CS520: Discontinuous solutions (Ch. 10)

Uri Ascher

Department of Computer Science
University of British Columbia
ascher@cs.ubc.ca
people.cs.ubc.ca/∼ascher/520.html

OUTLINE

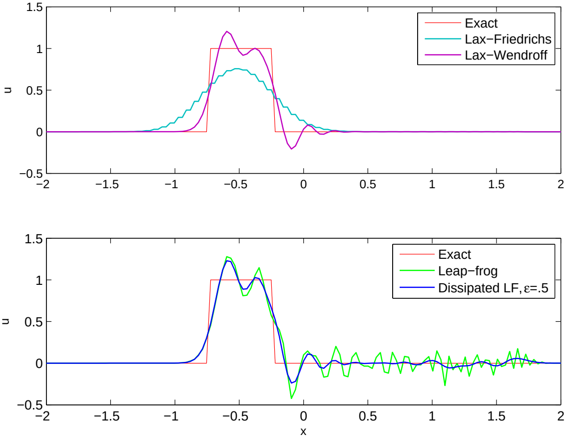
- These slides cover a highly abbreviated version of **Chapter 10**.
- We'll consider, mainly by examples:
  - Constant coefficient hyperbolic PDEs
  - Conservations laws: approximating the Burgers equation
  - What sort of schemes are there?

Constant coefficient advection

- Recall that a discontinuity in initial value function $u_0(x)$ propagates along characteristic in (t, x).
- We saw various methods:
  - Lax-Wendroff
  - Leap-frog
  - dissipated Leap-frog
  - Lax-Friedrichs
  - Upwind

Example: $u_t = u_x$ with u_0 a square wave

u_0 square wave on $[.25, .75]$, $k = .5h$, $h = .01\pi$.



EXAMPLE: $u_t = u_x$ WITH u_0 A SQUARE WAVE

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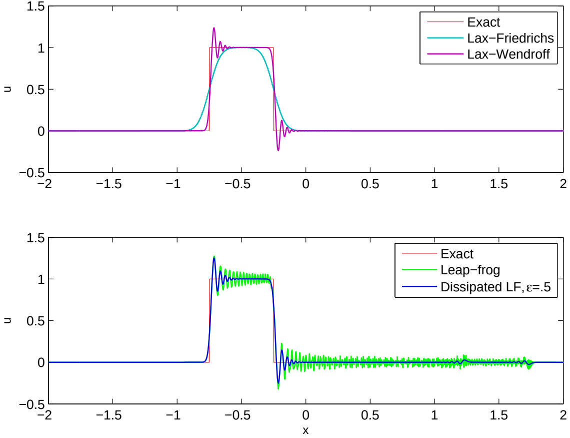

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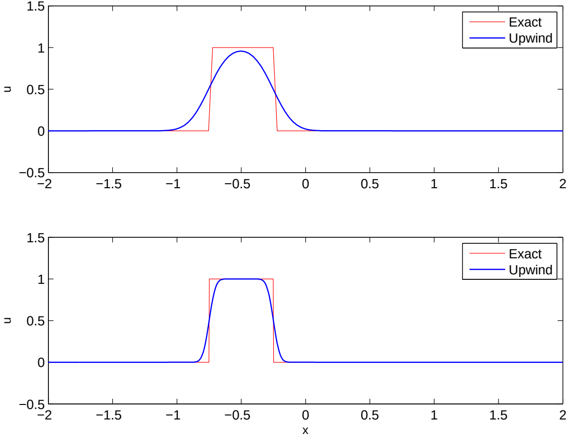

Observations

- Both dissipative and non-dissipative schemes of order (2,2) display annoying overshoots. (Note Gibbs phenomenon).
- The large error in the dissipative ones is more localized.
- These overshoots can become much more troublesome for nonlinear problems.
- Both Lax-Friedrichs and upwind are monotone and there are no overshoots.
- However, monotone schemes are only 1st order accurate and feature significant artificial viscosity/diffusion, especially Lax-Friedrichs.
- Observe similar behaviour for parabolic problems with small diffusion term ("almost hyperbolic").

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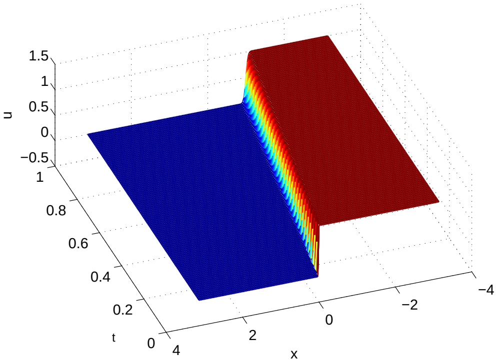
EXAMPLE: $u_t + au_x = \sigma u_{xx}$ WITH $0 < \sigma \ll 1$

$\sigma = 1.\text{e-}3, \;\; a = \text{-}1; \; u(t, -\pi) = 1, \;\; u(t, \pi) = 0;$
$u_0 = 1 \text{ if } x \le 0, \;\; u_0 = 0 \text{ otherwise }.$
Crank-Nicolson (CN), $k = .0001, \;\; h = .0001\pi$.



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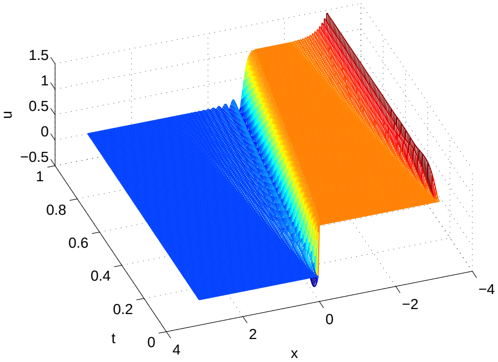

EXAMPLE: $u_t + au_x = \sigma u_{xx}$ WITH $0 < \sigma \ll 1$

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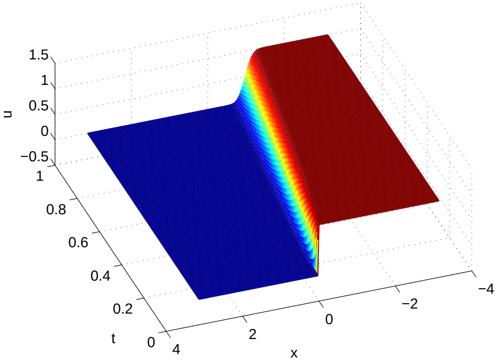
Upwind, $k = .01$, $h = .01\pi$.



Constant coefficient system

$$\mathbf{u}_t + A\mathbf{u}_x = \mathbf{0},$$

where A is diagonalizable with **real** eigenvalues

$$T^{-1}AT = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_s).$$

- Lax-Friedrichs can be extended directly.
- Define $|\Lambda| = \mathrm{diag}\,(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_s|)$, and then $|A| = T|\Lambda|T^{-1}$. The upwind method can be written as

$$\mathbf{v}_j^{n+1} = \mathbf{v}_j^n - \frac{\mu}{2}A(\mathbf{v}_{j+1}^n - \mathbf{v}_{j-1}^n) + \frac{\mu}{2}|A|(\mathbf{v}_{j+1}^n - 2\mathbf{v}_j^n + \mathbf{v}_{j-1}^n).$$

- Alternatively for upwind,

$$\mathbf{v}_j^{n+1} = \mathbf{v}_j^n - \mu[A^+D_- + A^-D_+]\mathbf{v}_j^n, \quad \text{where}$$

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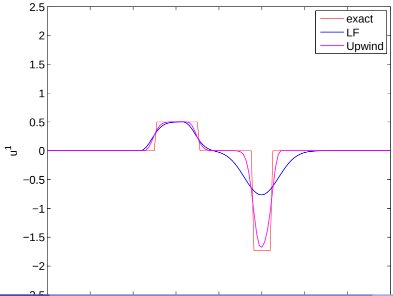
Example: two advection equations mixed

$$\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -0.1 \end{pmatrix}, \quad T = \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix}, \quad A = T\Lambda T^{-1}.$$

Periodic BC on $[-\pi, \pi]$ and initial conditions $\mathbf{u}_0 = T\mathbf{w}_0$, with

$$w_0^1(x) = \begin{cases} 1 & .25 \le x < .75 \\ 0 & \text{otherwise} \end{cases}, \quad w_0^2(x) = \begin{cases} 2 & .5 \le x < .7 \\ 0 & \text{otherwise} \end{cases}$$



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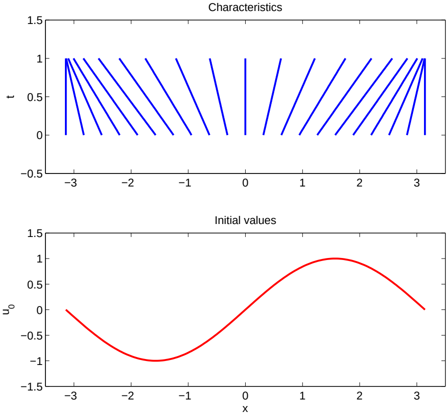
The inviscid Burgers equation

$$u_t + \frac{1}{2}(u^2)_x = 0,$$

is a scalar conservation law with $f(u) = \frac{1}{2}u^2, \;\; a(u) = u$.



Upwind discretization

$$u_t + \frac{1}{2}(u^2)_x = 0,$$

hence $f(u) = \frac{1}{2}u^2$, $a(u) = u$. Characteristic curves are straight lines, but where they meet a shock discontinuity forms.

- Discretize **conservation form**!

$$v_j^{n+1} = v_j^n - \frac{\mu}{2}\begin{cases}[(v_{j+1}^n)^2 - (v_j^n)^2] & \text{if } v_j^n < 0 \\ [(v_j^n)^2 - (v_{j-1}^n)^2] & \text{if } v_j^n \geq 0\end{cases}.$$

- **Do not** discretize "advection form"

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Example

Shock may be located at wrong place unless discretizing conservation form: