CS520: Dispersion and dissipation (Ch. 7)

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Outline

- Dispersion in the PDE
- Numerical dispersion
- Dispersion and dissipativity
- The classical wave equation

PDE DISPERSION

- Consider hyperbolic PDEs with smooth solutions.
- May require long time integration and conservation of physical quantities such as energy, Hamiltonian.
- Note lack of dissipation in PDE.
- Get dispersion when waves associated with different wave numbers travel at different speed.
- Consider (yes, again) the simplest advection equation first, with a special initial value function:

$$u_t + au_x = 0,$$
$$u(0,x) = u_0(x) = e^{-\imath\xi x}.$$

Solution:

$$u(t,x) = e^{\imath\xi(at-x)}.$$

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Dispersion relation, phase velocity, group velocity

- More generally (not just for advection), for $u_0(x) = e^{-\imath \xi x}$,

$$u(t,x) = e^{\imath(\omega t - \xi x)},$$

  where ω = frequency.

- Three important basic definitions:

$$\begin{aligned} \omega &= \omega(\xi) && \text{Dispersion relation} \\ c(\xi) &= \frac{\omega(\xi)}{\xi} && \text{Phase velocity} \\ C(\xi) &= \frac{d\omega(\xi)}{d\xi} && \text{Group velocity.} \end{aligned}$$

- For advection, $\omega = a\xi$ so $C = c = a$. The advection PDE is non-dispersive: phase velocity is independent of wave number.

Example: linearized KdV

- Consider the constant coefficient PDE

$$u_t + \rho u_x + \nu u_{xxx} = 0.$$

- **Dispersion relation**: plug in $u(t,x) = e^{\imath(\omega t - \xi x)}$, obtaining

$$\omega = \rho\xi - \nu\xi^3.$$

- **Phase velocity**:

$$c = \rho - \nu\xi^2$$

- Here phase velocity depends on the wave number, so this is a dispersive PDE. Different waves travel at different speeds.
- **Group velocity**:

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Dispersion in advection semi-discretization

- Consider dispersion in the semi-discretization of the non-dispersive advection equation: Plug $v_j(t) = e^{\imath(\omega t - \xi j h)}$ in

$$\frac{dv_j}{dt} + \frac{a}{2h} D_0 v_j = 0.$$

- Obtain dispersion relation

$$\omega = \frac{a}{h} \sin(\xi h).$$

- So, phase velocity

$$c = a \frac{\sin(\xi h)}{\xi h},$$

  group velocity

$$C = a \cos(\xi h).$$

Dispersion in advection semi-discretization

- Semi-discretization

$$\frac{dv_j}{dt} + \frac{a}{2h} D_0 v_j = 0.$$

- Dispersion relation $\omega = \frac{a}{h}\sin(\xi h)$.
  Phase velocity $c = a\frac{\sin(\xi h)}{\xi h}$.
  Group velocity $C = a\cos(\xi h)$.
- Thus, the semi-discretization is dispersive although the PDE isn't!
- Low wave numbers: $C \approx c \approx a$. So, no difficulty here.
- High wave numbers: some waves nearly stationary. These are parasitic (spurious) waves: difficulty here.

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Dispersion in advection full discretization

- All full discretizations we have seen for the advection equation exhibit numerical dispersion!
- (After all, they are not meant to approximate high wave number solution components well.)
- Trouble may be delayed, though not fully eliminated, when using higher order methods.
- The big difference is that dissipative methods dampen high wave number components, hence the spurious waves are dampened too. For instance, expect more trouble with leap-frog than with Lax-Wendroff.

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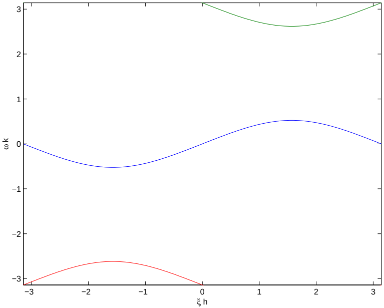
Dispersion in leap-frog

- Plug $v_j^n = e^{\imath(\omega nk - \xi jh)}$ into $v_j^{n+1} = v_j^{n-1} - \mu a(v_{j+1}^n - v_{j-1}^n)$:

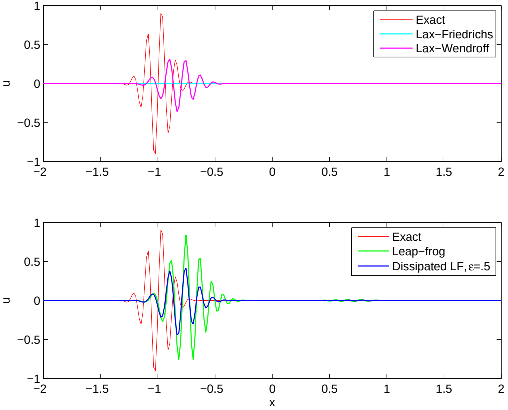
$$\sin(\omega k) = \mu a \sin(\xi h).$$

$\mu a = .5$:



$u_t = u_x, \; u_0(x) = \sin(\eta x) e^{-\eta x^2}.$

Set $\eta = 50, \; \mu = 0.5, \; h = .005\pi.$



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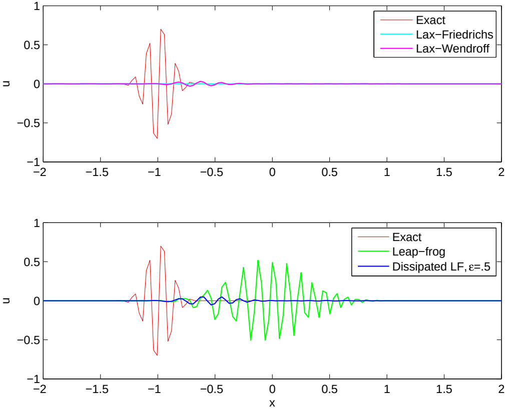

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The (nonlinear) wave equation

- The PDE is 2nd order in time and space:

$$\phi_{tt} = c^2\phi_{xx} - V'(\phi), \quad x_0 \le x \le x_{J+1}, \; t > 0,$$

$c > 0$ is a constant, $V(\phi)$ is smooth, $V'(\phi) \equiv \frac{dV(\phi)}{d\phi}$.

- Initial conditions

$$\phi(0, x) = \phi_0(x), \quad \phi_t(0, x) = \phi_1(x).$$

- Boundary conditions: periodic on $[x_0, x_{J+1}]$ or Dirichlet:

$$\phi(t, x_0) = \phi(t, x_{J+1}) = 0.$$

- May also have absorbing, or radiating BC, designed to ensure that spurious waves do not propagate back into domain.
- For $V' \equiv 0$ can write $\mathbf{u}_t + \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \mathbf{u}_x = \mathbf{0}, \qquad \mathbf{u} = \begin{pmatrix} \phi_t \\ -c\phi_x \end{pmatrix}.$
- All in all a far more civilized problem than advection.

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THE WAVE EQUATION

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- Characteristic curves: $\frac{dx}{dt} = \pm c$.
- For the linear case $V' \equiv 0$, the solution of the Cauchy problem is

$$\phi(t,x) = \frac{1}{2}\left[\phi_0(x-ct) + \phi_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \phi_1(\xi)d\xi.$$

- So, it is easy to construct exact solutions if $\phi_t(0,x) = 0$.
- For periodic BC on $[-L, L]$, obtain $\phi(2lL, x) = \phi(0, x)$ for any integer l; for Dirichlet BC on $[-L, L]$, obtain $\phi(4lL, x) = \phi(0, x)$ for any integer l.

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Hamiltonian semi-discretization

- Centred semi-discretization

$$\frac{d^2 v_j}{dt^2} = \frac{c^2}{h^2}(v_{j-1} - 2v_j + v_{j+1}) - V'(v_j), \quad j = 1, \ldots, J.$$

- Can write this as

$$\frac{d^2 \mathbf{v}}{dt^2} = -B\mathbf{v} - V'(\mathbf{v}),$$

  the matrix B is symmetric positive definite.
- This is a (separable) Hamiltonian system with

$$H(\mathbf{v}, \mathbf{w}) = \frac{1}{2}\mathbf{w}^T\mathbf{w} + \frac{1}{2}\mathbf{v}^T B\mathbf{v} + V(\mathbf{v}),$$

  (so $w_j = \frac{dv_j}{dt}$, $j = 1, \ldots, J$).
- This semi-discretization yields a symplectic map, and it makes sense to discretize it in time using a symplectic method.

Leap-frog redux

- Apply the symplectic, explicit Verlet method

$$\frac{v_j^{n+1} - v_j^n}{k} = w_j^{n+1/2}, \quad j = 1, \ldots, J,$$
$$\frac{w_j^{n+1/2} - w_j^{n-1/2}}{k} = \frac{c^2}{h^2}\left(v_{j-1}^n - 2v_j^n + v_{j+1}^n\right) - V'(v_j^n).$$

- Eliminate the w's, obtaining (for $j = 1, 2, \ldots, J$)

$$v_j^{n+1} - 2v_j^n + v_j^{n-1} = c^2\mu^2(v_{j-1}^n - 2v_j^n + v_{j+1}^n) - k^2 V'(v_j^n).$$

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- The leap-frog method is a favourite method for integrating the classical wave equation (for variable $c(x)$, too).
- But what about dispersion?

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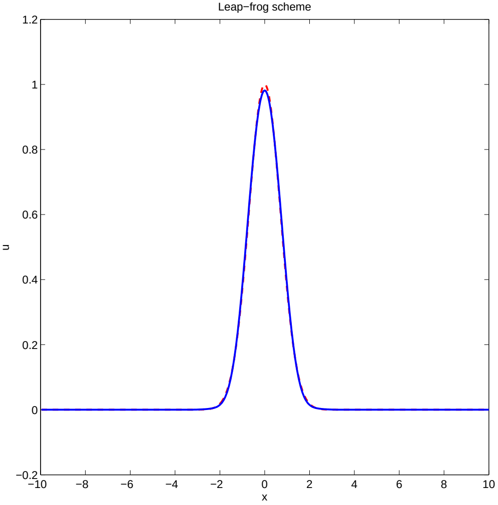
Leap-frog example: $V' \equiv 0,\ \alpha = 1,\ \phi_t(0) \equiv 0$

$c = 1,\ \phi(0,x) = e^{-\alpha x^2},\ -10 \le x \le 10;\ t_f = 400,\ k = .02, h = .04.$



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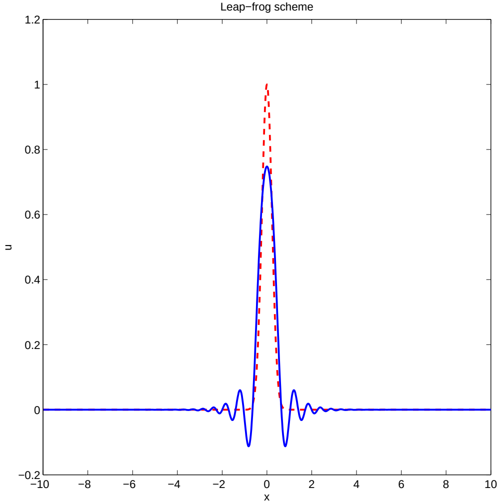
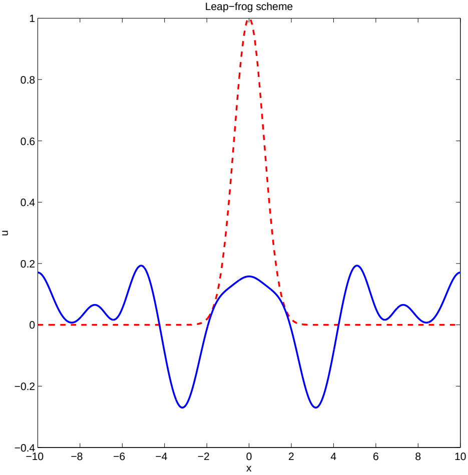


Leap-frog example: $V'(\phi) = \sin(\phi)$, $\alpha = 1$

The sine-Gordon eqn: is solution qualitatively correct?

$c = 1$, $\phi(0, x) = e^{-\alpha x^2}$, $-10 \leq x \leq 10$; $t_f = 400$, $k = .005$, $h = .01$.



Spectral methods

- An important class of high order methods, introduced here in an anecdotal fashion.
- For the problem $u_{xx} = q(x)$ apply Fourier transform as a solver, rather than a theoretical tool to analyze stability.
- On a uniform mesh, use fast Fourier transform (FFT). Solve pointwise in Fourier space ξ, then return to x using IFFT.
- This gives a very high order of accuracy if the BC are periodic, good for numerical dispersion.
- For our PDE problem with periodic boundary conditions, use leap-frog in time:

$$\mathbf{v}^{n+1} = \mathbf{v}^{n-1} + 2\mathbf{v}^n + k^2\mathcal{F}^{-1}\left(-\xi^2\mathcal{F}(\mathbf{v}^n)\right) - k^2V'(\mathbf{v}^n).$$

- See Section 7.4 for more.

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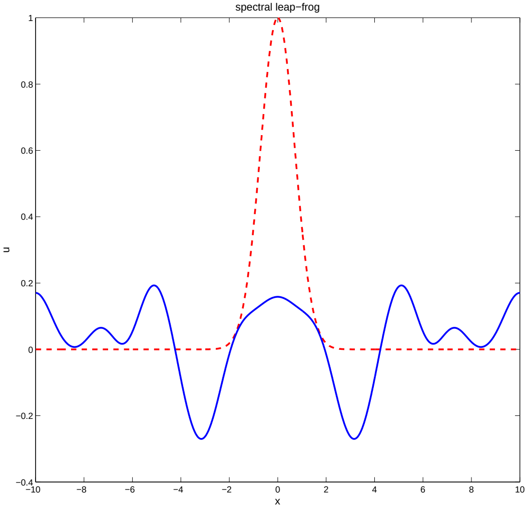
Spectral l-f: $V'(\phi) = \sin(\phi)$, $\alpha = 1$, $c = 1$

$\phi(0,x) = e^{-\alpha x^2}$, $-10 \le x \le 10$; $t_f = 400$, $k = .002$, $J = 2000$ $(h = .01)$.



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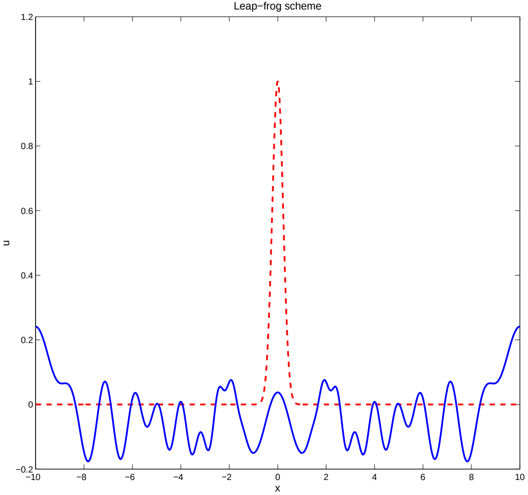

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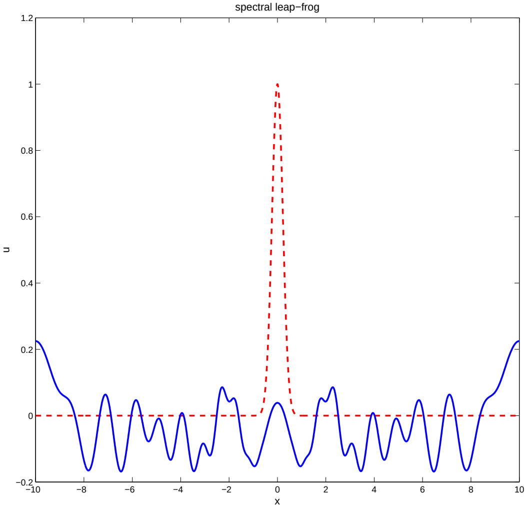

SPECTRAL L-F: $V'(\phi) = \sin(\phi)$, $\alpha = 10$, $c = 1$

$\phi(0,x) = e^{-\alpha x^2}$, $t_f = 400$, $k = .001$, $J = 4000$ ($h = .005$).