CPSC 403/542
Assignment 3 - Solutions

1. (a) Letting $y_1 = u$ and $y_2 = \varepsilon u' - au$ we get

$$\begin{aligned} \varepsilon y_1' &= ay_1 + y_2 \\ y_2' &= b(t)y_1 + q(t). \end{aligned}$$

   (b) Letting $\varepsilon \to 0$ we get

$$\begin{aligned} 0 &= ay_1 + y_2 \\ y_2' &= b(t)y_1 + q(t). \end{aligned}$$

   Since y_1 is given in terms of y_2 as $y_1(t) = -a^{-1}y_2(t)$, this is a semi-explicit index-1 DAE.

   (c) Plugging the expression for y_1 into the ODE for y_2 we get a scalar ODE, necessitating only one boundary condition.

   To understand which boundary condition is kept, consider the equation $\varepsilon y_1' = ay_1 + y_2$. Here y_2 is like an inhomogeneity, so the stability of the ODE for y_1 depends on the sign of a. If $a < 0$ then the IVP is stable, so we prescribe $y_1(0) = b_1$. If $a > 0$ then the terminal value ODE is stable, so we prescribe $y_1(1) = b_2$.

   Note: It is the sign of a, *not* b/a, which matters: just look at the eigenvalues of the matrix $\begin{pmatrix} a/\epsilon & 1/\epsilon \\ b & 0 \end{pmatrix}$, which are $a/\epsilon + O(1)$ and $O(1)$.

2. (a) First note that upon multiplying (3) by $H(\mathbf{x}_\nu)$ we get (2). Now, let R span the *orthogonal subspace*, i.e. R is $(l - m) \times l$ such that $HR^T = 0$. Multiplying (3) by R we see that $R\mathbf{x}_{\nu+1} = R\mathbf{x}_\nu$. Thus, the deviation that is not strictly implied by (2) is 0, which is minimal.

   (b) Use $F = H^T(HH^T)^{-1}$ (which indeed makes the invariant set attracting) and apply forward Euler with stepsize $h = 1/\gamma$. Note that F is independent of the discretization stepsize!

3. The following reference numbers relate to the text. In Example 9.7 our index-3 DAE is reduced into an ODE by two differentiations (utilizing (9.23) to eliminate λ). Note that our initial conditions, obtained by setting $\varepsilon = 0$ in the values of Figure 9.2, are *consistent*, satisfying both (9.21e) and the hidden constraint (9.22).

   Setting $(y_1, y_2, y_3, y_4)^T = \mathbf{y} = (q_1, q_2, v_1, v_2)^T$ we therefore obtain the unstabilized ODE

$$\mathbf{y}' = \mathbf{f}(\mathbf{y})$$

where

$$\mathbf{f} = \begin{pmatrix} y_3 \\ y_4 \\ -(y_3^2 + y_4^2 - y_2 g)y_1 \\ -(y_3^2 + y_4^2 - y_2 g)y_2 - g \end{pmatrix}$$

with the initial conditions

$$\mathbf{y}(0) = (1, 0, 0, 0)^T.$$

If we integrate this initial value ODE using the MATLAB code ode45 with standard options then the drift $r^2 - 1$ is seen to grow in time as expected. This can be fixed by stabilization. But for the purpose of this exercise the accuracy even in the unstabilized formulation is sufficient to conclude that the profiles for q_1 and q_2 correspond to those of Figure 9.2 with the high oscillation filtered out. Thus, a much larger step size can be taken for a pointwise accurate solution in the limit $\varepsilon \to 0$, i.e. for the high index DAE, than what is possible for the highly oscillatory ODE when $0 < \varepsilon \ll 1$.

Continuing beyond the question, it may be tempting in general to switch to the reduced equations in order to efficiently obtain a qualitative picture of the solution with rapidly oscillating components filtered out. Unfortunately, however, life can be more complicated than the example of this exercise suggests, and the solution of the reduced problem does not always provide a qualitatively correct picture of the limit oscillatory solution as $\varepsilon \to 0$. A counter-example is discussed in Exercise 10.14 of the text.