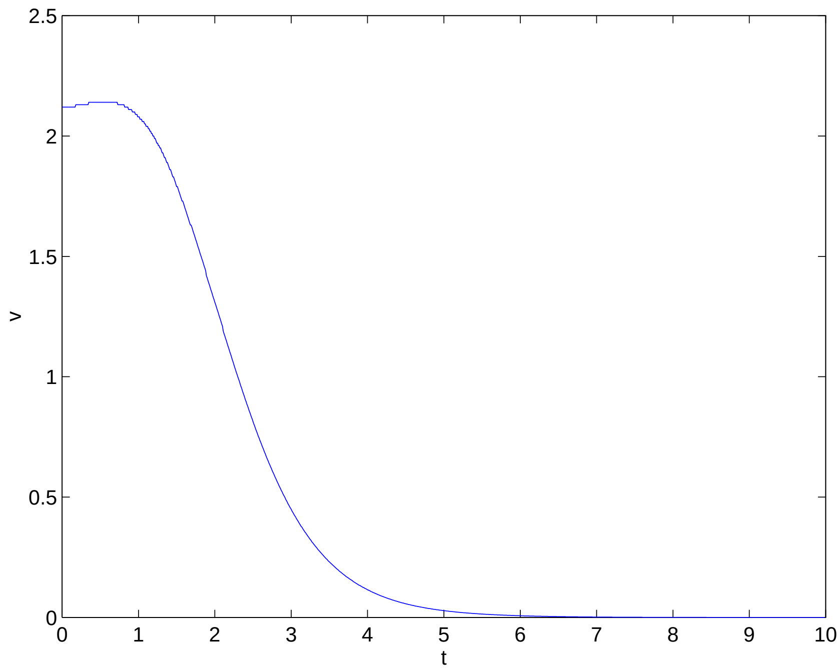

Figure 0.1: Nontrivial soltion for problem on semi-infinite interval

(b) The famous van der Pol equation

$$u'' = (1-u^2)u' - u$$

can be integrated using, e.g. the Matlab default IVP solver starting from, say, $u(0) = u'(0) = 1$. The trajectory gets attracted to the limit cycle, and the figure below is produced.

If we integrate this IVP up to $t = 100$ then the obtained solution value, $u(0) = 2.0078, u'(0) = -.056$, can be assumed to be on the limit cycle to our working accuracy.

Next we want to find the period. Obviously, we cannot simply specify $u(0) = u(b), u'(0) = u'(b)$, because b is unknown and the problem is thus

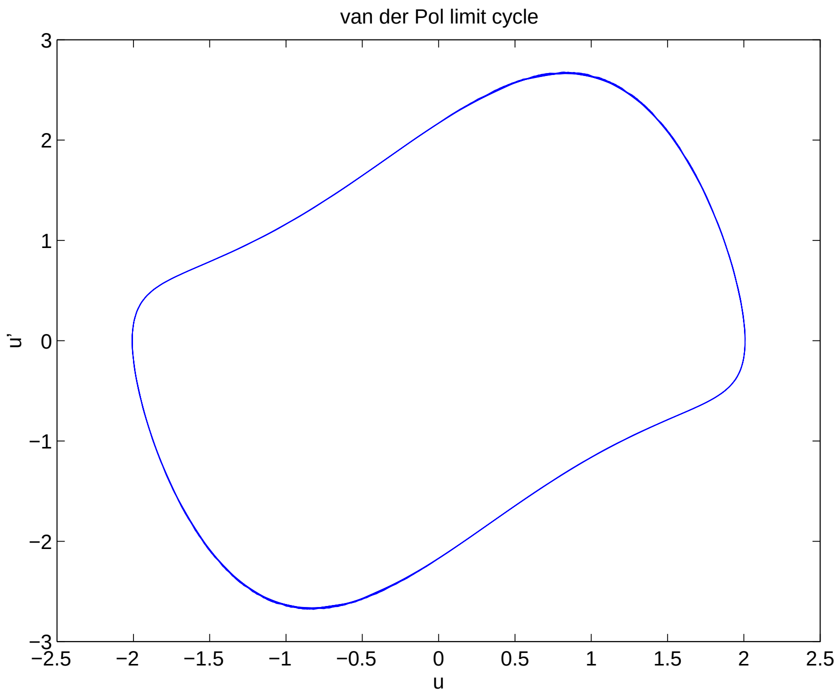

Figure 0.2: Limit cycle of the van der Pol equations

underdetermined. We first redefine the independent variable: $\tau = t/b$, where b is the period that we want to find. Denoting by $\dot{u}$ the derivative of u with respect to τ, we get the system

$$\begin{aligned} \dot{y}_1 &= by_2 \\ \dot{y}_2 &= b[(1-y_1^2)y_2 - y_1] \\ \dot{b} &= 0 \end{aligned}$$

$$y_1(0) = 2.0078,\ y_1(1) = 2.0078,\ y_2(0) = -.056$$

This is solved, starting from an appropriate initial guess (the initial guess $y_1 = y_1(0)\cos(\pi\tau)$, $y_2 = y_1(0)\sin(\pi\tau)$, $b = 10$, worked for me), and the resulting solution is checked to make sure it makes sense (i.e. check that $y_2(1) = y_2(0)$ and $b > 0$). This yields the period

$$b \approx 6.6633$$

Comment: This approach is definitely ad hoc and hard to automate. There are better, more general, systematic approaches to finding periodic solutions, which are not covered in the text.

Note that here too, there is a trivial solution $b = 0$ to avoid.

(c) This problem is simple. Obtain, using a general-purpose code,

$$\lambda \approx 3.456287\,.$$