CPSC 403/542
Assignment 2

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Due: Wed, Feb 14, 2001.

1. (a) Show that a general Runge-Kutta scheme (4.7) as written in class can be written in the form

$$\mathbf{K}_i = \mathbf{f}\left(t_{n-1} + c_i h, \mathbf{y}_{n-1} + h\sum_{j=1}^{s} a_{ij}\mathbf{K}_j\right), \qquad 1 \le i \le s, \tag{1a}$$

$$\mathbf{y}_n = \mathbf{y}_{n-1} + h\sum_{i=1}^{s} b_i \mathbf{K}_i. \tag{1b}$$

   What is the relationship between $\mathbf{K_i}$ and $\mathbf{Y_i}$ of (4.7)?

   (b) Prove: *The one-step method*

$$\mathbf{y}_n = \mathbf{y}_{n-1} + h\boldsymbol{\psi}(t_{n-1}, \mathbf{y}_{n-1}, h)$$

   is 0-stable if $\boldsymbol{\psi}$ satisfies a Lipschitz condition in $\mathbf{y}$.

2. The following classical example from astronomy gives a strong motivation to integrate initial value ODEs with error control.

   Consider two bodies of masses $\mu = 0.012277471$ and $\hat{\mu} = 1 - \mu$ (earth and sun) in a planar motion, and a third body of negligible mass (moon) moving in the same plane. The motion is governed by the equations

$$\begin{aligned} u_1'' &= u_1 + 2u_2' - \hat{\mu}\frac{u_1 + \mu}{D_1} - \mu\frac{u_1 - \hat{\mu}}{D_2} \\ u_2'' &= u_2 - 2u_1' - \hat{\mu}\frac{u_2}{D_1} - \mu\frac{u_2}{D_2} \\ D_1 &= ((u_1 + \mu)^2 + u_2^2)^{3/2} \\ D_2 &= ((u_1 - \hat{\mu})^2 + u_2^2)^{3/2} \end{aligned}$$

   Starting with the initial conditions

$$u_1(0) = 0.994,\ u_2(0) = 0,\ u_1'(0) = 0,\ u_2'(0) = -2.00158510637908252240537862224$$

   the solution is periodic with period < 17.1. Note that $D_1 = 0$ at $(-\mu, 0)$ and $D_2 = 0$ at $(\hat{\mu}, 0)$, so we need to be careful when the orbit passes near these singularity points.

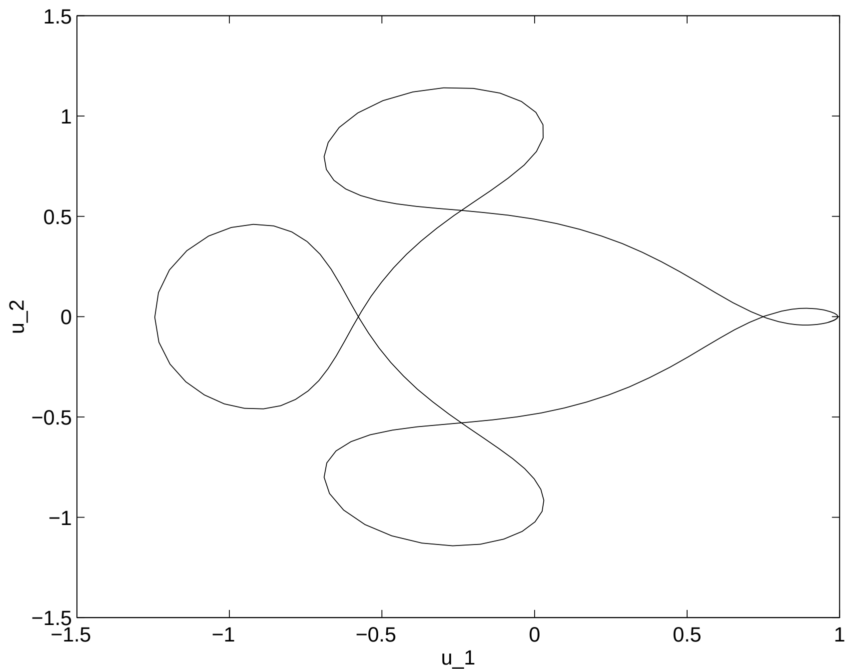

Figure 0.1: Astronomical orbit using the Runge-Kutta Fehlberg method.

The orbit is depicted in the attached figure. It was obtained using the Fehlberg embedded pair with a local error tolerance $1.e-6$. This necessitated 204 time steps.

Using the classical Runge-Kutta method of order 4, integrate this problem on $[0, 17.1]$ with a *uniform* step size, using 100, 1000, 10000 and 20000 steps. Plot the orbit for each case. How many uniform steps are needed before the orbit appears to be *qualitatively* correct?

3. Let y_{n-1} be an approximate solution of $y' = f(t, y)$ at t_{n-1}, and consider finding an approximate solution y_n at $t_n = t_{n-1} + h$. If $y_n = y_n^{1,1}$ is obtained by forward Euler then it can be shown that the error has a series expansion in h,

$$y(t_n) - y_n = C_1 h + C_2 h^2 + \ldots + C_j h^j + \ldots$$

where C_j depends on $y^{(j+1)}(t)$, but not on h.

Thus, we can apply **extrapolation**: Let $y_n^{k,1}$ be the approximation obtained by applying forward Euler k times with step size h/k, starting from y_{n-1} at t_{n-1}. Then

$$y(t_n) - y_n^{k,1} = C_1 \frac{h}{k} + C_2 (\frac{h}{k})^2 + \ldots + C_j (\frac{h}{k})^j + \ldots .$$

Thus, for instance,

$$y_n^{2,2} = 2y_n^{2,1} - y_n^{1,1}$$

gives a second order method because

$$y(t_n) - y_n^{2,2} = \sum_{j\geq 2}(2^{1-j} - 1)C_j h^j.$$

(a) Show that $y_n^{2,2}$ is equivalent to the explicit midpoint method.

(b) Derive a $3rd$ order method $y_n^{3,3}$ by applying another extrapolation step.

(c) Show that these three methods $y_n^{k,k}$ are all explicit Runge-Kutta methods. Figure out how many function evaluations are involved in obtaining $y_n^{3,3}$ and advise if this method should be used in general.

(d) Demonstrate performance of these three methods (i.e., calculate errors and convergence rates) for the problem

$$y' = -5ty^2 + \frac{5}{t} - \frac{1}{t^2}, \qquad y(1) = 1,$$

for $1 \leq t \leq 25$.

(e) Show that in general there is no limit to the order that explicit Runge-Kutta methods may achieve, provided sufficiently many stages are allowed.

4. (a) Show that the only k-step method of order k which has the stiff decay property is the k-step BDF method.

(b) For those folks taking CS542: Is it possible to design a strongly stable linear multistep method of order 7 which has stiff decay? Please justify your guess.