CPSC520: Solutions to Assignment 1, 2012

1. (a) We have to look at

$$|e^{P(\imath\xi)t}| = \left| e^{\left(p_m \imath^m \xi^m + \sum_{j=1}^{m-1} p_j \imath^j \xi^j\right)t} \right|$$

for $t > 0,\ -\infty < \xi < \infty$. Now, for ξ sufficiently large the first term in the exponent dominates. Further, if $\Re(\imath^m p_m)$ is positive then we have for very large ξ that $|e^{P(\imath\xi)t}| \approx e^{\Re(\imath^m p_m)\xi^m t}$ grows unboundedly. Hence the Cauchy problem is ill-posed.

(b) Here we have only odd derivatives, and this yields

$$\begin{aligned} P(\imath\xi) &= \imath p_1 \xi + \imath^3 p_3 \xi^3 + \imath^5 p_5 \xi^5 + \ldots \\ &= \imath(p_1\xi - p_3\xi^3 + p_5\xi^5 + \ldots) \equiv \imath q, \end{aligned}$$

where q is real. Hence $|e^{P(\imath\xi)t}| = |e^{\imath qt}| = 1$.

This solution operator is not a smoother because higher wave numbers are not attenuated. Furthermore, letting $t \leftarrow -t$ gives $|e^{-\imath qt}| = 1$. Hence integrating backwards in time is a well-posed problem.

2. (a) Note first that the transformation $x = Ee^y$ (i.e. $y = \log(x/E)$) takes $[0,\infty)$ to $(-\infty,\infty)$. Likewise, when $t \le T$ we have $s \ge 0$, and the terminal-value problem becomes an initial-value problem.

Next, $\partial_t = -\frac{\sigma^2}{2}\partial_s$, $\partial_x = \frac{1}{x}\partial_y$, $\partial_{xx} = \frac{1}{x^2}(\partial_{yy} - \partial_y)$. The given PDE becomes

$$-\frac{\sigma^2}{2}v_s + \frac{\sigma^2}{2}(v_{yy} - v_y) + rv_y - rv = 0.$$

The desired result follows by defining $\kappa = 2r/\sigma^2$. The transformation of the initial conditions is straightforward by substitution.

(b) Straightforward.

(c) For w we have a well-posed initial-value problem as per the class notes. Now, the transformation from u to w is well-conditioned (i.e., it and its inverse are bounded), so the same applies to the original formulation.

3. The amplification factor is

$$g(\zeta) = \cos(\zeta) - \imath\mu a \sin(\zeta).$$

Thus, assuming $\mu|a| \le 1$ we have

$$|g(\zeta)|^2 = \cos^2(\zeta) + \mu^2 a^2 \sin^2(\zeta) \le \cos^2(\zeta) + \sin^2(\zeta) = 1$$

for any ζ.

4. This is the advection equation $u_t + au_x = 0$, with $a = -2$. Here are the results:

η	h	$-a\mu$	Error in (1.15a)	Error in (1.15b)	Error in (1.15c)
2	$.1\pi$	0.8	2.0e-1	1.2	7.4e-2
	$.01\pi$	0.8	2.4e-2	1.0e-1	9.4e-4
	$.001\pi$	0.8	2.5e-3	*	9.5e-6

We observe:

- The error in the method (1.15a) looks like $O(k) + O(h)$, and the error in (1.15c) looks like $O(k^2) + O(h^2)$.
- The error in the unstable method (1.15b) looks large yet sort of OK for larger k values but blows up for smaller k when more time steps are taken to reach $t = 1$.
- Upon carrying additional experiments with different η, the errors are larger in absolute value than those obtained for the slowly varying u_0 with $\eta = 1$ and smaller than those for the rapidly varying $u_0(x)$ with $\eta = 10$.

5. (a) We have f evaluated at 3 arguments, namely y_n, $y_{n+1/2}$ and y_{n+1}. Hence there are three stages. Note also

$$4y_{n+1/2} - y_n = 3y_n + k(f(y_n)) + f(y_{n+1/2})).$$

Hence we get the tableau

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \hline & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}$$

Since A is lower triangular but its diagonal elements are not all zero, it is diagonally implicit.

(b) For $y_{n+1/2}$ it is the trapezoidal rule which is 2nd order. The third stage is the same as y_{n+1}. For y_{n+1} the order is obviuosly 2 because it is composed of two second order methods. (This can also be verified directly by the tableau and (2.13).)

(c) Substituting $f = \lambda y,\ z = k\lambda$ in (5a) yields $y_{n+1/2} \approx -y_n$ for z large. Then into (5b) this yields

$$R(z) \approx 5/z \to 0 \quad \text{as } z \to -\infty.$$

Stiff decay follows similarly.

(d) Consider

$$y' = -1000y, \quad y_n = 1,$$

and use $k = .1$, say. The BDF2 method yields $y_{n+1} \geq 0$ that is close to 0 but still nonnegative. The trapezoidal method would yield a negative y_{n+1}.

Now consider the system

$$y_1' = -1000 y_1, \quad y_2' = \log(y_1).$$

The BDF2 method will complete the step successfully, whereas the featured method will get stuck, being unable to evaluate $\mathbf{y}_{n+1/2}$.