CPSC 520 Assignment 1

Due Thursday, Sep. 20, 2012

1. Consider the Cauchy problem for the constant coefficient PDE

$$u_t = P(\partial_x)u, \quad P(\partial_x) = \sum_{j=1}^{m} p_j \frac{\partial^j}{\partial x^j}.$$

   (a) Assuming that p_m is a complex scalar, show that if $\imath^m p_m$ has a positive real part, then the problem cannot be well-posed.

   (b) Assuming that p_j are all real, m is odd, and $p_{2l} = 0,\ l = 0, 1, \ldots, (m-1)/2$, show that

$$|e^{P(\imath\xi)t}| = 1, \quad -\infty < \xi < \infty.$$

   What does this imply regarding the smoothing properties of the solution operator? Does integrating backward in time lead to a well-posed problem?

2. The celebrated Black-Scholes model for the pricing of stock options is central in mathematical finance. The PDE is given by

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0, \quad 0 < x < \infty,\ t \leq T. \tag{1}$$

   For the sake of completeness let us add that u is the sought value of the option under consideration, t is time, x is the current value of the underlying asset, r is the interest rate, σ the volatility of the underlying asset, T the expiry date and E is the exercise price. In general, r and σ may vary, but here they are assumed to be known constants, as are E and T.

   For the European call option we have the terminal condition

$$u(T, x) = \max(x - E, 0), \tag{2a}$$

   and the boundary conditions

$$u(t, 0) = 0, \quad u(t, x) \sim x - Ee^{-r(T-t)} \ \text{ as } x \to \infty. \tag{2b}$$

(a) Show that the transformation

$$x = Ee^y, \quad t = T - \frac{2s}{\sigma^2}, \quad u = Ev(s, y),$$

results in the initial value PDE

$$\begin{aligned} v_s &= v_{yy} + (\kappa - 1)v_y - \kappa v, \quad -\infty < y < \infty, \\ v(0, y) &= \max(e^y - 1, 0), \end{aligned} \tag{3}$$

where $\kappa = \frac{2r}{\sigma^2}$.

(b) Show further that transforming

$$\begin{aligned} v &= e^{\gamma y + \beta s} w(s, y), \quad \text{where} \\ \gamma = (1 - \kappa)/2, &\qquad \beta = -(\kappa + 1)^2/4, \end{aligned}$$

yields the PDE problem

$$\begin{aligned} w_s &= w_{yy}, \quad -\infty < y < \infty, \ s \geq 0, \\ w(0, y) &= \max(e^{\frac{1}{2}(\kappa+1)y} - e^{\frac{1}{2}(\kappa-1)y}, 0). \end{aligned} \tag{4}$$

(c) Prove that the terminal-value PDE (1)-(2) is well-posed.

[Note that the solution of (4), and therefore also of (3) and (1)-(2), can be specified exactly in terms of the integral

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\zeta^2/2} d\zeta.$$

However, you don't need this for the purpose of the present exercise.]

3. Consider the advection equation

$$u_t + au_x = 0,$$

and recall that the consistent scheme (1.15b) is unconditionally unstable. The **Lax-Friedrichs** scheme is a variation:

$$v_j^{n+1} = \frac{1}{2}(v_{j-1}^n + v_{j+1}^n) - \frac{\mu a}{2}(v_{j+1}^n - v_{j-1}^n).$$

Show that the Lax-Friedrichs scheme is stable, provided that the CFL condition holds.

4. Carry out calculations using the three difference schemes (1.15) introduced in class and in the text for the problem

$$\begin{aligned} u_t &= 2u_x, \\ u(0, x) = u_0(x) &= \sin(\eta x), \end{aligned}$$

with periodic boundary conditions on $[-\pi, \pi]$. Set $\eta = 2,\ \mu = 0.4$, and employ the three spatial step sizes $h = .1\pi,\ .01\pi$ and $.001\pi$. Record the maximum errors at $t = 1$ using the three schemes. Try also $\eta = 1$ and $\eta = 10$ to see trends, but do not report the obtained errors. What are your observations?

5. The TR-BDF2 is a one-step method for the ODE $y' = f(t, y)$ consisting of applying first the trapezoidal scheme over half a step $k/2$ to approximate the midpoint value, and then the BDF2 scheme over one step:

$$y_{n+1/2} = y_n + \frac{k}{4}(f(y_n) + f(y_{n+1/2})), \tag{5a}$$

$$y_{n+1} = \frac{1}{3}[4y_{n+1/2} - y_n + kf(y_{n+1})]. \tag{5b}$$

One advantage is that only two systems of the original size need be solved per time step.

(a) Write the method (5) as a Runge-Kutta method in standard tableau form (i.e. find A and $\mathbf{b}$). This is an instance of a *diagonally implicit Runge-Kutta* (DIRK) method: please explain this name.

(b) Show that both the order and the stage order equal 2.

(c) Show that the stability function satisfies $R(-\infty) = 0$: this method is L-stable and has stiff decay.

(d) Can you construct an example where this method would fail where the BDF2 method would not?