CPSC 403/542
Assignment 3

U. M. Ascher
Due: Mon, March 12, 2001.

1. Consider the two-point boundary value problem

$$\varepsilon u'' = au' + b(t)u + q(t)$$
$$u(0) = b_1, \quad u(1) = b_2 \qquad (1)$$

where $a \neq 0$ is a constant and b, q are continuous functions, all $O(1)$ in magnitude, and $0 < \varepsilon \ll 1$.

   (a) Write the ODE in first order form for the variables $y_1 = u$ and $y_2 = \varepsilon u' - au$.

   (b) Letting $\varepsilon \to 0$, show that the limit system is an index-1 DAE.

   (c) Show that only one of the boundary conditions in (1) is needed to determine the reduced solution (i.e. the solution of the DAE).
   Optional: Which one?

2. Consider the problem of solving an *underdetermined* system of nonlinear equations,

$$\mathbf{h}(\mathbf{x}) = \mathbf{0}$$

where

$$H = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}$$

is a rectangular $m \times l$ matrix ($m \leq l$) having a full row rank m for any $\mathbf{x}$.

A linearization a la Newton gives the iteration

$$\mathbf{0} = \mathbf{h}(\mathbf{x}_\nu) + H(\mathbf{x}_\nu)(\mathbf{x}_{\nu+1} - \mathbf{x}_\nu) \qquad \nu = 0, 1, \ldots . \qquad (2)$$

This is a linear, underdetermined system for $\mathbf{x}_{\nu+1}$, i.e. there are many solutions $\mathbf{x}_{\nu+1}$ to (2). Of course we cannot have our sequence of iterates $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_\nu, \ldots)$ horse around because of this underdetermination.

A common, numerically sound way for pinning down $\mathbf{x}_{\nu+1}$ is to minimize the deviation from the current iterate, $|\mathbf{x}_{\nu+1} - \mathbf{x}_\nu|_2$, subject to (2) holding. This gives the iteration

$$\mathbf{x}_{\nu+1} = \mathbf{x}_\nu - H^T(HH^T)^{-1}\mathbf{h}(\mathbf{x}_\nu) \qquad (3)$$

where $H = H(\mathbf{x}_\nu)$.

(a) *Optional*: Verify that the solution of (3) indeed minimizes $|\mathbf{x}_{\nu+1} - \mathbf{x}_\nu|_2$ subject to satisfying (2).

(b) Another approach is to consider $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ as defining an invariant set, and to define a dynamical system whose trajectories get attracted to this invariant set. Thus, starting with $\mathbf{x}' = \mathbf{0}$ (the independence of $\mathbf{x}$ on the variable t that we have just dreamt up), we modify according to §9.2 of the text to

$$\mathbf{x}' = -\gamma F(\mathbf{x})\mathbf{h}(\mathbf{x}) \tag{4}$$

where $\gamma > 0$ is a parameter. The initial value $\mathbf{x}_0$ for this ODE system is anyone's guess.

Show that by a suitable choice of F and a discretization scheme for (4), the constrained Newton method (3) is obtained.

Optional: Can you think of a better method?

3. Set $\varepsilon = 0$ in Example 9.8 of the text, and solve the resulting DAE numerically. (Take $g = 1$.) You may use any (justifiable) means you like, including index reduction and/or use of an appropriate software package. Plot the solution and compare with Figure 9.2. Discuss.