CPSC 520 Assignment 4

Due Thursday, Nov. 8, 2012

Please email your programs in one zipped file to `cs520@cs.ubc.ca`, with the message title mentioning Assignment 4 and your name.

1. Write a short program which uses the forward Euler, the backward Euler, and the trapezoidal *or* midpoint methods to integrate a linear, scalar ODE with a known solution, using a fixed step size $k = b/N$, and which finds the maximum error. Apply your program to the ODE

$$\frac{dy}{dt} = (\cos t)y, \qquad 0 \leq t \leq b,$$

with $y(0) = 1$. The exact solution is

$$y(t) = e^{\sin t}.$$

Verify those entries given in Table 1 and complete the missing ones. Make as many

b	N	Forward Euler	Backward Euler	Trapezoidal	Midpoint
1	10	.35e-1	.36e-1	.29e-2	.22e-2
	20	.18e-1	.18e-1	.61e-3	.51e-3
10	100				
	200				
100	1000	2.46	25.90	.42e-2	.26e-2
	2000				
1000	1000				
	10000	2.72	1.79e+11	.42e-2	.26e-2
	20000				
	100000	2.49	29.77	.42e-4	.26e-4

Table 1: Maximum errors for long interval integration of $y' = (\cos t)y$.

(useful) observations as you can about the results in the complete table. Attempt to provide explanations. [Hint: Plotting these solution curves for $b = 20$, $N = 10b$, say, may help.]

2. Consider the following modification of the notorious FPU problem. There are variables $q_1, \ldots, q_{2\hat{m}}$ and $p_1, \ldots, p_{2\hat{m}}$ in which the associated Hamiltonian is written as

$$\begin{aligned} H(\mathbf{q}, \mathbf{p}) &= \frac{1}{4}\left[2\sum_{i=1}^{2\hat{m}} p_i^2 + 2\omega^2 \sum_{i=1}^{\hat{m}} q_{\hat{m}+i}^2 + (q_1 - q_{\hat{m}+1})^4 + (q_{\hat{m}} + q_{2\hat{m}})^4 \right. \\ &+ \left. \sum_{i=1}^{\hat{m}-1} (q_{i+1} - q_{\hat{m}+1+i} - q_i - q_{\hat{m}+i})^4 \right]. \end{aligned}$$

The parameter ω relates to a stiff spring constant and is large. This Hamiltonian is conserved as usual by the solution of the corresponding Hamiltonian system. In addition, let

$$I_i = \frac{1}{2}(p_{\hat{m}+i}^2 + \omega^2 q_{\hat{m}+i}^2).$$

Then it turns out that there is an exchange of energies such that the total oscillatory energy

$$I = \sum_{i=1}^{\hat{m}} I_i,$$

is an **adiabatic invariant**, satisfying

$$I(\mathbf{q}(t), \mathbf{p}(t)) = I(\mathbf{q}(0), \mathbf{p}(0)) + O(\omega^{-1})$$

for exponentially long times t.

Next we choose $\hat{m} = 3$ (yielding an ODE system of size $m = 12$), $\omega = 100$, $\mathbf{q}(0) = (1, 0, 0, \omega^{-1}, 0, 0)^T$, $\mathbf{p}(0) = (1, 0, 0, 1, 0, 0)^T$, and integrate from $t = 0$ to $t = 500$ using RK4 with a constant step size $k = .00025$. The resulting Hamiltonian error is a mere 6.8×10^{-6}, and the oscillatory energies are recorded in Figure 1.

(a) Derive the ODE system that corresponds to this Hamiltonian.

(b) Run MATLAB's `ode45` with default tolerances (abstol = 1.e-6, reltol = 1.e-3), and plot the energies corresponding to those in Figure 1. Comment on the observed results.

(c) Find sufficiently small absolute and relative error tolerances for `ode45` to obtain the qualitatively correct phase plot of Figure 1 in less than one CPU hour (say).

3. The advection equation with variable coefficient

$$u_t + a(x)u_x = 0, \quad a(x) = -(.2 + \sin^2(x - 5)),$$

for $0 \le x \le 2\pi$, $t \ge 0$, with periodic BC and $u_0(x) = e^{-100(x-5)^2}$, has the solution depicted in Figure 2.

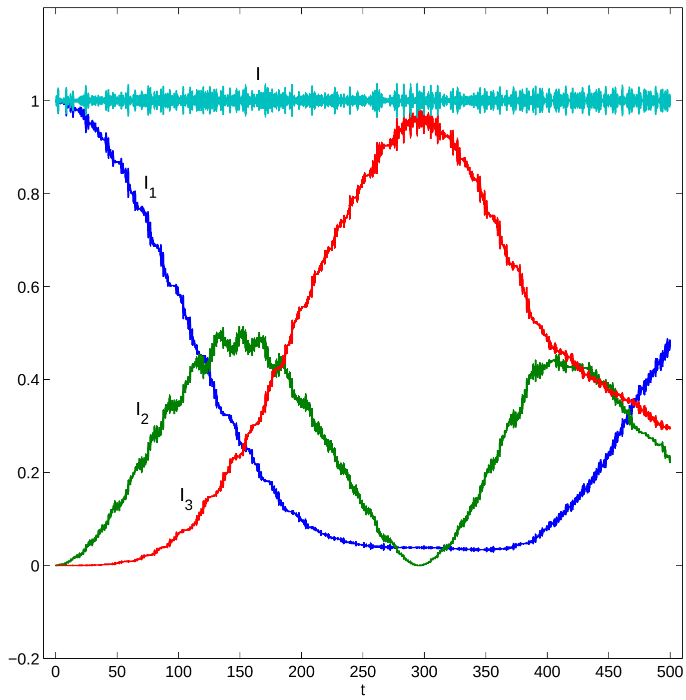

Figure 1: Oscillatory energies for the Fermi-Pasta-Ulam (FPU) problem. Note that the trend of I stays almost constant in time.

Write a program that solves this problem using the the leap-frog method with $k = h/4$. Implement also a method of order $(4, 4)$ which utilizes the centred 5-point, 4th order scheme derived in Section 3.1.1, together with RK4 in time: for the latter you may set $k = h$.

Try your program for $J = 128$, $J = 192$ and $J = 256$, using $h = 2\pi/J$ and producing a total of 6 plots. Explain your observations.

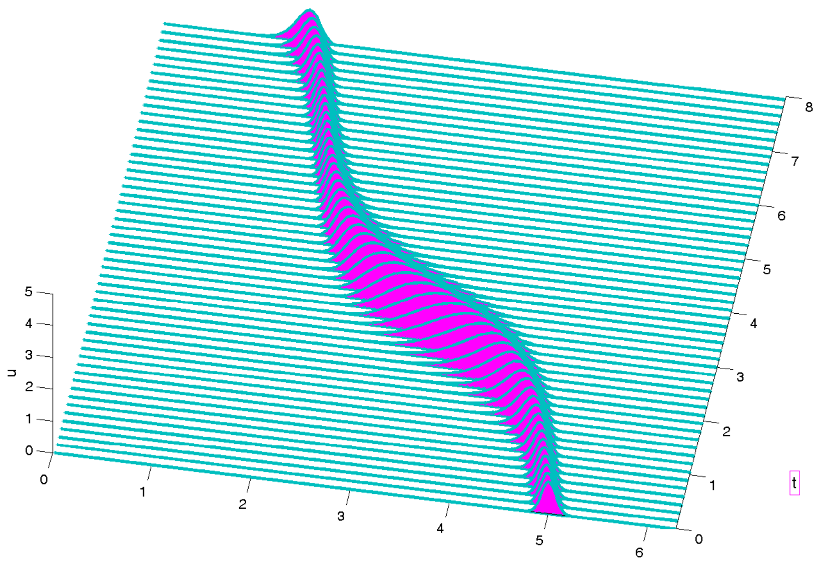

Figure 2: Solution of an advection problem with variable wave speed.