Formulations of an Extended NaDSet

by

Paul C. Gilmore
and
George K. Tskinis

Technical Report 91-15
August 1991

Department of Computer Science
University of British Columbia
Vancouver, B.C.
CANADA V6T 1Z2

e-mail addresses: <gilmore@cs.ubc.ca> <tsiknis@cs.ubc.ca>

FORMULATIONS of an EXTENDED NaDSet

by

Paul C Gilmore & George K. Tsiknis

Technical Report TR 91-15

August, 1991

ABSTRACT

NaDSet, a Natural Deduction based Set theory and logic, of this paper is an extension of an earlier logic of the same name. It and some of its applications have been described in earlier papers. A proof of the consistency and completeness of NaDSet is provided elsewhere. In all these earlier papers NaDSet has been formulated as a Gentzen sequent calculus similar to the formulation LK by Gentzen of classical first order logic, although it was claimed that any natural deduction formalization of first order logic, such as Gentzen's natural deduction formulation NK, could be simply extended to be a formalization of NaDSet. This is indeed the case for the method of semantic tableaux of Beth or for Smullyan's version of the tableaux, but the extensions needed for other formalizations, including NK and the intutionistic version NJ, require some care. The consistency of NaDSet is dependant upon restricting its axioms to those of the form $A \rightarrow A$, where A is an atomic formula; an equivalent restriction for the natural deduction formulation is not obvious. The main purpose of this paper is to describe the needed restriction and to prove the equivalence of the resulting natural deduction logic with the Gentzen sequent calculus formulation for both the intuitionistic and the classical versions of NaDSet. Additionally the paper provides a brief sketch of the motivation for NaDSet and some of its proven and potential applications.

The authors gratefully acknowledges support from the Natural Science and Engineering Research Council of Canada.

1. INTRODUCTION

NaDSet, a Natural Deduction based Set theory and logic, and some of its applications have been described in the earlier papers [Gilmore89] and [Gilmore&Tsiknis91a,b,c] as well as in the thesis [Tsiknis91]; it is an extension of the logic described in [Gilmore 71,80,86]. A consistency proof has been provided in [Gilmore90] and a completeness proof in [Gilmore91]. In all these papers the logic was formulated as a Gentzen sequent calculus similar to the formulation LK of classical first order logic [Gentzen35] [Szabo69], although it was claimed that any natural deduction formalization of first order logic, such as the formulation NK of natural deduction also described in [Gentzen35], as well as those presented in [Jaskowski34], [Fitch 52], [Beth55], [Prawitz65], or [Smullyan68], could be simply extended to be a formalization of NaDSet. This is indeed the case for the semantic tableaux formalization described in [Beth55] and for Smullyan's version of the tableaux described in [Smullyan68]; but the extensions needed for the other formalizations require some care. The consistency of NaDSet is dependant upon restricting its axioms to those of the form $A \rightarrow A$, where A is an atomic formula; an equivalent restriction for the natural deduction formulation is not obvious. The purpose of this paper is to describe the needed restriction and to prove the equivalence of the resulting natural deduction logic with the Gentzen sequent calculus formulation for both the intuitionistic and the classical versions of NaDSet.

Intuitionistic first order logic is easier to present as a natural deduction logic than is the classical, simply because additional assumptions are needed for classical negation. For this reason the intuitionistic form of NaDSet is treated first. The intuionistic sequent calculus formulation of NaDSet, extending Gentzen's notation is denoted by LSJ; its sequents are restricted to having at most one formula appearing in the succedent. The natural deduction presentation of this intuitionistic logic is denoted by NSJ. The classical sequent calculus formulation is denoted by LSK and a natural deduction presentation of this same logic is denoted by NSK.

In section 2 the elementary and logical syntax for LSK is described; LSJ is then LSK with the succedents of its sequents restricted to having at most one formula. In section 3, NSJ is defined, and in section 4 the equivalence of LSJ and NSJ is proved. Finally in section 5, the equivalence of LSK and NSK is proved.

In the remainder of this introduction, motivations for NaDSet and some of its proven and potential applications are sketched.

1.1. Why a New Logic is Needed

Mathematics has traditionally used a process of abstraction to generalize and simplify structures: A property of objects is regarded as an object that may itself have properties. The traditional set theories are attempts to codify acceptable abstractions to ensure that undesirable conclusions are not drawn from sound premisses. But the concern of these set theories with what sets may correctly exist has given them an ad hoc character which may account for why "[they] have never been of particular interest to mathematicians. They now function mainly as a talismen to ward off evil" [Gray84].

The limitations imposed on set abstraction by the traditional set theories make them unsuitable for some applications in both mathematics and computer science. Consider the following example from [Feferman84]: Define a B structure to be a triple $< B, \cong, \otimes >$, where B is a set on which an identity $\cong$ and a binary, commutative, and associative operation $\otimes$ are defined. Define $\mathbb{B}$ to be the set of all B structures. Then $< \mathbb{B}, \cong, \otimes >$ is itself a B structure when $\cong$ is interpreted as isomorphism and $\otimes$ as Cartesian product. Hence $< \mathbb{B}, \cong, \otimes >$ is a member of the set $\mathbb{B}$. But this conclusion cannot be expressed in a traditional set theory because the restrictions on set abstraction do not permit a triple to be a member of one of its elements. A proof of the conclusion can be given in NaDSet [Gilmore89]. Similarly, it can be proved that the set of all categories can itself be a category [Gilmore&Tsiknis91b] [Tsiknis91], a result that cannot be proved in traditional set theories [MacLane71], [Feferman77].

The need for abstractions in computer science not available within traditional set theories has also been argued. For example, [Scott70] describes the problems of self-application that can arise when interpreting programming languages and proposes a solution that has led to the development of domain theory as a means for providing denotational semantics. In Scott's foreword to [Stoy77], he concludes "For the future the problems of an adequate proof theory and of explaining non-determinism loom very large." In [Gilmore&Tsiknis91a] NaDSet is used to provide semantics for a toy language used in [Stoy77] to demonstrate Scott's methods, and an approach to domain theory is described that is more fully developed in [Gilmore&Tsiknis91c].

In a recent paper [Scott91] Scott has suggested that a new language for programming systems semantics might be developed within an intuitionistic set theory. The intuitionistic version of NaDSet provides such a set theory.

Horn clause programming, as introduced in Prolog, provides a computational model, but not a deductive model, for its programs. In NaDSet, the definition of a predicate by Horn clauses is an abstraction term that is complete in the sense that two predicates with different but equivalent definitions can be proved identical without additional axioms [Gilmore&Tsiknis91a]. In short, a proof theory for the semantics of Horn clause programming can be provided. For this reason, NaDSet may suggest extensions to Prolog that incorporate second order concepts.

The increasing levels of abstraction required for the conceptual models used in enterprise modelling for database design, knowledge engineering and object-oriented systems, demands a logic within which such abstractions can be defined as objects and reasoned about. [Gilmore87a, 87b,88] describes applications of the earlier form of NaDSet to some of these problems.

Both the earlier and the present version of NaDSet offer an elementary resolution of the paradoxes of set theory which is described in 1.2.1 and 1.2.2; briefly, they suggest that the underlying source of the paradoxes is an abuse of use and mention. [Sellars63a,b] suggested a similar source.

1.2. Features of NaDSet

Classical first order logic provides a formalization of two of the three fundamental concepts of modern logic, namely truth functions and quantification. In classical set theories the third fundamental concept, namely abstraction, is formalized by adding axioms to first order logic. In NaDSet the three concepts are formalized in the same manner, namely through inference rules in a natural deduction presentation of the logic. This is the first of four distinguishing features of NaDSet which will be discussed.

1.2.1. Natural Deduction based Set Theory

Natural deduction presentations of logic, but in particular the Gentsen sequent calculus, provide a transparent formalization of the traditional reductionist semantics of [Tarski36], in which the truth value of a complex formula depends upon the truth values of simpler formulas, and eventually upon the truth values of atomic sentences. Formalizing a set theory with abstraction terms in this way has the effect of replacing an unrestricted comprehension axiom scheme by a comprehension inference rule. Although natural deduction presentations of first order logic can be seen as only pedagogically superior presentations of earlier formulations of the logic [Quine51] [Church56], the natural deduction presentation of set theory provided in NaDSet has no correspondingly simple formulation in terms of the earlier logics. That a natural deduction logic is used for the theory is essential to its presentation, and not just a pedagogically convenient device. For example, unlike classical first order logic, the deductions formalized in the inference rules of NaDSet cannot be represented in terms of the classical conditional.

The replacement of an unrestricted comprehension axiom scheme by a comprehension inference rule in a natural decution presentation of logic is not novel to NaDSet; for example, several of the theories described in [Schütte60] or the set theory of Fitch described in [Prawitz65] or [Fitch 52] have this feature. It is also implicit in the description of the logic described in section 21 of [Church41] from which the later papers apparently derive their inspiration. This replacement is, however, not enough to ensure consistency; the theory described in [Gilmore68], for example, is inconsistent because of an improper definition of 'atomic formula'.

The interpretation of atomic formulas is critical for the reductionist semantics of Tarski. A second distinguishing feature of NaDSet is its interpretation of its atomic formulas.

1.2.2. A Nominalist Interpretation of Atomic Formulas

In NaDSet, only names of sets, not sets may be members of sets. To emphasize that this interpretation is distinct from the interpretation of atomic formulas in classical set theory, ':' is used in place of '$\in$' to denote the membership relationship. For example, the atomic formula

(i) $\{u \mid \sim u{:}u\}{:}C$

is true in an interpretation if the term '$\{u \mid \sim u{:}u\}$' is in the set assigned to 'C', and is false otherwise. Note that the term '$\{u \mid \sim u{:}u\}$' is being mentioned in the formula while 'C' is being used.

To avoid confusions of use and mention warned against in [Tarski36] and [Church56], NaDSet must be in effect a second order logic. The first order domain for the logic is the set $\mathbb{D}$ of all closed terms, as defined in 2.1 below. For example, the term '$\{u \mid \sim u{:}u\}$' is a member of $\mathbb{D}$. The second order domain for the logic is the set of all subsets of $\mathbb{D}$. Thus if 'C' is a second order constant, then an interpretation will assign it a subset of $\mathbb{D}$, so that (i) will be true or false in the interpretation. A fuller treatment of the semantics of NaDSet is provided in [Gilmore89, 91].

Although NaDSet is in effect a second order logic, the elementary syntax requires only one kind of quantifiable variable and quantifier for both the first order and second order domains. This is the third distinguishing feature of NadSet

1.2.3. First and Second Order Quantifiers are Combined

In [Gentzen35,38], [Szabo69], and [Prawitz65], a syntactic distinction is drawn between free and bound variables; substitutions of terms can thereby be greatly simplified since a free variable can never become bound. In NaDSet the practice of [Prawitz65] is followed in calling free variables parameters. Thus an occurrence of a parameter in a formula or term of NaDSet plays the role of a variable not bound by a quantifier or an abstraction term.

In an interpretation of NaDSet, first order parameters are assigned members of $\mathbb{D}$, while second order parameters are assigned subsets of $\mathbb{D}$. Thus the second order character of NaDSet is maintained without having distinct quantifiable variables and quantifiers for first and second order domains, as was the case with the earlier version of NaDSet.

1.2.4. A Generalized Abstraction

The term '{u | ~u:u }' introduced in 1.2.2 is a typical abstraction term for a set theory that admits such terms; they take the form {**v** | **F** }, where **v** is a variable, and **F** is a formula in which the variable may have a free occurrence. The term is understood to represent the set of **v** satisfying **F**. In NaDSet, however, v may be replaced by any term in which there is at least one free occurrence of a variable and there are no occurrences of parameters (2.1.3 below). A term satisfying these conditions is the ordered pair term defined for variables **u** and **v** that are distinct from w as follows:

<u,v> for {w | (u:C $\wedge$ v:C)}

Since the variables **u** and **v** occur free in {w | (**u**:C $\wedge$ **v**:C)} and no parameters occur, the term satisfies the restriction and can be used to form new terms, such as, for example, the Cartesian product of two sets **A** and **B**:

[AxB] for {<u,v> | (u:**A** $\wedge$ v:**B**)}

That a simple term like {w | (**u**:C $\wedge$ **v**:C)} has the desired properties of the ordered pair is demonstrated in [Gilmore89].

The inference rules for the introduction of abstraction terms such as these are natural generalizations of the inference rules for abstraction terms of the form {**v** | **F**}. These abstraction rules determine what are appropriate uses of abstraction terms in mathematical arguments, rather than determine what sets may consistently coexist. For example, the arguments Russell used to show that the empty set is a member of the Russell set and that the universal set is not, are arguments that can be shown to be correct in NaDSet, while the arguments demonstrating that the Russell set is and is not a member of itself cannot be justified in NaDSet. Thus, while the traditional set theories have concentrated on answering the question What sets can consistently exist?, NaDSet, like other natural deduction logics, provides an answer to the question What constitutes a sound argument?

2. NaDSet

The elementary syntax for the logic is described in 2.1, while the logical syntax or proof theory is described in 2.2 as a Gentzen sequent calculus LSK. To maintain ease of comparison with the natural deduction logics described in [Prawitz65], which are variants of the logics NJ and NK of [Gentzen35], $\perp$ (contradiction), $\supset$, $\wedge$, and $\vee$ will be taken to be the primitive logical connectives of NaDSet. Negation ~ is defined:

~F for (F $\supset$ $\perp$)

2.1. Elementary Syntax

Five different kinds of sytactical objects are used in the elementary syntax, namely variables, first and second order constants, and first and second order parameters. It is assumed that there are denumerably many objects of each kind, and that any object of one kind is distinct from any object of any other kind.

2.1.1. Elementary Terms

- A variable is a term. The single occurrence of the variable in the term is a free occurrence in the term.
- Any parameter or constant is a term. No variable has a free occurrence in the term.

2.1.2. Formulas

- $\perp$ is a formula in which no variable has a free occurrence.
- If **r** and **s** are any terms, then **r**:**s** is a formula. A free occurrence of a variable in **r** or in **s**, is a free occurrence of the variable in the formula.
- If **F** and **G** are formulas then (**F**$\supset$**G**), (**F**$\wedge$**G**), and (**F**$\vee$**G**) are formulas. A free occurrence of a variable in **F** or in **G** is a free occurrence in each of these formulas.
- If **F** is a formula and **v** a variable, then $\forall$**vF** and $\exists$**vF** are formulas. A free occurrence of a variable other than **v** in **F**, is a free occurrence in $\forall$**vF** and $\exists$**vF**; no occurrence of **v** is free in $\forall$**vF** or $\exists$**vF**.

2.1.3. Abstraction Terms

- Let **t** be any term in which there is at least one free occurrence of a variable and no occurrence of a parameter. Let **F** be any formula. Then {**t** | **F**} is an abstraction term. A free occurrence of a variable in **F** which does not also have a free occurrence in **t**, is a free occurrence in {**t** | **F**}. A variable with a free occurrence in **t** has no free occurrence in {**t** | **F**}.
- An abstraction term is a term.

2.1.4. First and Second Order Terms and Atomic and Closed Formulas

- A term is <u>first order</u> if no second order parameter occurs in it, and is otherwise <u>second order</u>.
- A formula **t:T** is <u>atomic</u> if **t** is first order, and **T** is a second order parameter or constant.
- $\perp$ is an <u>atomic</u> formula.
- A term or formula in which no variable has a free occurrence is said to be <u>closed</u>.

It is important to understand what are free and not free and bound occurrences of variables in a term $\{\mathbf{t} \mid \mathbf{F}\}$. Consider the formula:

$$<u,v>:\{<u,v> \mid u:v \wedge <v,w>:B\}$$

The first occurrence of each of the variables 'u' and 'v' in this formula are free occurrences; all other occurrences of these variables are not free. The single occurrence of the variable 'w' is a free occurrence. Therefore in the formula

$$[\forall u][\forall w](<u,v>:\{<u,v> \mid u:v \wedge <v,w>:B\})$$

only the first occurrence of 'v' is free.

2.2. <u>Logical Syntax</u>

A sequent in LSK takes the form $\Gamma \rightarrow \Theta$, where Γ and Θ are finite, possibly empty, sequences of closed formulas. The formulas Γ form the **antecedent** of the sequent, and the formulas of Θ the **succedent**. A sequent can be interpreted as asserting that one of the formulas of its antecedent is false, or one of the formulas of its succedent is true.

2.2.1. Axioms

$$\mathbf{A} \rightarrow \mathbf{A},$$

where **A** is a closed atomic formula, and

$$\perp\rightarrow$$

2.2.2. Propositional Rules

$$\frac{\Gamma, \mathbf{F} \rightarrow \mathbf{G}, \Theta}{\Gamma \rightarrow (\mathbf{F}\supset\mathbf{G}), \Theta} \qquad \frac{\Gamma \rightarrow \mathbf{F}, \Theta \qquad \Delta, \mathbf{G} \rightarrow \Lambda}{\Gamma, \Delta, (\mathbf{F}\supset\mathbf{G}) \rightarrow \Theta, \Lambda}$$

$$\frac{\Gamma \rightarrow \mathbf{F}, \Theta \qquad \Delta \rightarrow \mathbf{G}, \Lambda}{\Gamma, \Delta \rightarrow (\mathbf{F}\wedge\mathbf{G}), \Theta, \Lambda} \qquad \frac{\Gamma, \mathbf{F} \rightarrow \Theta}{\Gamma, (\mathbf{F}\wedge\mathbf{G}) \rightarrow \Theta} \qquad \frac{\Gamma, \mathbf{G} \rightarrow \Theta}{\Gamma, (\mathbf{F}\wedge\mathbf{G}) \rightarrow \Theta}$$

$$\frac{\Gamma \rightarrow \mathbf{F}, \Theta}{\Gamma \rightarrow (\mathbf{F} \vee \mathbf{G}), \Theta} \qquad \frac{\Gamma \rightarrow \mathbf{G}, \Theta}{\Gamma \rightarrow (\mathbf{F} \vee \mathbf{G}), \Theta} \qquad \frac{\Gamma, \mathbf{F} \rightarrow \Theta \qquad \Delta, \mathbf{G} \rightarrow \Lambda}{\Gamma, \Delta, (\mathbf{F} \vee \mathbf{G}) \rightarrow \Theta, \Lambda}$$

2.2.3. Quantification Rules

$$\frac{\Gamma \rightarrow [\mathbf{p}/\mathbf{u}]\mathbf{F}, \Theta}{\Gamma \rightarrow \forall \mathbf{u}\mathbf{F}, \Theta} \qquad \frac{\Gamma, [\mathbf{t}/\mathbf{u}]\mathbf{F} \rightarrow \Theta}{\Gamma, \forall \mathbf{u}\mathbf{F} \rightarrow \Theta}$$

$$\frac{\Gamma \rightarrow [\mathbf{r}/\mathbf{u}]\mathbf{F}, \Theta}{\Gamma \rightarrow \exists \mathbf{u}\mathbf{F}, \Theta} \qquad \frac{\Gamma, [\mathbf{p}/\mathbf{u}]\mathbf{F} \rightarrow \Theta}{\Gamma, \exists \mathbf{u}\mathbf{F} \rightarrow \Theta}$$

- **F** is any formula in which at most the variable **u** has a free occurrence.
- **p** is a parameter that does not occur in **F**, or in any formula of Γ or Θ of the first or fourth rules.
- **t** is any closed term.

2.2.4. Abstraction Rules

$$\frac{\Gamma \rightarrow [\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{F}, \Theta}{\Gamma \rightarrow [\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{t}{:}\{\mathbf{t} \mid \mathbf{F}\}, \Theta} \qquad \frac{\Gamma, [\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{F} \rightarrow \Theta}{\Gamma, [\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{t}{:}\{\mathbf{t} \mid \mathbf{F}\} \rightarrow \Theta}$$

- $\underline{\mathbf{u}}$ is a sequence of the distinct variables with free occurrences in the term **t**.
- **F** is a formula in which no variable, other than one of $\underline{\mathbf{u}}$, has a free occurrence.
- $\underline{\mathbf{r}}$ is a sequence of closed terms, one for each variable in $\underline{\mathbf{u}}$.
- $[\underline{\mathbf{r}}/\underline{\mathbf{u}}]$ is a simultaneous substitution operator that replaces each occurrence of the variables in $\underline{\mathbf{u}}$ with the corresponding terms in $\underline{\mathbf{r}}$.

2.2.5. Structural Rules

Thinning

$$\frac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow \mathbf{F}, \Theta} \qquad \frac{\Gamma \rightarrow \Theta}{\Gamma, \mathbf{F} \rightarrow \Theta}$$

where **F** is any closed formula.

Contraction

$$\frac{\Gamma, \rightarrow \mathbf{F}, \mathbf{F}, \Theta}{\Gamma \rightarrow \mathbf{F}, \Theta} \qquad \frac{\Gamma, \mathbf{F}, \mathbf{F} \rightarrow \Theta}{\Gamma, \mathbf{F} \rightarrow \Theta}$$

Interchange

$$\frac{\Gamma \rightarrow \mathbf{F}, \mathbf{G}, \Theta}{\Gamma \rightarrow \mathbf{G}, \mathbf{F}, \Theta} \qquad \frac{\Gamma, \mathbf{F}, \mathbf{G} \rightarrow \Theta}{\Gamma, \mathbf{G}, \mathbf{F} \rightarrow \Theta}$$

2.2.6. Cut Rule

$$\frac{\Gamma \rightarrow \Theta, \mathbf{F} \qquad \mathbf{F}, \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda}$$

Although the cut rule has been shown to be redundant in [Gilmore91], it will be maintained as a rule of deduction here.

Essential to the consistency of NaDSet is the restriction of the axioms $\mathbf{A} \rightarrow \mathbf{A}$ to those formulas $\mathbf{A}$ that are atomic. This is not an ad hoc restriction on the formulation of NaDSet to achieve consistency, but is a consequence of Tarski's reductionist formulation of truth as described in 1.2.1.

The rules on the left affecting formulas in the succedent of sequents are called <u>succedent</u> rules, while those on the right affecting formulas in the antecedent are called <u>antecedent</u> rules. The propositional, quantification and abstraction succedent rules will be denoted respectively by $\rightarrow\supset$, $\rightarrow\wedge$, $\rightarrow\vee$, $\rightarrow\forall$, and $\rightarrow\{\}$, while the corresponding antecedent rules are denoted respectively by $\supset\rightarrow$, $\wedge\rightarrow$, $\vee\rightarrow$, $\forall\rightarrow$, and $\{\}\rightarrow$. The structural and cut rules will be referred to by name.

2.3. <u>LSK and LSJ</u>

In the intuitionistic sequent calculus formulation of NaDSet, denoted by LSJ, a succeedent of a sequent may contain at most one formula. Note that only an application of the succedent thinning rule can have a premiss that satisfies this restriction and a conclusion that does not. The classical sequent calculus formulation of NaDSet, with the given unrestricted rules is denoted by LSK.

2.4. <u>Theorem</u>

If Γ, $\sim\mathbf{F} \rightarrow \Theta$ is derivable in LSK, then so is $\Gamma \rightarrow \Theta, \mathbf{F}$.

Proof:

The formula ~**F** can be introduced into the antecedent of a derivable sequent either through antecedent thinning, or through an application of ⊃→ in which the second premiss of the rule is the axiom ⊥ →. By removing all applications of both these rules from a deduction of Γ, ~**F** → Θ, either a deduction of Γ → Θ or a deduction of Γ → Θ, **F**, ... , **F** is obtained. A derivation of Γ → Θ, **F** can therefore be obtained by succedent thinning or succedent contraction.

End of Proof

The theorem will be used in later sections. A similar theorem converting ~**F** in the succedent to **F** in the antecedent can also be proved, but is not needed here.

3. NSJ and NSK

The intutionistic natural deduction logic NSJ chosen for this second formulation of NaDSet is an extention of Gentzen's intuitionistic first order logic NJ. The classical natural deduction logic NSK is an extension of Prawitz's classical logic, which is a variant of Gentzen's.

Essential to the consistency of the sequent calculus presentation of NaDSet is the restriction of the axioms **A** → **A** to those formula **A** that are atomic. The point made in 1.2.1 and 1.2.2 cannot be emphasized too strongly: This is not an ad hoc restriction on the formulation of NaDSet to achieve consistency, but rather is a consequence of Tarski's reductionist formulation of truth. The corresponding restriction in NSJ and NSK is more complicated to describe. It is defined in 3.1.4 and illustrated with examples in 3.1.5.

3.1. The Logical Syntax for NSJ and NSK

A natural deduction presentation of a deduction of a formula from assumptions is a tree of formulas with each leaf formula being an assumption, and the root formula being the conclusion of the deduction. Each formula that is not an assumption is the conclusion of one of the inference rules given below from premisses appearing above the conclusion in the tree. Some inference rules can discharge an assumption. For these rules the assumption is indicated with a '+', while the conclusion of the rule is marked with a '-' to indicate that the assumption is discharged by the rule. When more than one assumption appears in a deduction and is discharged, such as in any deduction in which the ∨E rule is used, assumptions are numbered and the conclusion of a rule that discharges one or more of them is marked with the numbers of the assumptions being discharged.

The inference rules for the natural deduction presentation consist of an introduction (I) and an elimination (E) rule for each logical connective, quantifier, and for abstraction terms, and a single rule for the logical constant $\bot$ that is classified as an introduction rule.

3.1.1. Propositional Rules

$$\supset\mathrm{I}\quad \frac{\begin{array}{ll}\mathbf{F} & [+]\\ \mathbf{G} & \end{array}}{\begin{array}{ll}(\mathbf{F}\supset\mathbf{G}) & [-]\end{array}} \qquad\qquad \supset\mathrm{E}\quad \frac{\mathbf{F}\quad(\mathbf{F}\supset\mathbf{G})}{\mathbf{G}}$$

The assumption **F** may be absent.

$$\wedge\mathrm{I}\quad \frac{\mathbf{F}\quad\mathbf{G}}{(\mathbf{F}\wedge\mathbf{G})} \qquad\qquad \wedge\mathrm{E}\quad \frac{(\mathbf{F}\wedge\mathbf{G})}{\mathbf{F}}\quad \frac{(\mathbf{F}\wedge\mathbf{G})}{\mathbf{G}}$$

$$\vee\mathrm{I}\quad \frac{\mathbf{F}}{(\mathbf{F}\vee\mathbf{G})} \qquad\qquad \vee\mathrm{E}\quad \frac{(\mathbf{F}\vee\mathbf{G})\quad \begin{array}{l}\mathbf{F}\,[+1]\\ \mathbf{H}\end{array}\quad \begin{array}{l}\mathbf{G}\,[+2]\\ \mathbf{H}\end{array}}{\mathbf{H}}\quad [-1,-2]$$

$$\bot\mathrm{J}\quad \frac{\bot}{\mathbf{F}}$$

This is the form of the $\bot$ rule for NSJ, the intuitionistic version. It is an I rule.

$$\bot\mathrm{K}\quad \frac{\begin{array}{ll}(\mathbf{F}\supset\bot) & [+]\\ \bot & \end{array}}{\begin{array}{ll}\mathbf{F} & [-]\end{array}}$$

This is the form of the $\bot$ rule for NSK, the classical version. It is an I rule.

3.1.2. Quantifier Rules

$$\forall\mathrm{I}\quad \frac{[\mathbf{p}/\mathbf{u}]\mathbf{F}}{\forall\mathbf{u}\mathbf{F}} \qquad\qquad \forall\mathrm{E}\quad \frac{\forall\mathbf{u}\mathbf{F}}{[\mathbf{r}/\mathbf{u}]\mathbf{F}}$$

Where **p** is a parameter not occurring in $\forall$u**F** or in any assumption on which **F** depends and **r** is any term.

$$\exists\mathrm{I}\quad \frac{[\mathbf{r}/\mathbf{u}]\mathbf{F}}{\exists\mathbf{u}\mathbf{F}} \qquad\qquad \exists\mathrm{E}\quad \frac{\exists\mathbf{u}\mathbf{F}\quad \begin{array}{ll}[\mathbf{p}/\mathbf{u}]\mathbf{F} & [+]\\ \mathbf{G} & \end{array}}{\begin{array}{ll}\mathbf{G} & [-]\end{array}}$$

Where **r** is any term and **p** is a parameter not occurring in $\exists$u**F**, **G**, or in any assumption on which **G** depends other than [**p**/**u**]**F**.

3.1.3. Abstraction Rules

$$\{\}\mathrm{I}\quad \frac{[\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{F}}{[\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{t}:\{\mathbf{t}|\mathbf{F}\}} \qquad\qquad \{\}\mathrm{E}\quad \frac{[\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{t}:\{\mathbf{t}|\mathbf{F}\}}{[\underline{\mathbf{r}}/\underline{\mathbf{u}}]\mathbf{F}}$$

Where $\underline{\mathbf{u}}$ is a sequence of the distinct variables with free occurrences in the term **t**, **F** is a formula in which only the variables $\underline{\mathbf{u}}$ have free occurrences and $\underline{\mathbf{r}}$ is a sequence of closed terms, one for each variable in $\underline{\mathbf{u}}$.

3.1.4. The restriction on deductions

The restriction on deductions in the natural deduction formulations of NaDSet corresponding to the restriction to atomic in the axioms of the sequent calculus formulation can be expressed:

A formula with an occurrence in a deduction that is minimum must be atomic

By a quasi-deduction is meant a deduction which may not satisfy this restriction.

The remainder of this section is devoted to defining the meaning of a minimum occurrence of a formula in a deduction, and related definitions. The definitions have been adapted from [Prawitz65].

The premiss **F** of the ⊃E rule, the premisses **H** of the ∨E rule, and the premiss **G** of the ∃E rule, are minor premisses of these rules; the premiss **(F ⊃ G)** of the ⊃E rule, the premiss **(F ∨ G)** of the ∨E rule, the premiss ∃uF of the ∃E rule, and all the premisses of the other rules are major premisses of these rules.

A branch of length n, n ≥ 1, of a deduction is a sequence of occurrences of formulas

$$\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$$

satisfying the following conditions:

1. $\mathbf{F}_1$ is an assumption of the deduction;
2. For each i, $1 < i \leq n$, $\mathbf{F}_i$ is the conclusion of an application of an inference rule for which $\mathbf{F}_{i-1}$ is a major premiss;
3. $\mathbf{F}_n$ is either a minor premiss of an application of ∨E, ⊃E, or ∃E, or is the occurrence of a formula at the root of the deduction. In the former case, the branch is called a minor branch, while in the latter it is called a major branch.

Let $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$ be a branch of a deduction. A formula occurrence $\mathbf{F}_i$ in the branch is a minimum occurrence if it is not the conclusion of an application of ∨E or ∃E and satisfies one of the following conditions:

1. $1 = i = n$, in which case $\mathbf{F}_1$ is necessarily an assumption of the deduction; or

2. $1 = i < n$ and $\mathbf{F}_1$ is the premiss of an application of an I rule; or
3. $1 < i < n$ and $\mathbf{F}_i$ is the conclusion of an E rule and a premiss of an I rule; or
4. $1 < i = n$ and $\mathbf{F}_i$ is the conclusion of an application of an E rule.

A formula occurrence $\mathbf{F}_i$ in the branch is a maximum occurrence if it is the conclusion of an application of an introduction rule and a major premiss of an elimination rule.

The following observation is an immediate consequence of the definition of maximum: Let $\mathbf{F}_1$, $\mathbf{F}_2, \ldots, \mathbf{F}_n$ be a major branch for which $\mathbf{F}_1$ is a major premiss of an application of an introduction rule and $\mathbf{F}_n$ is the conclusion of an application of an elimination rule. Then necessarily for some i, $1 < i < n$, $\mathbf{F}_i$ is a maximum occurrence.

3.1.5. Example Deductions in LSJ and NSJ

Examples of deductions in the two formulations of NaDSet will be given in this section to illustrate the effect on deductions of the restrictions. Pairs of deductions are given, the first in the pair being a deduction in LSJ, and the second in NSJ.

Throughout this section **A**, **B**, and **C** denote atomic formulas, and **G** and **H** any formulas. The rules used in the deductions illustrated below are not explicitly stated since they can be inferred from the form of the formulas appearing in the deduction. Further, the horizontal bars used in the description of the rules will be omitted, except in the case of rules with multiple premisses.

3.1.5.1.

The NSJ formulation of the following LSJ deduction illustrates an application of the ⊃I rule without an assumption. It corresponds to an application of antecedent thinning in LSJ.

$\mathbf{A} \rightarrow \mathbf{A}$	$\mathbf{A}$ [+1]
$\rightarrow \mathbf{A} \supset \mathbf{A}$	$\mathbf{A} \supset \mathbf{A}$ [-1]
$\mathbf{F} \rightarrow \mathbf{A} \supset \mathbf{A}$	$\mathbf{F} \supset (\mathbf{A} \supset \mathbf{A})$
$\rightarrow \mathbf{F} \supset (\mathbf{A} \supset \mathbf{A})$	

The assumption **A** of the NSJ deduction is a minimum formula of the single branch of the deduction because it is the premiss of an application of the ⊃I rule.

3.1.5.2.

These deductions illustrate that some "obvious" theorems require longer deductions in the NaDSet formulations than they ordinarily do.

$$\begin{array}{ll}
A \rightarrow A & B \rightarrow B \\ \hline
\multicolumn{2}{l}{A, B \rightarrow A \wedge B} \\
A \wedge B \rightarrow A \wedge B & C \rightarrow C \\ \hline
\multicolumn{2}{l}{A \wedge B, A \wedge B \supset C \rightarrow C \quad C} \\
\multicolumn{2}{l}{A \wedge B \rightarrow (A \wedge B \supset C) \supset C} \\
\multicolumn{2}{l}{\rightarrow A \wedge B \supset ((A \wedge B \supset C) \supset C)}
\end{array}$$

$$\begin{array}{lll}
A \wedge B\ [+1] & A \wedge B\ [+1] & \\
A & B & \\ \hline
\multicolumn{2}{l}{A \wedge B} & (A \wedge B \supset C)\ [+2] \\ \hline
\multicolumn{3}{l}{(A \wedge B \supset C) \supset C\ \ [-2]} \\
\multicolumn{3}{l}{A \wedge B \supset ((A \wedge B \supset C) \supset C)\ \ [-1]}
\end{array}$$

There are three branches in the NSJ deduction. The first two end in the minor premiss **A ∧ B** of the application of ⊃E, while the third begins with the major premiss of that application and ends with the conclusion of the deduction. The occurrences of **A** and **B** in the first two branches are minimum, while the occurrence of **C** in the third is minimum.

3.1.5.3.

The treatment of ∨ differs significantly in the two formulations, as illustrated in this example. The LSJ deduction is given first:

$$\begin{array}{llll}
A \rightarrow A & B \rightarrow B & A \rightarrow A & B \rightarrow B \\
A \rightarrow (A \vee B) & B \rightarrow (A \vee B) & A \rightarrow (A \vee B) & B \rightarrow (A \vee B) \\ \hline
\multicolumn{2}{c}{(A \vee B) \rightarrow (A \vee B)} & \multicolumn{2}{c}{(A \vee B) \rightarrow (A \vee B)} \\
\multicolumn{2}{l}{G \wedge (A \vee B) \rightarrow (A \vee B)} & \multicolumn{2}{l}{H \wedge (A \vee B) \rightarrow (A \vee B)} \\ \hline
\multicolumn{4}{l}{(G \wedge (A \vee B)) \vee (H \wedge (A \vee B)) \rightarrow (A \vee B)} \\
\multicolumn{4}{l}{\rightarrow (G \wedge (A \vee B)) \vee (H \wedge (A \vee B)) \supset (A \vee B)}
\end{array}$$

This deduction takes the following form in NSJ:

$$\begin{array}{llllllll}
 & G \wedge (A \vee B)\ [+1] & A\ [+2] & B\ [+3] & H \wedge (A \vee B)\ [+4] & A\ [+5] & B\ [+6] \\
 & (A \vee B) & (A \vee B) & (A \vee B) & (A \vee B) & (A \vee B) & (A \vee B) \\ \hline
(G \wedge (A \vee B)) \vee (H \wedge (A \vee B))\ [+7] & \multicolumn{3}{l}{(A \vee B)\ \ [-2,-3]} & \multicolumn{3}{r}{(A \vee B)\ \ [-5,-6]} \\ \hline
\multicolumn{7}{l}{\qquad (A \vee B)\ \ [-1,-4]} \\
\multicolumn{7}{l}{(G \wedge (A \vee B)) \vee (H \wedge (A \vee B)) \supset (A \vee B)\ \ [-7]}
\end{array}$$

There are three applications of ∨E in this deduction corresponding to the three applications of ∨→ in LSJ. The first two each have an occurrence of **(A ∨ B)** as major premiss, and two occurrences of

LSJ. The first two each have an occurrence of $(\mathbf{A} \vee \mathbf{B})$ as major premiss, and two occurrences of the same formula as minor premisses. The third has the assumption [+7] as major premiss and two occurrences of $(\mathbf{A} \vee \mathbf{B})$ as minor premisses. The last rule applied is $\supset$I.

The 6 minor branches of the deduction are:

Two of each of $\mathbf{A}$, $(\mathbf{A} \vee \mathbf{B})$ and $\mathbf{B}$, $(\mathbf{A} \vee \mathbf{B})$, and

$\mathbf{G} \wedge (\mathbf{A} \vee \mathbf{B})$, $(\mathbf{A} \vee \mathbf{B})$, $(\mathbf{A} \vee \mathbf{B})$ and $\mathbf{H} \wedge (\mathbf{A} \vee \mathbf{B})$, $(\mathbf{A} \vee \mathbf{B})$, $(\mathbf{A} \vee \mathbf{B})$

The single major branch is:

$(\mathbf{G} \wedge (\mathbf{A} \vee \mathbf{B})) \vee (\mathbf{H} \wedge (\mathbf{A} \vee \mathbf{B}))$, $(\mathbf{A} \vee \mathbf{B})$, $(\mathbf{G} \wedge (\mathbf{A} \vee \mathbf{B})) \vee (\mathbf{H} \wedge (\mathbf{A} \vee \mathbf{B})) \supset (\mathbf{A} \vee \mathbf{B})$

In each of the first 4 minor branches, one of $\mathbf{A}$ or $\mathbf{B}$ is minimum. In each of the remaining minor branches, only the last formula can be minimum since it is the conclusion of an E rule. But recall that such conclusions are specifically excluded from being minimum, so that neither of these two minor branches have a minimum. In the single major branch $(\mathbf{A} \vee \mathbf{B})$ is the conclusion of an E rule and the major premiss of an I rule, but it may not be a minimum since it is the conclusion of an $\vee$E rule. Therefore each occurrence of a minimum formula in the NSJ deduction is atomic.

3.1.5.4.

The treatment of the existential quantifier is also different in the two formulations, as illustrated in this example. Here D is used as a second order constant, and p as a first order parameter, so that p:D is atomic.

$p{:}D \rightarrow p{:}D$		$[p/x](\mathbf{G} \wedge x{:}D)$ [+1]
$p{:}D \rightarrow \exists x(x{:}D)$		$[p/x]\mathbf{G} \wedge p{:}D$
$[p/x]\mathbf{G} \wedge p{:}D \rightarrow \exists x(x{:}D)$	$\exists x(\mathbf{G} \wedge x{:}D)$ [+2]	$p{:}D$
$[p/x](\mathbf{G} \wedge x{:}D) \rightarrow \exists x(x{:}D)$	————————	
$\exists x(\mathbf{G} \wedge x{:}D) \rightarrow \exists x(x{:}D)$	$p{:}D$ [-1]	
$\rightarrow \exists x(\mathbf{G} \wedge x{:}D) \supset \exists x(x{:}D)$	$\exists x(x{:}D)$	
	$\exists x(\mathbf{G} \wedge x{:}D) \supset \exists x(x{:}D)$ [-2]	

The single minor branch in the NSJ deduction begins with the assumption +1 and ends with the minor premiss p:D of the application of $\exists$E. The occurrence of p:D in this branch is minimum. The single major branch begins with the assumption +2. In this branch p:D is the conclusion of the application of $\exists$E and the major premiss of an application of $\exists$I. However, since it is the conclusion of the application of $\exists$E it is not minimum, although it is atomic.

3.1.5.5.

The use of the abstraction rules has not yet been illustrated. This is done in the following "deductions" which at the same time demonstrate why the restrictions on the deductions are necessary. The example is an adaptation of Curry's paradoxical combinator Y. [Curry58] The following abbreviation is used in the "deductions":

$$Y \textbf{ for } \{x \mid x{:}x \supset A\}$$

$$\begin{array}{ll}
\underline{Y{:}Y \to Y{:}Y \qquad A \to A} & \\
Y{:}Y,\ Y{:}Y \supset A \to A & \underline{Y{:}Y \to Y{:}Y \qquad A \to A} \\
Y{:}Y,\ Y{:}Y \to A & Y{:}Y,\ Y{:}Y \supset A \to A \\
Y{:}Y \to A & Y{:}Y,\ Y{:}Y \to A \\
\to Y{:}Y \supset A & Y{:}Y \to A \\
\to Y{:}Y & \\
\hline
\to A &
\end{array}$$

Thus without the restriction, every atomic formula is derivable. Further, since the sequent $\bot \to \bot$ is derivable, the atomic formula A may be replaced by $\bot$ so that $\bot$ is also derivable. This is, however, <u>not</u> a deduction in LSJ because Y:Y is not an atomic formula.

The corresponding natural deduction formulation of this "deduction" follows:

$$\begin{array}{ll}
Y{:}Y\ [+1] & \\
\underline{Y{:}Y \supset A \qquad Y{:}Y\ [+1]} & \\
A & \underline{Y{:}Y \supset A \qquad Y{:}Y\ [+2]} \\
Y{:}Y \supset A\ [-1] & A \\
Y{:}Y & Y{:}Y \supset A\ [-2] \\
\hline
A &
\end{array}$$

The branches of the deduction are:

Y:Y, Y:Y ⊃ A, A, Y:Y ⊃ A, Y:Y

Y:Y

Y:Y ⊃ A, A, Y:Y ⊃ A

Y:Y

The occurrence of A in the first and third branches is minimum, as is the occurrence of Y:Y in the second and the fourth. But Y:Y is not atomic.

4. LSJ and NSJ are EQUIVALENT

A sequent $\Gamma \rightarrow \mathbf{H}$ is said to be derivable in NSJ if there is a deduction of **H** from the premisses Γ; a sequent $\Gamma \rightarrow$ is said to be derivable in NSJ if there is a deduction of $\perp$ from the premisses Γ. To prove that LSJ and NSJ are equivalent, it is sufficient to prove that a sequent $\Gamma \rightarrow \mathbf{H}$ is derivable in NSJ if and only if it is derivable in LSJ. The proof of the "if" result is by induction on the size of deductions in LSJ, where the size of a deduction is the number of applications of the logical rules 2.2.2, 2.2.3, and 2.2.4, of succedent thinning and of cut; antecedent thinning and the contraction and interchange rules are not counted. The proof of the "only if" result is by induction on the size of deductions in NSJ, where the size of a deduction is the number of applications of the rules 3.1.1, 3.1.2, and 3.1.3.

4.1. Theorem

If $\Gamma \rightarrow \mathbf{H}$ is derivable in LSJ, then it is derivable in NSJ.

Proof: By induction on the size k of the deduction of $\Gamma \rightarrow \mathbf{H}$. If k=0, the deduction consists of an axiom $\mathbf{A} \rightarrow \mathbf{A}$ or $\perp\rightarrow$. In the first case, the corresponding deduction of size 0 in NSJ consists of the single assumption **A**, while in the second case of the single assumption $\perp$; these deductions satisfy the minimum formula condition since both **A** and $\perp$ are atomic.

Assume the conclusion is true for deductions of size less than k, and consider a deduction of size k. Consider the last rule applied in the deduction. It may be assumed that for each premiss $\Delta \rightarrow \mathbf{F}$ of the rule an NSJ deduction of $\Delta \rightarrow \mathbf{F}$ has been given. If the rule is a one premiss rule, then Σ denotes the given NSJ deduction of the first premiss, excluding assumptions and conclusion. If the rule is a two premiss rule, then Σ_1 denotes the given NSJ deduction of the first premiss, excluding assumptions and conclusion, while Σ_2 denotes the given NSJ deduction of the second, excluding assumptions and conclusion. It is sufficient to prove that an NSJ deduction of the conclusion of the rule can be constructed from Σ if the rule has a single premiss, and from Σ_1 and Σ_2 if the rule has two premisses.

Consider now the possibilities for the last rule applied in the deduction of $\Gamma \rightarrow \mathbf{H}$. Proofs will be provided for some of the cases, since proofs for the other cases are similar.

$\rightarrow\supset$: Consider the following quasi-deduction:

$$\dfrac{\begin{array}{l}\Gamma\ \ \mathbf{F}[+1]\\ \quad\ \Sigma\\ \quad\ \mathbf{G}\end{array}}{(\mathbf{F}\supset\mathbf{G})}\ [-1]$$

Here the deduction ending in **G** is the given NSJ deduction of Γ, **F** $\rightarrow$ **G**. An occurrence of a formula in the quasi-deduction is minimum if and only if it is minimum in the NSJ deduction of Γ, **F** $\rightarrow$ **G**. Therefore the quasi-deduction is a deduction.

$\supset\rightarrow$: Consider the following quasi-deduction which will be denoted by Π:

$$\begin{array}{l}\dfrac{\begin{array}{c}\Gamma\\ \Sigma_1\\ \hline \mathbf{F}\end{array}\quad (\mathbf{F}\supset\mathbf{G})}{\mathbf{G}}\qquad \Delta\\ \qquad\quad \Sigma_2\\ \qquad\quad \overline{\ \mathbf{H}\ }\end{array}$$

Π is to be interpreted as follows. The deduction ending in the minor premiss **F** is the given NSJ deduction of $\Gamma \rightarrow$ **F**. The deduction below the application of $\supset$E is the given NSJ deduction of Δ, **G** $\rightarrow$ **H**. The quasi-deduction with conclusion **G** is repeated once for every (not discharged) occurrence of the assumption **G** in the NSJ deduction of Δ, **G** $\rightarrow$ **H**.

The branches of Π are:

a. the branches of the NSJ deduction of $\Gamma \rightarrow$ **F**;

b. the branches of the NSJ deduction of Δ, **G** $\rightarrow$ **H** that start with an assumption in Δ; and

c. the branches that start with (**F**$\supset$**G**) and **G** and continue through Σ_2, possibly to **H**.

Since no occurrence of the assumption (**F**$\supset$**G**) is minimum in Π, a minimum occurrence in this quasi-deduction is necessarily minimum in the branches (a) or (b). Therefore, Π is an NSJ deduction.

$\rightarrow\wedge$: Consider the following NSJ quasi-deduction:

$$\dfrac{\begin{array}{c} \Gamma \\ \Sigma_1 \\ \overline{\mathbf{F}} \end{array} \qquad \begin{array}{c} \Delta \\ \Sigma_2 \\ \overline{\mathbf{G}} \end{array}}{(\mathbf{F} \wedge \mathbf{G})}$$

An occurrence of a formula in this quasi-deduction is minimum if and only if it is minimum in the NSJ deduction of $\Gamma \rightarrow \mathbf{F}$, or in the NSJ deduction of $\Gamma \rightarrow \mathbf{G}$. Therefore it is an NSJ deduction.

$\vee\rightarrow$: Consider the following NSJ quasi-deduction:

$$\dfrac{\begin{array}{c} \Gamma \quad \mathbf{F}[+1] \\ \Sigma_1 \\ \overline{\mathbf{H}} \end{array} \qquad \begin{array}{c} \Delta \quad \mathbf{G}[+2] \\ \Sigma_2 \\ \overline{\mathbf{H}} \end{array} \qquad (\mathbf{F} \vee \mathbf{G})}{\mathrm{H}[-1,-2]}$$

Since the last occurrence of **H** is not minimum, the branch beginning with $(\mathbf{F} \vee \mathbf{G})$ has no minimum formula. Every other branch is a branch of the given NSJ deduction of Γ, $\mathbf{F} \rightarrow \mathbf{H}$, or of the given deduction of Δ, $\mathbf{G} \rightarrow \mathbf{H}$. Therefore the quasi-deduction is a deduction.

$\exists\rightarrow$: Consider the following NSJ quasi-deduction:

$$\dfrac{\exists \mathbf{u}\mathbf{F} \qquad \begin{array}{c} \Gamma \quad [\mathbf{p}/\mathbf{u}]\mathbf{F}\,[+1] \\ \Sigma \\ \mathbf{H} \end{array}}{\mathrm{H}}$$

The branch beginning with $\exists \mathbf{u}\mathbf{F}$ has no minimum formula. Every other branch is a branch in the NSJ deduction of Γ, $[\mathbf{r}/\mathbf{u}]\mathbf{F} \rightarrow \mathbf{H}$. Therefore the quasi-deduction is a deduction.

Succedent thinning: An application of this rule results in the following NSJ deduction:

$$\begin{array}{c} \Gamma \\ \Sigma \\ \overline{\bot} \\ \overline{\mathbf{H}} \end{array}$$

Cut Rule: Consider the following NSJ quasi-deduction:

$$\begin{array}{ll} \Gamma & \\ \Sigma_1 & \\ \hline \mathbf{F} & \Delta \\ \Sigma_2 & \\ \hline \mathbf{H} & \end{array}$$

If the indicated occurrence of **F** is minimum, then it is minimum either in the NSJ deduction of $\Gamma \rightarrow \mathbf{F}$, or in the NSJ deduction of $\Delta, \mathbf{F} \rightarrow \mathbf{H}$. Therefore, this is an NSJ deduction.

End of Proof of theorem 4.1.

4.2. Theorem

If $\Gamma \rightarrow \mathbf{H}$ is derivable in NSJ, then it is derivable in LSJ.

Proof:

By induction on the size k of the NSJ deduction of $\Gamma \rightarrow \mathbf{H}$.

If k=0, the NSJ deduction consists of a single atomic assumption **A**, and the corresponding LSJ deduction is the axiom $\mathbf{A} \rightarrow \mathbf{A}$..

Assume the theorem for deductions of size less than k and consider an NSJ deduction of $\Gamma \rightarrow \mathbf{H}$ of size k. Three special cases will be considered:

A) The last rule applied in the deduction is an introduction rule,

B) The first rule applied in a major branch of the deduction is an elimination rule,.

C) The last rule applied in the deduction is an elimination rule and the first rule applied in

a major branch is an introduction rule.

Since every NSJ deduction has at least one major branch, one of these three cases is true of the given deduction of $\Gamma \rightarrow \mathbf{H}$.

A) The last rule applied in the deduction is an introduction rule

It is sufficient to consider only the cases $\supset$I and $\forall$I, since the remaining cases are similar.

$\supset$I: In this case the NSJ deduction takes the form

$$\begin{array}{ll} \Gamma & \mathbf{F}[+1] \\ & \Sigma_1 \\ & \mathbf{G} \\ \hline & (\mathbf{F} \supset \mathbf{G})\ [-1] \end{array}$$

A minimum formula of the quasi-deduction ending in **G** is necessarily a minimum formula of the full deduction. Therefore the quasi-deduction is a deduction of size less than k, and hence by the induction assumption, there is an LSJ deduction of Γ, **F** $\rightarrow$ **G**. An LSJ deduction of $\Gamma \rightarrow$ (**F**$\supset$**G**) is then obtained by one application of $\rightarrow\supset$.

$\perp$J: In this case the NSJ deduction takes the form

$$\cfrac{\begin{array}{c}\Gamma\\ \Sigma\\ \perp\end{array}}{\mathbf{F}}$$

Since the $\perp$J rule is an introduction rule, a minimum formula of the deduction is necessarily a minimum formula of the quasi-deduction standing above the conclusion **F**. This quasi-deduction is therefore a deduction of size less than k. There is, therefore, an LSJ deduction of $\Gamma \rightarrow \perp$. The axiom $\perp\rightarrow$, one application of cut, and one application of succedent thinning yields an LSJ deduction of $\Gamma\rightarrow$ **F**.

$\forall$I: In this case the NSJ deduction takes the form

$$\cfrac{\begin{array}{l}\Gamma\\ \Sigma\\ [\mathbf{p}/\mathbf{u}]\mathbf{F}\end{array}}{\forall\mathbf{uF}}$$

A minimum formula of the quasi-deduction ending in **[p/u]F** is necessarily a minimum formula of the full deduction. Therefore the quasi-deduction is a deduction of size less than k, and hence by the induction assumption, there is an LSJ deduction of $\Gamma \rightarrow$ **[p/u]F**. An LSJ deduction of $\Gamma \rightarrow \forall$**uF** is obtained by one application of $\rightarrow\forall$.

B) <u>The first rule applied in a major branch of the deduction is an elimination rule</u>

It is sufficient to consider only the cases $\supset$E, $\wedge$E, $\vee$E, and $\exists$E since the remaining cases are similar.

$\supset$E: In this case the NSJ deduction takes the form:

$$\begin{array}{ll}\cfrac{\begin{array}{l}\Gamma\\ \Sigma_1\\ \mathbf{F}\quad(\mathbf{F}\supset\mathbf{G})\end{array}}{\mathbf{G}} & \begin{array}{l}\Delta\\ \Sigma_2\end{array}\\ \quad\mathbf{H} & \end{array}$$

A minimum formula of the quasi-deduction ending in **F** is necessarily a minimum formula of the full deduction. Therefore the quasi-deduction is a deduction of size less than k, and hence by the induction assumption, there is an LSJ deduction of $\Gamma \rightarrow \mathbf{F}$. Further, a minimum formula of the quasi-deduction with assumptions **G** and Δ and end formula **H** is necessarily a minimum formula of the full deduction. Therefore the quasi-deduction is a deduction of size less than k, and hence by the induction assumption, there is an LSJ deduction of $\Delta, \mathbf{G} \rightarrow \mathbf{H}$. One application of $\supset\rightarrow$ yields an LSJ deduction of $\Gamma, \Delta, (\mathbf{F}\supset\mathbf{G}) \rightarrow \mathbf{H}$.

$\wedge$**E:** In this case the NSJ deduction takes the form:

$$
\begin{array}{ll}
\dfrac{(\mathbf{F}\wedge\mathbf{G})}{\mathbf{F}} & \Gamma \\
\Sigma & \\
\mathbf{H} &
\end{array}
$$

Consider the quasi-deduction beginning with the assumptions **F** and Γ, and ending in the conclusion **H**. It is necessarily a deduction because $\wedge$E is an elimination rule. Consequently, by the induction assumption there is an LSJ deduction of $\Gamma, \mathbf{F} \rightarrow \mathbf{H}$ and therefore by one application of $\wedge\rightarrow$, an LSJ deduction of $\Gamma, (\mathbf{F}\wedge\mathbf{G}) \rightarrow \mathbf{H}$.

$\vee$E: In this case the NSJ deduction takes the form:

$$
\begin{array}{lllll}
\Gamma & \mathbf{F}[+1] & \Delta & \mathbf{G}[+2] & \\
 & \Sigma_1 & & \Sigma_2 & \\
 & \mathbf{E} & & \mathbf{E} & (\mathbf{F}\vee\mathbf{G}) \\
\hline
 & & & \mathbf{E}[-1,2] & \quad\Theta \\
 & & & \Sigma_3 & \\
 & & & \mathbf{H} &
\end{array}
$$

Consider the following quasi-deductions constructed from this deduction:

$$
\begin{array}{llllll}
\Gamma & \mathbf{F}[+1] & & \Delta & \mathbf{G}[+2] & \\
 & \Sigma_1 & & & \Sigma_2 & \\
 & \mathbf{E} & \Theta & & \mathbf{E} & \Theta \\
 & \Sigma_3 & & & \Sigma_3 & \\
 & \mathbf{H} & & & \mathbf{H} &
\end{array}
$$

If any formula other than **E** is a minimum formula in the quasi-deductions, then they are NSJ deductions. If **E** is a minimum formula in the quasi-deductions, then necessarily it is the conclusion of an elimination rule other than $\vee$E or $\exists$E applied in Σ_1 or Σ_2, so that it is also minimum in the original NSJ deduction. Therefore the quasi-deductions are NSJ deductions of size less than k of $\Gamma, \Theta, \mathbf{F} \rightarrow \mathbf{H}$ and $\Delta, \Theta, \mathbf{G} \rightarrow \mathbf{H}$ respectively. By the induction assumption there are LSJ deductions of these sequents, and therefore by one application of $\vee\rightarrow$ and applications of thinning, an LSJ deduction of the sequent $\Gamma, \Delta, \Theta, (\mathbf{F}\vee\mathbf{G}) \rightarrow \mathbf{H}$ is obtained.

$\exists$E: In this case the NSJ deduction takes the form:

$$
\begin{array}{lll}
\Gamma & [\mathbf{p}/\mathbf{u}]\mathbf{F}[+1] & \\
 & \Sigma_1 & \\
 & \mathbf{G} \quad \exists\mathbf{u}\mathbf{F} & \\
\hline
 & \mathbf{G}[-1] & \Delta \\
 & \Sigma_2 & \\
 & \mathbf{H} &
\end{array}
$$

Consider the following quasi-deduction constructed from this deduction:

$$
\begin{array}{lll}
\Gamma & [\mathbf{q}/\mathbf{u}]\mathbf{F}[+1] & \\
 & [\mathbf{q}/\mathbf{p}]\Sigma_1 & \\
 & \mathbf{G} & \Delta \\
 & \Sigma_2 & \\
 & \mathbf{H} &
\end{array}
$$

Here **q** is a parameter of the same order as **p** that is distinct from any parameter appearing in **G**, **H** or in any formula of Γ, Δ, Σ_1 or Σ_2. $[\mathbf{q}/\mathbf{p}]\Sigma_1$ is obtained from Σ_1 by replacing each occurrence of **p** in a formula by **q**. An argument similar to the argument applied in the $\vee$E case demonstrates that this quasi-deduction is an NSJ deduction. By the induction assumption there is an LSJ deduction of $\Gamma, \Delta, [\mathbf{q}/\mathbf{u}]\mathbf{F} \rightarrow \mathbf{H}$, and therefore by one application of $\exists\rightarrow$ of the sequent $\Gamma, \Delta, \exists\mathbf{u}\mathbf{F} \rightarrow \mathbf{H}$.

C) The lst rule applied in the deduction is an elimination rule and the first rule applied in a major branch of the deduction is an introduction rule.

In this case the given major branch contains a maximum formula. If **F** is the maximum formula, then the NSJ deduction takes the form:

$$
\begin{array}{ll}
\Gamma & \\
\Sigma_1 & \\
\mathbf{F} & \Delta \\
\Sigma_2 & \\
\mathbf{H} &
\end{array}
$$

which is an NSJ deduction of $\Gamma, \Delta \rightarrow \mathbf{H}$. But since **F** is a maximum formula of the major branch, it cannot be a minimum formula in either of the following NSJ deductions:

$$
\begin{array}{l}
\Gamma \\
\Sigma_1 \\
\mathbf{F}
\end{array}
\qquad\qquad
\begin{array}{ll}
\mathbf{F} & \Delta \\
\Sigma_2 & \\
\mathbf{H} &
\end{array}
$$

Since each of these NSJ deductions is of size less than k, there are LSJ deductions for the sequents $\Gamma \rightarrow \mathbf{F}$ and $\mathbf{F}, \Delta \rightarrow \mathbf{H}$, and hence by cut for the sequent $\Gamma, \Delta \rightarrow \mathbf{H}$.

End of Proof

Corollary

If $\Gamma \to \mathbf{H}$ is derivable in NSK, then it is derivable in LSK.

Proof:

The case $\perp$J of section (A) must instead consider an application of $\perp$K. The deduction takes the form:

$$\begin{array}{llll} \Gamma & (\mathbf{H}\supset\perp) & [+1] & \sim\Theta \\ & \Sigma & & \\ & \perp & & \\ \hline & \mathbf{H} & [-1] & \end{array}$$

From the argument in the $\perp$J case, it may be concluded that Γ, $(\mathbf{H}\supset\perp) \to \perp$ is derivable in LSK. By theorem 2.4, $\Gamma \to \perp$, $\mathbf{H}$ is derivable in LSK, so that a derivation of $\Gamma \to \mathbf{H}$ can then obtained from the axiom $\perp \to$ by one application of cut.

End of Proof

5. LSK and NSKare EQUIVALENT

The equivalence of the natural deduction formulation NSK of the classical NaDSet with its sequent calculus formulation LSK cannot be expressed in the simple manner that it was for the intuitionistic forms, apparently because the "natural" logic for a natural deduction presentation is the intuitionistic. A simple and intutitive presentation of classical NaDSet, that is LSK, is best achieved through semantic tableaux [Beth55] or the equivalent semantic trees [Smullyan68]. The equivalence of these presentations relies essentially on the following conclusion from theorem 2.4: A sequent $\Gamma \to \Theta$ is derivable in LSK if and only if Γ, $\sim\Theta \to$ is derivable, where $\sim\Theta$ is obtained from Θ by replacing each formula $\mathbf{F}$ of Θ with its negation $(\mathbf{F}\supset\perp)$.

The equivalence of LSK and NSK is expressed in terms of the derivability of sequents Γ, $\sim\Theta \to \perp$. The first half of the equivalence is stated in the following theorem:

5.1. Theorem

If $\Gamma \to \Theta$ is derivable in LSK, then Γ, $\sim\Theta \to \perp$ is derivable in NSK.

Proof:

By induction on the size k of the LSK deduction of $\Gamma \to \Theta$. If k=0, $\Gamma \to \Theta$ is an axiom $\mathbf{A} \to \mathbf{A}$ or $\perp \to$. In the first case $\mathbf{A}$, $(\mathbf{A}\supset\perp) \to \perp$ is derivable from $\mathbf{A} \to \mathbf{A}$ and $\perp \to$ by one application of $\supset\to$ and one application of antecedent thinning. In the second case $\perp \to \perp$ is an axiom or is derivable from $\perp \to$ by one application of antecedent thinning.

Assume the theorem for deductions of size less than k and consider a deduction of length k. Consider the last rule applied in the deduction. It may be assumed that for each premiss $\Delta \to \Lambda$

of the rule an NSK deduction of $\Delta \rightarrow \Lambda$ is given. The symbols Σ, Σ_1 and Σ_2 are used here in the same manner as in the proof of theorem 4.1. Proofs will be given only for the cases $\rightarrow\supset$, $\supset\rightarrow$ and cut, since the other cases are similar.

$\rightarrow\supset$: Consider the following quasi-deduction of $\Gamma \rightarrow (\mathbf{F}\supset\mathbf{G}), \Theta$:

$$
\begin{array}{llllll}
\Gamma & \mathbf{F}\,[+1] & (\mathbf{G}\supset\bot) & [+2] & \sim\Theta & \\
 & & \Sigma & & & \\
 & & \bot & & & \\
\cline{3-3}
 & & \mathbf{G} & [-2] & & \text{by } \bot\mathrm{K} \\
\cline{3-4}
 & & (\mathbf{F}\supset\mathbf{G})\,[-1] & & ((\mathbf{F}\supset\mathbf{G})\supset\bot) & \\
\cline{3-5}
 & & & \bot & & \\
\end{array}
$$

The quasi-deduction ending at the first occurrence of $\bot$ is an NSK deduction of $\Gamma, \mathbf{F}, (\mathbf{G}\supset\bot), \sim\Theta \rightarrow \bot$ obtained from the LSK deduction of $\Gamma, \mathbf{F} \rightarrow \mathbf{G}, \Theta$ postulated by the induction assumption. Since the rule $\bot_K$ is an introduction rule, any minimum occurrence of a formula in the above quasi-deduction is necessarily a minimum occurrence in the deduction of Γ, $\mathbf{F}, (\mathbf{G}\supset\bot), \sim\Theta \rightarrow \bot$, or is the last occurrence of $\bot$. Therefore, the above quasi-deduction is an NSK deduction of $\Gamma, ((\mathbf{F}\supset\mathbf{G})\supset\bot), \sim\Theta \rightarrow \bot$.

$\supset\rightarrow$: Consider the following quasi-deduction of $\Gamma, \Delta, (\mathbf{F}\supset\mathbf{G}), \sim\Theta, \sim\Lambda \rightarrow \bot$:

$$
\begin{array}{lllll}
\Gamma & (\mathbf{F}\supset\bot) & [+1] & \sim\Theta & \\
 & \Sigma_1 & & & \\
\cline{2-2}
 & \bot & & & \\
\cline{2-2}
 & \mathbf{F} & [-1] & (\mathbf{F}\supset\mathbf{G}) & \\
\cline{2-4}
\Delta & & \mathbf{G} & \sim\Lambda & \\
 & & \Sigma_2 & & \\
\cline{3-3}
 & & \bot & & \\
\end{array}
$$

where the pieces denoted by Σ_1 and Σ_2 together with the premisses above them and the conclusions are NSK deductions of the sequents $\Gamma, (\mathbf{F}\supset\bot), \sim\Theta \rightarrow \bot$ and $\Delta, \mathbf{G}, \sim\Lambda \rightarrow \bot$, respectively, obtained from the LSK deductions of $\Gamma \rightarrow \mathbf{F}, \Theta$ and $\Delta, \mathbf{G} \rightarrow \Lambda$ postulated by the induction assumption. By an argument similar to that presented in the corresponding case of theorem 4.1, it can be shown that the above quasi-deduction is an NSK deduction.

Cut: The quasi derivation in this case has the form

$$\begin{array}{lllll}
\Gamma & (\mathbf{F}\supset\bot) & [+1] & \sim\Theta & \\
 & \Sigma_1 & & & \\
 & \overline{} & & & \\
 & \bot & & & \\
 & \overline{} & & & \\
\Delta & \mathbf{F} & [-1] & \sim\Lambda & \text{by } \bot K \\
 & \Sigma_2 & & & \\
 & \bot & & &
\end{array}$$

A minimum occurrence of a formula in this quasi-dedution is necessarily minimum in either the deduction ending with the first $\bot$, or in the deduction beginning with the assumptions Δ, **F**, and $\sim\Lambda$. Therefore, the above quasi-deduction is an NSK deduction.

End of Proof

5.2. Theorem

If $\Gamma, \sim\Theta \rightarrow \bot$ is derivable in NSK, then $\Gamma \rightarrow \Theta$ is derivable in LSK.

Proof: Let $\Gamma, \sim\Theta \rightarrow \bot$ be derivable in NSK. By the corollary to theorem 4.2, the sequent is derivable in LSK. An LSK derivation of $\Gamma \rightarrow \bot, \Theta$ can then be obtained from theorem 2.4. A derivation of $\Gamma \rightarrow \Theta$ can then be obtained from the axiom $\bot \rightarrow$ by one application of cut.

End of Proof

REFERENCES

The numbers in parentheses refer to the date of publication. (xx) is the year 19xx.

Beth, E.W.

(55) Semantic Entailment and Formal Derivability, *Mededelingen van de Koninklijke Nederlandse Akademie der Wetenschappen, Afdeling Letterkunde, Nieuwe Reeks*, 18, no.13, 309-342.

Church, Alonzo

(41) The Calculi of Lambda Conversion, Princeton University Press.

(56) Introduction to Mathematical Logic, Vol I, Princeton University Press.

Curry, Haskell B.

(58) Combinatory Logic, Vol I, North-Holland

Feferman, Solomon

(77) Categorical Foundations and Foundations of Category Theory, Logic, Foundations of Mathematics and Computability Theory, Editors Butts and Hintikka, D. Reidel, 149-169.

(84) Towards Useful Type-Free Theories, I, *Journal of Symbolic Logic*, March, 75-111.

Fitch, Frederick B.

(52) Symbolic Logic: An Introduction, Ronald Press, New York.

Gentzen, Gerhard

(35) Untersuchungen über das logische Schliessen, *Mathematische Zeitschrift*, 39, 176-210, 405-431.

(38) Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheories, *Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften*, New Series, No. 4, 19-44.

Gilmore, P.C. (Paul C.),

(68) A Formalized Naive Set Theory, a paper presented at the Summer Conference on Intuitionism and Proof Theory, Buffalo, New York.

(71) A Consistent Naive Set Theory: Foundations for a Formal Theory of Computation, IBM Research Report RC 3413, June 22.

(80) Combining Unrestricted Abstraction with Universal Quantification, To H.B. Curry: Essays on Combinatorial Logic, Lambda Calculus and Formalism, Editors J.P. Seldin, J.R. Hindley, Academic Press, 99-123. This is a revised version of [Gilmore71].

(86) Natural Deduction Based Set Theories: A New Resolution of the Old Paradoxes, *Journal of Symbolic Logic*, 51, 393-411.

(87a) The SET Conceptual Model and the Domain Graph Method of Table Design, UBC Computer Science Department Technical Report 87-7.

(87b) Justifications and Applications of the SET Conceptual Model, UBC Computer Science Department Technical Report 87-9.

(88) A Foundation for the Entity Relationship Approach: How and Why, Proceedings of the 6th Entity Relationship Conference, editor S.T. March, North-Holland 95-113.

(89) How Many Real Numbers are There?, UBC Computer Science Department Technical Report 89-7, Revised August, 1991.

(90) The Consistency of an Extended NaDSet, Institute for Language, Logic and Information Prepublication Series for Mathematical Logic and Foundations ML-90-11, University of Amsterdam, 35pp.

(91) On the Consistency and Completeness of an Extended NaDset, UBC Computer Science Department Technical Report 91-17.

Gilmore, Paul C. & Tsiknis, George K.

(91a) Logical Foundations for Programming Semantics, a paper presented to the Sixth

Workshop on the Mathematical Foundations of Programming Semantics, Kingston, Ontario, Canada, May 15-19, 1990. To appear in Theoretical Computer Science.

(91b) A Formalization of Category Theory in NaDSet, a paper presented to the Sixth Workshop on the Mathematical Foundations of Programming Semantics, Kingston, Ontario, Canada, May 15-19, 1990. To appear in Theoretical Computer Science.

(91c) The Lambda Calculus in NaDSet, In Preparation.

Gray, J.W.

(84) Editor, Mathematical Applications of Category Theory, *Contemporary Mathematics*, 30, American Mathematical Society.

Prawitz, Dag

(65) Natural Deduction, A Proof-Theoretical Study, Stockholm Studies in Philosphy 3, Almquist & Wiksell, Stockholm.

Schütte, K

(60) Beweisetheorie, Springer-Verlag.

Sellars, Wilfred

(63a) Abstract Entitites, *Rev. of Metaphysics,* Vol 16, 625-671

(63b) Classes as Abstract Entitites and the Russell Paradox, *Rev. of Metaphysics,* Vol 17, 67-90

Smullyan, Raymond

(68) First Order Logic, Springer-Verlag.

Szabo, M.E. (editor)

(69) The Collected Papers of Gerhard Gentzen, North-Holland

Tarski, Alfred

(36) Der Wahrheitsbegriff in den formalisierten Sprachen, *Studia Philosophica,* vol. 1. pp261-405. English translation appears in Logic, Semantics, Metamathematics, Papers from 1923 to 1938, Oxford University Press, 152-278, 1956.

Tsiknis, George K.

(91) Applications of a Natural Deduction Based Set Theory, UBC Department of Computer Science PhD thesis.