CPSC 403/542
Assignment 4

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Due: Fri, Mar 30, 2001.

1. Consider an ODE system of size m

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}) \tag{1}$$

where $\mathbf{f}$ has bounded first and second partial derivatives, subject to initial conditions

$$\mathbf{y}(0) = \mathbf{c} \tag{2}$$

or boundary conditions

$$B_0\mathbf{y}(0) + B_b\mathbf{y}(b) = \mathbf{b}\,. \tag{3}$$

It is often important to determine the *sensitivity* of the problem with respect to the data $\mathbf{c}$ or $\mathbf{b}$. For instance, if we change c_j to $c_j + \epsilon$ for some j, $1 \leq j \leq m$, where $|\epsilon| \ll 1$, and call the solution of the perturbed problem $\hat{\mathbf{y}}(t)$, what can be said about $|\hat{\mathbf{y}}(t) - \mathbf{y}(t)|$ for $t \geq 0$?

(a) Denoting the solution of (1),(2) as $\mathbf{y}(t; \mathbf{c})$, define the $m \times m$ matrix function

$$Y(t) = \frac{\partial \mathbf{y}(t; \mathbf{c})}{\partial \mathbf{c}}\,.$$

Show that Y satisfies the initial value problem

$$\begin{aligned} Y' &= A(t)Y \\ Y(0) &= I \end{aligned}$$

where $A = \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(t, \mathbf{y}(t; \mathbf{c}))$.

(b) Let $\hat{\mathbf{y}}(t)$ satisfy (1) and

$$\hat{\mathbf{y}}(0) = \mathbf{c} + \epsilon\mathbf{d}$$

where $|\mathbf{d}| = 1$ and $|\epsilon| \ll 1$. Show that

$$\hat{\mathbf{y}}(t) = \mathbf{y}(t) + \epsilon Y(t)\mathbf{d} + O(\epsilon^2)\,.$$

In particular, what can you say about the sensitivity of the problem with respect to the j-th component of the initial value vector $\mathbf{c}$?

(c) Answer questions analogous to (a) and (b) above regarding the sensitivity of the boundary value problem (1),(3) with respect to the boundary values $\mathbf{b}$, assuming that $\mathbf{y}(t)$ is an isolated solution. How would a bound on $\|\hat{\mathbf{y}} - \mathbf{y}\|_\infty = \max_{0 \le t \le b} |\hat{\mathbf{y}}(t) - \mathbf{y}(t)|$ relate to the stability constant κ of (6.13) of the text?

2. It can be shown that the error when applying the trapezoidal scheme to a sufficiently smooth BVP

$$\begin{aligned} \mathbf{y}' &= \mathbf{f}(t, \mathbf{y}), \qquad 0 < t < b \\ & \mathbf{g}(\mathbf{y}(0), \mathbf{y}(b)) = \mathbf{0} \end{aligned}$$

has the expansion

$$\mathbf{e}_n = \mathbf{y}(t_n) - \mathbf{y}_n = \sum_{j=1}^{l} \mathbf{c}_j h_n^{2j} + O(h^{2l+1})$$

where $h = \max_n h_n$ on a mesh π which satisfies

$$h / \min_n h_n \le \textit{constant}.$$

The functions $\mathbf{c}_j$ are independent of the mesh π. Just how large l is depends on the smoothness of the problem, and we assume $l \ge 3$.

(a) Construct a method of order 6 using extrapolation, based on the trapezoidal scheme.

(b) Write a code implementing this extrapolation method on a general mesh, optionally estimating the error as well. Apply your code to the problem of Examples 8.1 and 8.3, using the same parameter values and meshes. Compare with collocation at 3 Gaussian points (Example 8.3). What are your conclusions?

3. Use your code from the previous exercise, or any available software, to solve the following problems to about 5-digit accuracy.

(a) Find a nontrivial solution for the problem

$$\begin{aligned} v'' + \frac{4}{t} v' + (tv - 1)v &= 0, \qquad 0 < t < \infty \\ v'(0) &= 0, \qquad v(\infty) = 0 \end{aligned}$$

of Exercise 7.4.

(b) Find the attracting limit cycle and the period of the van der Pol equation

$$u'' = (1 - u^2)u' - u \,.$$

(c) Solve the system of Example 8.5,

$$
\begin{aligned}
&u'' + \lambda e^u = 0 \\
&\lambda' = 0 \\
&w' = u^2 \\
&u(0) = u(1) = 0,\ w(0) = 0,\ w(1) = \mu^2
\end{aligned}
$$

for $\mu = 1$. What is the corresponding value of λ?