CPSC 520 Assignment 2

Due Wednesday, Oct. 3, 2012

1. Consider the method

$$\begin{aligned} y_{n+\theta} &= y_n + k\theta f(t_n, y_n), \\ y_{n+1} &= y_n + kf(t_{n+\theta}, y_{n+\theta}), \end{aligned}$$

   where $0.5 \le \theta \le 1$ is a parameter.

   (a) Show that this is a 2-stage RK method. Write it in tableau form.

   (b) Show that the method is 1st order accurate, unless $\theta = 0.5$ when it becomes 2nd order accurate (and is then called *explicit midpoint*).

   (c) Show that the domain of absolute stability contains a segment of the imaginary axis (i.e. not just the origin) iff $\theta > 0.5$.

2. Consider the special case of the test equation, $y' = \lambda y$, where λ is real, $\lambda \le 0$. If $y(0) = 1$ then the exact solution $y(t) = e^{\lambda t}$ remains nonnegative and decays monotonically as t increases. Let us call a discretization method **nonnegative** for a step size k if $y_n \ge 0$ implies $y_{n+1} \ge 0$. Show the following:

   (a) If in addition $z = \lambda k$ is in the absolute stability region, then we have **monotonicity**

$$y_n \ge y_{n+1} \ge 0.$$

   This guarantees that the qualitatively unpleasant oscillations that the trapezoidal method produces in Figure 2.9 of the text will not arise.

   (b) The forward Euler method is nonnegative only when $z \ge -1$. (NB Always $z \le 0$.)

   (c) The backward Euler method is unconditionally nonnegative.

   (d) The trapezoidal method is only conditionally nonnegative even though it is A-stable. Find its non-negativity condition.

   (e) Find the non-negativity condition for the TR-BDF2 method of Question 5 of Assignment 1. Is it unconditionally nonnegative?

   (f) Can a *symmetric* RK method be unconditionally nonnegative? Justify if not, or give an example if yes.
   [This last item is harder than the rest.]

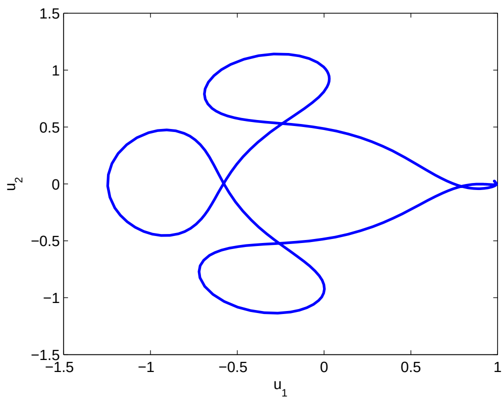

Figure 1: Astronomical orbit using `ode45`.

3. Consider two bodies of masses $\mu = 0.012277471$ and $\hat{\mu} = 1 - \mu$ (earth and sun) in a planar motion, and a third body of negligible mass (moon) moving in the same plane. The motion is governed by the equations

$$\begin{aligned} u_1'' &= u_1 + 2u_2' - \hat{\mu}\frac{u_1 + \mu}{D_1} - \mu\frac{u_1 - \hat{\mu}}{D_2}, \\ u_2'' &= u_2 - 2u_1' - \hat{\mu}\frac{u_2}{D_1} - \mu\frac{u_2}{D_2}, \\ D_1 &= ((u_1 + \mu)^2 + u_2^2)^{3/2}, \\ D_2 &= ((u_1 - \hat{\mu})^2 + u_2^2)^{3/2}. \end{aligned}$$

Starting with the initial conditions

$$\begin{aligned} &u_1(0) = 0.994,\ u_2(0) = 0,\ u_1'(0) = 0, \\ &u_2'(0) = -2.00158510637908252240537862224, \end{aligned}$$

the solution is periodic with period < 17.1. Note that $D_1 = 0$ at $(-\mu, 0)$ and $D_2 = 0$ at $(\hat{\mu}, 0)$, so we need to be careful when the orbit passes near these singularity points. The orbit is depicted in Figure 1.

(a) Write this ODE system in first order form, $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$.

(b) Run MATLAB's `ode45` with default tolerances (which, incidentally, are: absolute error tolerance of 1.e-6 and relative error tolerance of 1.e-3), to integrate the problem on the interval $[0, 17.1]$. You should be able to produce a similar plot to that in Figure 1. How many time steps were required? What were the largest step size and the smallest step sizes used?

(c) Integrate the same problem using the classical fourth order method RK4 with a uniform step size. Plot the solution as in Figure 1 using $1,000$, $5,000$ and

10, 000 uniform steps. For which of these do you no longer observe a difference that is apparent to the naked eye? Discuss your observations relatively to the performance of `ode45`.

4. Consider the following boundary value ODE

$$\begin{aligned} -(au')' &= q, \quad 0 < x < 1, \\ u(0) = 0, & \qquad u'(1) = 0, \end{aligned}$$

where $a(x) > 0$ and $q(x)$ are known, smooth functions. It is well-known (recall Section 3.1.3 of the text) that u also minimizes

$$T = \int_0^1 \left[a(u')^2 - 2uq\right] dx,$$

over all functions with bounded first derivatives that satisfy the essential BC $u(0) = 0$. Consider next discretizing the integral on a generally *nonuniform* mesh

$$0 = x_0 < x_1 < \cdots < x_J = 1.$$

Set $h_i = x_{i+1} - x_i,\ i = 0, 1, 2, \ldots, J - 1$.

   (a) Show that, applying the midpoint rule for the first term and the trapezoidal rule for the second term in T for each subinterval, one obtains the problem of minimizing

$$T_h = \sum_{i=0}^{J-1} a(x_{i+1/2}) \frac{(v_{i+1} - v_i)^2}{h_i} - h_i \left(q(x_i)v_i + q(x_{i+1})v_{i+1}\right)$$

   with $v_0 = 0$. Thus, $T_h(u) = T(u) + \mathcal{O}(h^2)$, where $h = \max_i h_i$.

   (b) Obtain the necessary conditions

$$a(x_{j+1/2}) \frac{v_j - v_{j+1}}{h_j} + a(x_{j-1/2}) \frac{v_j - v_{j-1}}{h_{j-1}} = \frac{h_j + h_{j-1}}{2} q(x_j)$$

   for $j = 1, \ldots, J$, where we set $v_{J+1} = v_J$.

   (c) Show that upon writing the above as a linear system of equations $A\mathbf{v} = \mathbf{q}$ the matrix A is tridiagonal, symmetric and positive definite despite the arbitrary non-uniformity of the mesh.

   (d) Convince yourself by running a computational example using a nonuniform mesh that the solution $\mathbf{v}$ is 2nd order accurate. Then, optionally, try to prove it.

5. For the problem and notation of Exercise 4 define the **hat function**

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_{i-1}} & x_{i-1} \le x < x_i, \\ \frac{x_{i+1} - x}{h_i} & x_i \le x < x_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that ϕ_i is piecewise linear, and that any piecewise linear function w on this mesh that satisfies $w(0) = 0$ can be written as

$$w(x) = \sum_{i=1}^{J} w(x_i)\phi_i(x).$$

(b) Derive the Galerkin finite element method (see Section 3.1.4) for the boundary value ODE. Show that the stiffness matrix A is tridiagonal, symmetric and positive definite. How does this method relate to the method of Exercise 4?