**The Blocking Probability of Spider-Web Networks**

by

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Technical Report 90-18
June 1990

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We determine the limiting behaviour of the blocking probability for spider-web networks, a class of crossbar switching networks proposed by Ikeno. We use a probabilistic model proposed by the author, in which the busy links always form disjoint routes through the network. We show that if the occupancy probability is below the threshold $2-\sqrt{2} = 0.5857...$, then the blocking probability tends to zero, whereas above this threshold it tends to one. This provides a theoretical explanation for results observed empirically in simulations by Bassalygo, Neiman and Vvedenskaya.

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This research was partially supported by an NSERC Operating Grant and an ASI Fellowship Award.

## 1. Introduction

This paper is devoted to the proof of a single result concerning the performance of a certain type of switching network in a certain probabilistic situation. The type of network will be defined more precisely in Section 2, and the probabilistic situation in Section 3. In Section 4 we shall survey previous work and formulate our result. Section 5 will indicate the methods to be used in the proof, which will occupy Sections 6–9. Some possible extendions to the result will be given in Section 10.

## 2. Switching Networks

The result described in Section 1 is motivated by questions concerning a certain type of switching network called a "spider-web" network. We shall now describe this network, together with another, called a "series-parallel" network, which will provide an interesting comparison.

The basic components of the networks we shall consider are $2 \times 2$ "crossbars". Such a crossbar has four terminals, two called "inlets" and two called "outlets". It contains four single-pole single-throw switches, one for connecting each inlet to each outlet. It can assume any of seven "configurations" (see Figure 2.1 (a–g)). In configuration (a), no switches are closed. In configurations (b–e), one switch is closed, connecting one of the inlets to one of the outlets. In configurations (f) and (g), two switches are closed, connecting the inlets in a one-to-one fashion with the outlets. The configurations (f) and (g) will have special significance for us, and will be called "orientations". The orientation (f) will be called "straight" and the orientation (g) will be called "crossed".

A "spider-web" network has, for some integer $k \geq 3$, $2k - 1$ "stages" each containing $2^{k-1}$ crossbars. The inlets of the crossbars in the first stage are the "inputs" of the network. For $1 \leq j \leq 2k-2$, the outlets of the crossbars in the $j$-th stage are connected by "links" to the inlets of the crossbars in the $(j+1)$-st stage according to a "shuffle" pattern (see Figure 2.2). The outlets of the crossbars in the last stage are the "outputs" of the network. The inputs, links and outputs are collectively called "vertices", and they form $2k$ "ranks", with the inputs forming the 0-th rank and the outputs forming the $(2k-1)$-st. Throughout this paper we shall assume that $k$ is odd, and set $k = 2h + 1$. (The case of even $k$ is similar.)

An "omega" network is constructed in the same way, except that there are only $k$ stages. (Each rank of links is connected according to a shuffle pattern.) It is not hard to see that the "inverse omega", which is constructed in same way, except that each rank of links is connected according to an "inverse shuffle pattern" (see Figure 2.3), is "isomorphic" to the omega network. (This means that there are one-to-one correspondences between crossbars and crossbars and between vertices and vertices, such that all relationships, such as incidences between vertices and crossbars, are preserved.) The omega and inverse omega are examples of a type of network known as a "banyan", in which there is a unique path (comprising $k$ crossbars alternating with $k-1$ links) between each input and each output.

A "series-parallel" network is constructed in the same way as a spider-web network, except that the links in the right half of the network (the $k$-th through $(2k-2)$-nd ranks) are connected according to an inverse shuffle pattern. A series-parallel network has an obvious mirror-image symmetry: it is isomorphic to the network obtained from it by reversing the order of the stages and the order of the ranks, inverting each connection pattern, and exchanging the roles of inlets and of inputs and outputs.

The series-parallel network also has a "recursive" structure: it can be decomposed into an initial stage, two central subnetworks (each of which is a series-parallel network with $2k-3$ stages), and a final stage. This recursive structure has been exploited by Beneš [B] to show that the series-parallel network is rearrangeable, and by Ikeno [I] and Pippenger [P1] for the calculation of blocking probabilities. (A network is "rearrangeable" if, given any one-to-one correspondence between inputs and outputs, there exists an assignment of configurations to the crossbars that connects, through disjoint routes from the inputs to the outputs, each input to its corresponding output. The calculation of blocking probabilities will be discussed in the next section.)

The spider-web network is isomorphic to its mirror image (though this is not as obvious as for the series-parallel network; see Pippenger [P2], Appendix), and we shall frequently exploit this symmetry below. It does not have a recursive structure analogous to that of the series-parallel network, and the question of whether it is rearangeable remains open. It does, however, have a non-recursive decomposition that will be useful. Specifically, a spider-web network with $2k-1$ stages can be decomposed into a "left part" comprising the first $k-1$ stages, a "middle" stage, and a "right part" comprising the last $k-1$ stages. The left part itself can be decomposed into two "left sectors", each of which is isomorphic to an omega network with $k-1$ stages. Similarly, the right part can be decomposed into two "right sectors", each of which is isomorphic to an omega network with $k-1$ stages (see Figure 2.4).

An omega network with an even number $2h$ of stages (an in particular, each of the four sectors in a spider-web network with $2k-1=4h+1$ stages) also has a decomposition that will be useful. Specifically, it can be decomposed into a "left part" comprising the first $h$ stages, and a "right part" comprising the last $h$ stages. Similarly, the right part can be decomposed into $2^h$ "right blocks", each of which is isomorphic to an omega network with $h$ stages (see Figure 2.5).

We shall need to introduce some further terminology for subnetworks of a spider-web network with $2k-1$ stages. The first $h$ stages will be called the "initial part", and the left blocks of the left sectors will be called "initial blocks". The last $h$ stages will be called the "final part", and the right blocks of the right sectors will be called "final blocks". The remaining $k = 2h+1$ stages will be called the "central part", the right blocks of the left sectors will be called "left central blocks" and the left blocks of the right sectors will be called "right central blocks". (The failure of this terminology to coincide with the hierarchical decomposition of the network is a manifestation of the lack of a recursive structure in the network.)

When we refer to a subnetwork (such as a sector or block) we shall use the term "inlink" to refer to the inlets of the crossbars in the first stage of the subnetwork (as contrasted with the inputs of the network as a whole), and the term "outlink" to refer the outlets of the crossbars in the last stage of the subnetwork.

We conclude this section with the remark that spider-web series-parallel, and omega networks (and more generally, networks of a type called "rhyming" networks, studied by Takagi [T]) have elegant and economical implementations as "time-division" networks, though we have described them as "space-division" networks. Ramanan, Jordan and Sauer [RJS] describe this implementation for series-parallel networks, but it is not difficut to extend their ideas to the others we have mentioned.

### 3. Probabilistic Models

By a "state" of a network we shall mean an assignment of configurations to its crossbars such that the closed switches form a set of disjoint paths called "routes" from certain of its inputs to certain of its outputs. A vertex that lies on a route of a state is said to be "busy" in that state; otherwise, it is said to be "idle". A path is said to be "busy" if it contains a busy vertex; otherwise it is said to be "idle"

Suppose that the input $v$ and the output $w$ are both idle in some state. We shall say that $v$ and $w$ are "blocked" in that state if every path from $v$ to $w$ is busy; otherwise we shall say they are "linked".

The principal question with which we shall deal in this paper is: if a network is in a random state, and if the input $v$ and the output $w$ are idle in this state, what is the probability that $v$ and $w$ are blocked in this state? This of course begs the question of what we mean by a "random" state; once we have a probability distribution on the states, we may use it to define the "blocking probability"

$$P = \Pr(v, w \text{ blocked} \mid v \text{ idle}, w \text{ idle}),$$

and the complementary "linking probability"

$$Q = \Pr(v, w \text{ linked} \mid v \text{ idle}, w \text{ idle}).$$

The analysis of practical switching networks (such as spider-web networks) with realistic probability distributions (such as the equilibrium distribution resulting when the network is subjected to assumed "traffic" and operated according to an assumed "policy") presents insurmountable difficulties owing to the astronomically large number of states of the network. This circumstance has led to the use of approximate analyses that are based on an assumed probability distribution on the states. These assumed probability distributions are not justified by consideration of traffic and policy models, but the results they lead to may be substantiated (at least for small networks) by empirical results observed in simulations.

By a "pseudo-state" of a network we shall mean an assignment of one of two conditions, "busy" or "idle", to each vertex (input, link or output) in the network. A state always yields a pseudo-state, but the converse is false, since the busy vertices in a pseudo-state need not form disjoint routes from inputs to outputs. Nevertheless, a probability distribution on the pseudo-states of the network is sufficient to support the definition of blocking and linking probabilities.

The oldest and most frequently used probability distribution is that of Lee [L] and Le Gall [L1,L2]. In this model, which we shall call the "independent" model, each vertex is assumed to be idle or busy independently of all other vertices. The model is therefore completely specified by giving the probability of being busy (the "occupancy" probability) or the complementary probability of being idle (the "vacancy" probability) for each vertex. For the networks we consider, in which the number of vertices does not vary from rank to rank, it is natural to assume that the occupancy probability $p$ or the vacancy probability $q = 1 - p$ is the same for each vertex.

The independent model is very convenient for the calculation of blocking probabilities, but is subject to a significant objection, which we shall call the "continuity" objection. In a state, the busy vertices form disjoint routes connecting inputs to outputs, so that in particular the number of busy inlets equals the number of busy outlets for any crossbar.

The independent model does not reflect this constraint, and in fact distributes most of the probability over pseudo-states that do not correspond to states.

A more refined model, addressing this objection, was proposed by Pippenger [P1]. In this model, which we shall call the "coherent" model, each crossbar is independently assigned one of two possible orientations, straight or crossed, with equal probabilites. These orientations establish a set of disjoint routes from the inputs to the outputs. Each route is then independently "accepted" with probability $p$ or "rejected" with probability $q = 1 - p$. The vertices lying on accepted routes are considered busy, while those lying on rejected routes are considered idle. This establishes a configuration for each crossbar: a orientation (f) will become a configuration (a), (b), (c) or (f), and a orientation (g) will become a configuration (a), (d), (e) or (g). Clearly this model overcomes the continuity objection, while preserving much of the spirit of the independent model.

A model similar to the coherent model was proposed independently by Koverninskiĭ [K]. In this variant, each route is broken into two subroutes at its midpoint, and the two subroutes are accepted or rejected independently. This variant overcomes the continuity objection at all stages except the middle stage, while greatly simplifying the calculation of blocking probabilities. In this paper we shall use the coherent model despite its additional complications, since it most completely overcomes the continuity objection.

## 4. Previous Results

In 1959, Ikeno [I] determined the limiting value of the blocking probability in the independent model for series-parallel networks with many stages (see Figure 4.1). He also introduced spider-web networks, conjectured the limiting value of their blocking probability, and showed that this limiting probability is achieved by third type of network with a "random" interconnection pattern (see Figure 4.2).

In 1968, Takagi [T] showed that spider-web networks are optimal (in the sense of having the smallest blocking probability in the independent model) in a class of networks called "rhyming" networks. This class includes series-parallel networks, but not those with a random interconnection pattern.

In 1989, Pippenger showed that limiting value of the blocking probability conjectured by Ikeno is actually achieved by spider-web networks, and that this limiting value is optimal among all networks containing the same crossbars arranged in the same number of stages, but interconnected according to any pattern. Thus spider-web networks are "asymptotically" optimal in the independent model.

It should be noted that spider-web networks are optimal only asymptotically: Chung and Hwang [CH] have shown that there are networks with smaller blocking probability in the independent model, though their margin of superiority is asymptotically negligible (and they are not rhyming networks).

In 1975, Pippenger introduced the coherent model and used it to determine the limiting value of the blocking probability for series-parallel ntworks (see Figure 4.3). At about the same time, Koverninskiĭ [K] introduced his variant of the coherent model and, on the basis of simulations by Neiman and Vvedenskaya [NV] (see also Neiman [N], and Bassalygo, Neiman and Vvedenskaya [BNV]), conjectured the limiting value of the blocking probability for spider-web networks (see Figure 4.4). This conjecture is verified in the present paper.

The networks proposed by Chung and Hwang have not been evaluated in the coherent model, either analytically or empirically, but it should be noted that the result of Section 6 applies to these networks as well as to spider-web networks, so again their margin of superiority (if indeed they are superior) is asymptotically negligible.

## 5. Strategy and Tactics

The overall strategy of our proof will be the same as as the one used for the independent model in Pippenger [P2]. When the occupancy probability is above threshold, we derive an upper bound to the expected number of idle paths from an input to an output, and show that this tends to zero. It follows that the probability that there exists an idle route tends to zero also.

When the occupancy probability is below threshold, we can show that the expected number of idle paths tends to infinity. This does not, of course, imply that with high probability there exists an idle path. It would suffice to show as well that the variance of the number of idles paths was small (compared with the square of the expectation), but the overlaps among the paths render this impossible.

We thus divide the network into three zones, an initial zone, a central zone, and a final zone. We show that with high probability an idle input has access to many links at the interface between the initial and central zones, and that an idle output has access to many links at the interface between the central and final zones. The paths between the interfaces overlap only in very simple ways, and thus we can use the expectation-and-variance argument outlined above to show that with high probability there there is an idle path between accessible links in the two interfaces.

The tactics needed to implement this strategy differ considerably, however, from those sufficient for the independent model. The sweeping independence assumptions of the independent model make the calculations described above relatively easy. The coherent model, on the other hand, continually confronts us with problems of "weakly dependent random variables". There are three principal tactics we shall use to deal with these problems.

The first tactic is "parsimonious examination" of the state of the network. By this we meant that we shall regard the state of the network as initially unknown to us. If we wish to determine whether a particular event occurs, we shall examine only those aspects of the state that bear upon this question, and even these we shall examine in an adaptive sequential fashion, so that as much as possible of the state remains unexamined. In this way we shall preserve as much as possible such independence as exists within the coherent model to simplify the calculation of probabilities of other events. (This tactic is not new of course; it frequently occurs, for example, in the theory of random graphs, where disjoint sets of potential edges are examined in different part of an argument.)

The second tactic is one we call "separation and reconciliation". This tactic comes into play when even parsimonious examination fails to preserve enough independence for our calculations. If we wish to calculate the probability of the conjunction of two events, for example, we may "separate" the state into an "extended" state containing two independent "versions" of the state. If we calculate the probability of one event with respect to one version, and that of the other event with respect to the other version, then by the assumed independence of the versions, the probability of the conjunction is obtained by multiplication. We then attempt to "reconcile" the two versions into a single consistent state, without affecting the occurance of the events in question. With high probabiity we succeed in doing this, and we obtain an upper or lower bound to the probability of the conjunction by adding or subtracting the probability of failure in this process of reconciliation. (This tactic too has antecedents in the literature; it is an instance of what has come to be known as a "coupling" argument, as first used by Doeblin [D].)

The third tactic is the introduction of "phantom" traffic in the network, in addition to and independently of the "true" traffic. The phantom traffic assigns to certain vertices a "co-condition" ("co-busy" or "co-idle") in addition to their condition (busy or idle). This traffic is used to guide parsimonious examination and to govern the processes of separation and reconciliation.

## 6. The Blocking Regime

Throught this section (except for Lemma 6.2) we shall assume $q < \sqrt{2} - 1$. Our goal in this section is to prove the following theorem.

*Theorem 6.1:* We have

$$\Pr(v, w \text{ linked} \mid v \text{ idle}, w \text{ idle}) \to 0$$

as $k \to \infty$.

For the proof of Theorem 6.1 we shall need the following lemma and proposition.

*Lemma 6.2:* We have

$$\Pr(v \text{ idle}, w \text{ idle}) \to q^2$$

as $k \to \infty$.

*Proof:* We shall say that $v$ and $w$ are "coupled" if they lie on the same route. There are $2^{2k-1}$ paths originating at $v$, of which $2^{k-1}$ terminate at $w$. Since each of these paths is equally likely to be the route originating at $v$, we have

$$\Pr(v, w \text{ coupled}) = 2^{-k}.$$

If $v$ and $w$ are not coupled, the probability that they are both idle is $q^2$, since the routes on which they lie are independently rejected with probability $q$. If they are coupled, the probability that they are both idle is $q$, since this is the probability that the route on which they both lie is rejected. Thus we have

$$\begin{aligned}\Pr(v \text{ idle}, w \text{ idle}) &= q^2(1 - 2^{-k}) + q2^{-k} \\ &\to q^2\end{aligned}$$

as $k \to \infty$. $\triangle$

*Proposition 6.3:* We have

$$\Pr(v, w \text{ linked}) \to 0$$

as $k \to \infty$.

For the proof of Proposition 6.3 we shall need the following three lemmas.

Let $\pi$ be a path from $v$ to $w$. We shall say that $\pi$ is "coupled" if some route first passes through a crossbar on $\pi$, then passes through one or more crossbars not on $\pi$, and finally again passes through a crossbar on $\pi$.

*Lemma 6.4:* We have

$$\Pr(\pi \text{ coupled}) \leq \binom{k}{2} / 2^{k-1}.$$

*Proof:* Consider the crossbar $I$ in the $i$-th stage on $\pi$ and the crossbar $J$ on the $j$-th stage on $\pi$. There can be two distinct paths from $I$ to $J$ only if $j - i \geq k$, in which case there are exactly $2^{j-i-k}$ paths, one of which is included in $\pi$. Let $\varrho$ be such a path, not included in $\pi$. For a route to include $\varrho$, the $j - i - 1$ crossbars on $\varrho$ between $I$ and $J$ must have appropriate orientations, and this occurs with probability $1/2^{j-i-1}$. Thus the probability that $\pi$ is coupled by a route passing through $I$ and $J$ is at most $1/2^{k-1}$. Since there are $\binom{k}{2}$ ways of choosing $i \geq 1$ and $j \leq 2k - 1$ such that $j - i \geq k$, the lemma follows. $\triangle$

*Lemma 6.5:* We have

$$\Pr(\pi \text{ idle} \mid \pi \text{ not coupled}) = q\big((1+q)/2\big)^{2k-1}.$$

*Proof:* Let $u_0 = v, \cdots, u_{2k+1} = w$ be the successive vertices of $\pi$. When $\pi$ is not coupled, the events "$u_0$ idle", "$u_1$ idle", ..., "$u_{2k-1}$ idle" form a Markov chain, since a route can intersect $\pi$ only in a segment of consecutive vertices. Thus

$$\Pr(\pi \text{ idle} \mid \pi \text{ not coupled}) = \\ \Pr(u_0 \text{ idle} \mid \pi \text{ not coupled}) \prod_{1 \leq j \leq 2k-1} \Pr(u_j \text{ idle} \mid u_{j-1} \text{ idle}, \pi \text{ not coupled}).$$

For the first factor in the right-hand side we have $\Pr(u_0 \text{ idle} \mid \pi \text{ not coupled}) = \Pr(u_0 \text{ idle}) = q$, and for each of the remaining $2k - 1$ factors we have $\Pr(u_j \text{ idle} \mid u_{j-1} \text{ idle}, \pi \text{ not coupled}) = \Pr(u_j \text{ idle} \mid u_{j-1} \text{ idle}) = (1 + q)/2$, since the route through $u_j$ is either the one through $u_{j-1}$ (which is idle) or a new one (which is idle with probability $q$), and these alternatives each have equal probability. The lemma follows. $\triangle$

*Lemma 6.6:* We have

$$\Pr(\pi \text{ idle} \mid \pi \text{ coupled}) \leq q\big((1+q)/2\big)^{k}.$$

*Proof:* Let $u_0 = v, \ldots, u_k$ be the successive vertices of $\pi^*$, the subpath comprising the first $k + 1$ vertices of $\pi$. Since the first $k$ stages of the network form a banyan, a route can intersect $\pi^*$ only in a segment of consecutive vertices. Thus, even if $\pi$ is coupled, the events "$u_0$ idle", "$u_1$ idle", ..., "$u_k$ idle" form a Markov chain. Since "$\pi$ idle" implies "$\pi^*$ idle", we have

$$\Pr(\pi \text{ idle} \mid \pi \text{ coupled}) \leq \\ \Pr(u_0 \text{ idle} \mid \pi \text{ coupled}) \prod_{1 \leq j \leq k} \Pr(u_j \text{ idle} \mid u_{j-1} \text{ idle}, \pi \text{ coupled}).$$

Again the first factor is $q$ and the remaining $k$ factors are $(1+q)/2$, so the lemma follows. $\triangle$

*Proof of Proposition 6.3:* Let the random variable $T$ denote the number of idle paths from $v$ to $w$. We have (Markov's inequality)

$$\Pr(v, w \text{ linked}) \leq \operatorname{Ex}(T) = \sum_{\pi} \Pr(\pi \text{ idle}),$$

where the sum is over all paths from $v$ to $w$. By Lemmas 6.4, 6.5 and 6.6 we have

$$\begin{aligned}\Pr(\pi \text{ idle}) &= \Pr(\pi \text{ idle} \mid \pi \text{ not coupled}) + \Pr(\pi \text{ idle} \mid \pi \text{ coupled}) \Pr(\pi \text{ coupled}) \\ &\leq q\big((1+q)/2\big)^{2k-1} + q\big((1+q)/2\big)^{k} \binom{k}{2} / 2^{k-1}.\end{aligned}$$

Since there are $2^{k-1}$ choices for a path $\pi$ from $v$ to $w$ we have

$$\operatorname{Ex}(T) \leq 2^{k-1} q\big((1+q)/2\big)^{2k-1} + q\big((1+q)/2\big)^{k} \binom{k}{2}.$$

Since $q < \sqrt{2} - 1$, we have

$$2^{k-1} q\big((1+q)/2\big)^{2k-1} \to 0 \qquad \text{and} \qquad q\big((1+q)/2\big)^{k} \binom{k}{2} \to 0$$

as $k \to \infty$. The proposition follows. $\triangle$

*Proof of Theorem 6.1:* Since the event "$v, w$ linked" implies the events "$v$ idle" and "$w$ idle", we have

$$\Pr(v, w \text{ linked} \mid v \text{ idle}, w \text{ idle}) = \Pr(v, w \text{ linked}) / \Pr(v \text{ idle}, w \text{ idle}).$$

Thus Theorem 6.1 follows from Lemma 6.2 and Proposition 6.3. $\triangle$

## 7. The Linking Regime

Throughout the rest of this paper we shall assume $q > \sqrt{2} - 1$. Our goal in this section is to prove the following theorem.

*Theorem 7.1:* We have

$$\Pr(v, w \text{ linked} \mid v \text{ idle}, w \text{ idle}) \to 1$$

as $k \to \infty$.

For the proof of Theorem 7.1 we shall need Lemma 6.2 and the following proposition.

*Proposition 7.2:* We have

$$\Pr(v, w \text{ linked}) \to q^2$$

as $k \to \infty$.

The proof of Proposition 7.2 will be commenced in this section and completed in Section 9.

*Proof of Theorem 7.1:* Since the event "$v, w$ linked" implies the events "$v$ idle" and "$w$ idle", we have

$$\Pr(v, w \text{ linked} \mid v \text{ idle}, w \text{ idle}) = \Pr(v, w \text{ linked}) / \Pr(v \text{ idle}, w \text{ idle}).$$

Thus Theorem 7.1 follows from Lemma 6.2 and Proposition 7.2. $\triangle$

We shall use the technique of separation and reconciliation, so we begin by describing the extended probability space that we shall use. We divide the network into three "zones": an "initial" zone comprising the 0-th through the $h$-th ranks, a "central" zone comprising the $(h+1)$-st through the $(3h)$-th ranks, and a "final" zone comprising the $(3h+1)$-st through the $(4h+1)$-st ranks. To generate a random extended state of the network, we first independently assign with equal probability one of two orientations, straight or crossed, to each crossbar. In this way the vertices are partitioned into disjoint routes. We next break each route into three subroutes, called the initial, central and final subroutes, corresponding to the three zones of the network. (The breaks occur at the $(h+1)$-st and $(3h+1)$-st stages.) We then independently accept each subroute with probability $p$ and reject it with probability $q$. We declare the vertices lying on accepted subroutes to be busy and those on rejected subroutes to be idle. Finally, we assign each initial and each final subroute a "co-disposition" ("co-accepted" or "co-rejected") in addition to its disposition (accepted or rejected). (Central subroutes will not be assigned co-dispositions.) The initial subroute containing $v$ and the final subroute containing $w$ will be co-rejected. We choose a parameter $a$ (whose value will be specified later), and independently co-accept each initial subroute (except the one containing $v$), and each final subroute (except the one containing $w$), with probability $a$ (co-rejecting it with the complementary probability $b = 1 - a$). We declare the vertices lying on co-accepted subroutes to be co-busy and those on co-rejected subroutes to be co-idle.

The links in the $h$-th rank that lie on paths from $v$ to $w$ will be called "left interface links". There are $2^h$ such links; they are the outlinks of the initial block that contains $v$, and each left central block contains one left interface link. A left interface link $u$ will be called "accessible" if the path from $v$ to $u$ is idle, and "co-accessible" if that path is co-idle (that is, all the vertices of that path are co-idle). The links in the $(3h+1)$-st rank that lie on paths from $v$ to $w$ will be called "right interface links". There are $2^h$ such links; they are the inlinks of the final block that contains $w$, and each right central block contains one right interface link. A right interface link $u$ will be called "accessible" if the path from $u$ to $w$ is idle, and "co-accessible" if that path is co-idle.

Suppose that we are given a coherent state, and that we wish "separate" it into an extended state. We adopt the orientations of the crossbars from the coherent state. In this way the vertices are partitioned into disjoint routes. We break each route into three subroutes, and assign co-dispositions to its initial and final subroutes, as described above. We now accept or reject each of the three subroutes of a route according to the following rules. If the route passes through a co-accessible left interface link, the initial subroute is accepted or rejected according as the route was accepted or rejected, and the central and final subroutes are independently accepted with probability $p$ and rejected with probability $q$. Otherwise, if the route passes through a co-accessible right interface link, the final subroute is accepted or rejected according as the route was accepted or rejected, and the initial and central subroutes are independently accepted with probability $p$ and rejected with probability $q$. Otherwise the route passes through no co-accessible interface link; the central subroute is accepted or rejected according as the route was accepted or rejected, and the initial and final subroutes are independently accepted with probability $p$ and rejected with probability $q$. It is clear that this separation procedure takes a state with the coherent distribution into one with the extended distribution.

Now suppose that we are given an extended state and that we wish to "reconcile" it into a coherent state. We adopt the orientations of the crossbars from the extended state. In this way the vertices are partitioned into disjoint routes. We accept or reject each route according to the following rules. If the route passes through a co-accessible left interface link, we accept it or reject it according as its initial subroute is accepted or rejected. Otherwise, if the route passes through a co-accessible right interface link, we accept it or reject it according as its final subroute is accepted or rejected. Otherwise the route passes through no co-accessible interface link; we accept it or reject it according as its central subroute is accepted or rejected. This reconciliation process is a deterministic inverse of the separation process described above; it therefore takes a state with the extended distribution into one with the coherent distribution.

We shall now fix the parameters $a$ and $b$, together with some others that we shall need later. Since $q > \sqrt{2}-1$, we can choose $b < 1$ so that $t = qb > \sqrt{2}-1$. We then set $a = 1-b$ and $s = 1-t$. Since $1+t > \sqrt{2}$, we can choose $z > \sqrt{2}$ so that $z < 1+t$. Finally, since $b < 1$, we can choose $y < 2$ so that $y < 1 + b$. The parameters $a$, $b$, $s$, $t$, $y$ and $z$ will remain fixed throughout the rest of this paper.

## 8. Separation

Our goal in this section is to prove the following proposition.

We shall say that a vertex is "bi-idle" if it is idle and co-idle. We shall say that a path is "bi-idle" if all its vertices are bi-idle. We shall say that $v$ and $w$ are "bi-linked" in an extended state if there is a path $\pi$ from $v$ to $w$ comprising a bi-idle initial subpath, an idle central subpath, and a bi-idle final subpath.

*Proposition 8.1:* We have

$$\Pr(v, w \text{ bi-linked}) \to q^2$$

as $k \to \infty$.

For the proof of Proposition 8.1 we shall need Proposition 8.2 Corollary 8.4 and Proposition 8.5 below.

We shall say that a left interface link is "bi-accessible" if it is both accessible and co-accessible. We shall say that $v$ is "ample" if there are at least $Z = \lfloor z^h \rfloor$ bi-accessible left interface links; otherwise, we shall say that $v$ is "deficient".

*Proposition 8.2:* We have

$$\Pr(v \text{ ample}) \to q$$

as $k \to \infty$.

For the proof of Proposition 8.2 we shall need the following lemma.

*Lemma 8.3:* The conditional generating function for the number of bi-accessible left interface links, given that $v$ is idle, is $f^{(h)}(\zeta)$, the $h$-th iterate of $f(\zeta) = \zeta(s + t\zeta)$.

*Proof:* Let us broaden the term "bi-accessible" to include all vertices $u$ in the initial zone such that is a path from $v$ to $u$ all of whose vertices are bi-idle. For $0 \le j \le h$, let the random variable $T_j$ denote the number of bi-accessible vertices in the $j$-th rank. If $v$ is idle, then $T_0 = 1$. We seek to determine the generating function for $T_h$ under this condition.

Each initial subroute except the one containing $v$ is independently accepted with probability $p$ and co-accepted with probability $a$. If we say that an inital subroute is "bi-rejected" if it is both rejected and co-rejected, and that it is "amphi-accepted" if it is either accepted or co-accepted, then each initial subroute except the one containing $v$ is independently bi-rejected with probability $t = qb$ and amphi-accepted with the complementary probability $s = 1 - qb$.

If $u$ is a bi-accessible vertex in the $j$-th rank, where $0 \le j \le h-1$, then $u$ is an inlet of a crossbar in the $(j+1)$-st stage. The bi-rejected subroute through $u$ carries it to one bi-accessible link in the $(j+1)$-st rank. The crossbar in the $(j+1)$-st stage has another outlet in the $(j+1)$-st rank, and this link is bi-accessible if and only if the subroute through it is bi-rejected, which occurs with probability $t$. A bi-accessible vertex in the $j$-th rank therefore has either one or two "successors" in the $(j+1)$-st rank, and the generating function for the number of successors is $f(\zeta) = \zeta(s+t\zeta)$. If there are $m$ bi-accessible vertices in the $j$-th rank, the numbers of successors they have are independent, since the subroutes in question are distinct. thus the generating function for $T_{j+1}$ is $f(\zeta)^m$. Furthermore, the numbers $T_0, \ldots, T_h$ of bi-accessible vertices in successive ranks form a Markov chain, since the sets of subroutes in question are disjoint. Thus these numbers form a branching process, and if $f_j(\zeta)$ is the generating function for $T_j$, then $f_{j+1}(\zeta) = f\big(f_j(\zeta)\big)$ is the generating function for $T_{j+1}$. If we define $f^{(0)}(\eta) = \eta$ and $f^{(j+1)}(\eta) = f^{(j)}\big(f(\eta)\big)$, then since the generating function for $T_0$ is $f_0(\eta) = \eta$, it follows by induction on $j$ that the generating function for $T_j$ is $f^{(j)}(\eta)$. Taking $j = h$, the lemma follows. $\triangle$

*Proof of Proposition 8.2:* Since the event "$v$ ample" implies the event "$v$ idle", we have

$$\Pr(v \text{ ample}) = \Pr(v \text{ ample} \mid v \text{ idle}) \Pr(v \text{ idle}).$$

Since $\Pr(v \text{ idle}) = q$, it will suffice to show

$$\Pr(v \text{ ample} \mid v \text{ idle}) \to 1,$$

or equivalently

$$\Pr(v \text{ deficient} \mid v \text{ idle}) \to 0,$$

as $k \to \infty$.

Let the random variable $T$ denote the number of bi-accessible left interface links. By Lemma 8.3, the conditional generating function for $T$, given that $v$ is idle, is $f^{(h)}(\zeta)$. Thus we have (Bernstein's inequality)

$$\Pr(v \text{ deficient} \mid v \text{ idle}) = \Pr(T < Z \mid v \text{ idle}) \le f^{(h)}(\zeta)/\zeta^Z$$

for any $\zeta < 1$.

The function $f(\zeta) = \zeta(s + t\zeta)$ has an attractive fixed point at $\zeta = 0$ and a repulsive fized point at $\zeta = 1$ with derivative $f'(1) = 1 + t$. We shall take $\zeta = 1 - 1/z^h$. Since $z < 1 + t$, the iterate $f^{(h)}(1 - 1/z^h)$ is repulsed by the fixed point at unity, an attracted to the fixed point at zero, as $h \to \infty$. On the other hand, $1/(1-1/z^h)^Z$ remains bounded (indeed, tends to $e = 2.1718\ldots$) in this limit. Thus

$$f^{(h)}(\zeta)/\zeta^Z \to 0$$

for this choice of $\zeta < 1$, and the proposition follows. $\triangle$

We shall say that a right interface link is "bi-accessible" if it is both accessible and co-accessible. We shall say that $w$ is "ample" if there are at least $Z = \lfloor z^h \rfloor$ bi-accessible right interface links; otherwise, we shall say that $w$ is "deficient".

*Corollary 8.4:* We have

$$\Pr(w \text{ ample}) \to q$$

as $k \to \infty$.

*Proof:* The proof is the mirror image of that of Proposition 8.2. $\triangle$

Let $A$ be a set of $Z = \lfloor z^h \rfloor$ left interface links, and let $B$ be a set of $Z$ right interface links. We shall say that $A$ and $B$ are "joined" if there exists a path from some link $a \in A$ to some link $b \in B$ all of whose links (other than $a$ and $b$ themselves) are idle; otherwise we shall say that $A$ and $B$ are "severed". The central zone comprises $k = 2h+1$ successive stages (the $(h+1)$-st through $(3h+1)$-st stages), with pair of consecutive stages being interconnected according to a perfect shuffle. The central zone is therefore a "banyan": there is a unique path joining any left interface link to any right interface link. We shall say that a path from a link $a \in A$ to a link $b \in B$ is "good" if all of its links (except for $a$ and $b$ themselves) are idle.

*Proposition 8.5:* If $\#A = \#B = Z$, then

$$\Pr(A, B \text{ severed}) \to 0$$

as $k \to \infty$.

For the proof of Proposition 8.5 we shall need the following lemma and proposition.

We shall let the random variable $T$ denote the number of good paths $\tau$ from links $a \in A$ to links $b \in B$.

*Lemma 8.6:* We have

$$\mathrm{Ex}(T) = Z^2 q\big((1+q)/2\big)^{2h-1}.$$

*Proof:* We have

$$\mathrm{Ex}(T) = \sum_{\tau} \Pr(\tau \text{ good}),$$

where the sum is over all paths $\tau$ from a link $a \in A$ to a link $b \in B$. Let $u_1, \ldots, u_{2h}$ be the successive vertices of $\tau$, excluding $a$ and $b$. As in the proof of Lemma 6.5, the events "$u_1$ idle", "$u_2$ idle", ..., "$u_{2h}$ idle" form a Markov chain, and we have

$$\Pr(\tau \text{ good}) = \Pr(u_1 \text{ idle}) \prod_{2 \le j \le 2h} \Pr(u_j \text{ good} \mid u_{j-1} \text{ good}).$$

Thus we have $\Pr(\tau \text{ good}) = q\big((1+q)/2\big)^{2h-1}$. Since there are $Z$ ways to choose $a \in A$ and $Z$ ways to choose $b$ $B$, the lemma follows. $\triangle$

*Proposition 8.7:* We have

$$\begin{aligned}\mathrm{Var}(T) \le\ & Z^2 q\big((1+q)/2\big)^{2h-1} \\ & + 2Z^3 q^2 \big((1+q)/2\big)^{3h} \\ & + Z^2 h^2 q^2 \big((1+q)/2\big)^{2h-2}.\end{aligned}$$

For the proof of Proposition 8.7 we shall need the following three lemmas, corollary and proposition.

*Lemma 8.8:* We have

$$\mathrm{Var}(T) = \mathrm{Var}_{1,1}(T) + \mathrm{Var}_{1,2}(T) + \mathrm{Var}_{2,1}(T) + \mathrm{Var}_{2,2}(T),$$

where

$$\begin{aligned}\mathrm{Var}_{1,1}(T) &= \sum_{a=a', b=b'} \mathrm{Covar}(\tau, \tau'), \\ \mathrm{Var}_{1,2}(T) &= \sum_{a=a', b\ne b'} \mathrm{Covar}(\tau, \tau'), \\ \mathrm{Var}_{2,1}(T) &= \sum_{a\ne a', b=b'} \mathrm{Covar}(\tau, \tau'), \\ \mathrm{Var}_{2,2}(T) &= \sum_{a\ne a', b\ne b'} \mathrm{Covar}(\tau, \tau'),\end{aligned}$$

and

$$\mathrm{Covar}(\tau, \tau') = \Pr(\tau \text{ good}, \tau' \text{ good}) - \Pr(\tau \text{ good}) \Pr(\tau' \text{ good}).$$

*Proof:* Expanding the square in the definition $\mathrm{Var}(T) = \mathrm{Ex}\big(\big(T - \mathrm{Ex}(T)\big)^2\big)$, we have

$$\mathrm{Var}(T) = \sum_{\tau, \tau'} \mathrm{Covar}(\tau, \tau'),$$

where the sum is over all ordered pairs of paths $\tau, \tau'$ (identical or distinct) from links $a, a' \in A$ to links $b, b' \in B$. The lemma follows by breaking the sum into four parts according to whether the starting links $a$ and $a'$, and the ending links $b$ and $b'$, are identical or distinct. $\triangle$

*Lemma 8.9:* We have

$$\mathrm{Var}_{1,1}(T) = Z^2 q\big((1+q)/2\big)^{2h-1}.$$

*Proof:* If $a = a'$ and $b = b'$, then by the banyan property $\tau = \tau'$. Thus we have

$$\mathrm{Covar}(\tau, \tau') \leq \Pr(\tau \text{ good}, \tau' \text{ good}) = \Pr(\tau \text{ good}).$$

Summing over $a \in A$ and $b \in B$, we obtain $\mathrm{Var}_{1,1}(T) \leq \mathrm{Ex}(T)$. The lemma then follows from Lemma 8.6. $\triangle$

*Lemma 8.10:* We have

$$\mathrm{Var}_{1,2}(T) = Z^3 q^2 \big((1+q)/2\big)^{3h}.$$

*Proof:* Suppose that $\tau$ and $\tau'$ both begin at $a$, but end at $b$ and $b' \neq b$, respectively. Let $a, u_1, \ldots, u_{2h}, b$ be the successive links of $\tau$ and let $a, u'_1, \ldots, u'_{2h}, b'$ be the successive links of $\tau'$. If we consider a link $u^*$ in the $(2h)$-th rank, then there is a unique path from $u^*$ to the output $w$, since the last $k = 2h+1$ stages of the network form a banyan. If we were to have $u_h = u'_h = u^*$, there would be paths from $u^*$ through the distinct links $b$ and $b'$ to the output $w$, contradicting the banyan property. It follows that, for some $0 \leq i \leq h-1$, $\tau$ and $\tau'$ have their first $i$ links after $a$ in common, $u_1 = u'_1, \ldots, u_j = u'_i$, and their last $2h - i$ links before $b$ and $b'$ distinct, $u_{i+1} \neq u'_{i+1}, \ldots, u_{2h} \neq u'_{2h}$. If $i = 0$, then $\tau$ and $\tau'$ are disjoint (except for $a$), the events "$\tau$ good" and "$\tau'$ good" are independent, and $\mathrm{Covar}(\tau, \tau')$ vanishes. Suppose then that $i \geq 1$. Then a route can intersect the union of $\tau$ and $\tau'$ only in a segment of consecutive links in either $\tau$ or $\tau'$. Thus the events "$u_1$ idle", $\ldots$, "$u_i$ idle", "$u_{i+1}$ idle and $u'_{i+1}$ idle", $\ldots$, "$u_{2h}$ idle and $u'_{2h}$ idle" form a Markov chain. Furthermore, for $i+2 \leq j \leq 2h$, we have

$$\Pr(u_j \text{ idle}, u'_j \text{ idle} \mid u_{j-1} \text{ idle}, u'_{j-1} \text{ idle}) = \Pr(u_j \text{ idle} \mid u_{j-1} \text{ idle}) \Pr(u'_j \text{ idle} \mid u'_{j-1} \text{ idle}).$$

Thus

$$\begin{aligned}
\Pr(\tau \text{ good}, \tau' \text{ good}) = \Pr(u_1 \text{ idle}) &\prod_{2 \leq j \leq i} \Pr(u_j \text{ idle} \mid u_{j-1} \text{ idle}) \\
&\Pr(u_{i+1} \text{ idle}, u'_{i+1} \text{ idle} \mid u_i \text{ idle}) \\
&\prod_{i+2 \leq j \leq 2h} \Pr(u_j \text{ idle} \mid u_{j-1} \text{ idle}) \Pr(u'_j \text{ idle} \mid u'_{j-1} \text{ idle}).
\end{aligned}$$

We have $\Pr(u_1 \text{ idle}) = q$ and $\Pr(u_{i+1} \text{ idle}, u'_{i+1} \text{ idle} \mid u_i \text{ idle}) = q$, since the one of the two routes through $u_{i+1}$ and $u'_{i+1}$ is the one through $u_i$ (which is idle) and the other is a new one (which is idle with probability $q$). The remaining $(i-1)+2(2h-i) = 4h-i-1$ factors each equal $\left((1+q)/2\right)$. Since $i \leq h-1$, we have

$$\begin{aligned} \mathrm{Covar}(\tau, \tau') &\leq \Pr(\tau \text{ good}, \tau' \text{ good}) \\ &\leq q^2 \left((1+q)/2\right)^{3h}. \end{aligned}$$

Since there are $Z$ ways to choose $a \in A$ and $Z(Z-1) \leq Z^3$ ways to choose $b, b' \in B$ with $b \neq b'$, the lemma follows. $\Delta$

*Corollary 8.11:* We have

$$\mathrm{Var}_{2,1}(T) = Z^3 q^2 \left((1+q)/2\right)^{3h}.$$

*Proof:* The proof is the mirror image of that of Lemma 8.10. $\Delta$

*Proposition 8.12:* We have

$$\mathrm{Var}_{2,2}(T) = Z^2 h^2 q^2 \left((1+q)/2\right)^{2h-2}.$$

For the proof of Proposition 8.12 we shall need the following proposition.

Consider a path $\tau$ from $a$ to $b$, and a path $\tau'$ from $a' \neq a$ to $b' \neq b$. By the banyan property of the central zone, there exist a unique path $\sigma$ from $a$ to $b'$ and a unique path $\sigma'$ from $a'$ to $b$. Let $\varrho$ denote that portion of $\sigma$ that is disjoint from $\tau$ and $\tau'$, and let $\varrho'$ denote that portion of $\sigma'$ that is disjoint from $\tau$ and $\tau'$. The number of crossbars on $\varrho$ (including the last one on $\tau$ and the first one on $\tau'$) will be called the "distance" between $\tau$ and $\tau'$, and will be denoted $\Delta(\tau, \tau')$. The number of crossbars on $\varrho'$ is also $\Delta(\tau, \tau')$.

*Proposition 8.13:* We have

$$\sum_{\substack{\tau, \tau' \\ a \neq a', b \neq b'}} 2^{-\Delta(\tau, \tau')} \leq Z^2 h^2 / 8.$$

For the proof of Proposition 8.13 we shall need the following lemma and corollary.

The paths $\tau$ and $\sigma$ diverge at a crossbar $I$ within a block $C$ of the central zone having $a$ as an input. The number of crossbars on $\varrho$ contained within $C$ (including the one at which $\tau$ and $\sigma$ diverge) will be called the "distance" between $b$ and $b'$, and will be denoted $\Delta(b, b')$. (By the symmetry of the network, this number depends only on $b$ and $b'$, and not on $a$.)

*Lemma 8.14:* We have

$$\sum_{\substack{b,b' \in B \\ b \neq b'}} 2^{-\Delta(b,b')} \leq Zh/2.$$

*Proof:* Let us fix a path $\tau$ from link $a \in A$ to link $b \in B$, and vary the path $\sigma$ from link $a$ to $b' \in B$, where $b' \neq b$. These paths form a tree within the block $C$. In this tree, the path to $b$ can have at most one sibling $b'$ with $\Delta(b,b') = 1$, two with $\Delta(b,b') = 2$, and so forth through $2^{h-1}$ with $\Delta(b,b') = h$. Thus for any $b \in B$ we have

$$\sum_{b' \in B, b' \neq b} 2^{-\Delta(b,b')} \leq h/2.$$

Since there are $Z$ choices for $b \in B$, the lemma follows. $\triangle$

The paths $\tau$ and $\sigma'$ converge at a crossbar $J$ within a block $D$ of the central zone having $b$ as an output. The number of crossbars on $\varrho'$ contained within $D$ (including the one at which $\tau$ and $\sigma'$ converge) will be called the "distance" between $a$ and $a'$, and will be denoted $\Delta(a,a')$. (By the symmetry of the network, this number depends only on $a$ and $a'$, and not on $b$.)

*Corollary 8.15:* We have

$$\sum_{\substack{a,a' \in A \\ a \neq a'}} 2^{-\Delta(a,a')} \leq Zh/2.$$

*Proof:* The proof is the mirror image of that of Lemma 8.14. $\triangle$

*Proof of Proposition 8.13:* The $\Delta(\tau,\tau')$ crossbars on $\varrho$ (including the last one on $\tau$ and the first one on $\tau'$) fall into three classes: there are $\Delta(b,b')$ in the block $C$ of central zone containing $a$ as an input, there is one in the middle stage, and there are $\Delta(a,a')$ in the block $D'$ of the central zone containing $b'$ as an output. Thus we have

$$\Delta(\tau,\tau') = 1 + \Delta(a,a') + \Delta(b,b').$$

This identity allows us to factor the sum over $\tau$ and $\tau'$:

$$\sum_{\substack{\tau,\tau' \\ a \neq a', b \neq b'}} 2^{-\Delta(\tau,\tau')} = \Bigg( \sum_{\substack{a,a' \in A \\ a \neq a'}} 2^{-\Delta(a,a')} \Bigg) \Bigg( \sum_{\substack{b,b' \in B \\ b = b'}} 2^{-\Delta(b,b')} \Bigg) / 2.$$

The proposition now follows from Lemma 8.14 and Corollary 8.15. $\triangle$

*Proof of Proposition 8.12:* We shall say that $\tau$ and $\tau'$ are "coupled" if a subroute includes either $\varrho$ or $\varrho'$. This event occurs only if either the $\Delta(\tau,\tau') - 2$ crossbars on $\varrho$ (excluding the last one on $\tau$ and the first one on $\tau'$) or the $\Delta(\tau,\tau')-2$ crossbars on $\varrho'$ (excluding the last one on $\tau'$ and the first one on $\tau$) all have appropriate orientations. Thus we have

$$\Pr(\tau,\tau' \text{ coupled}) \leq 8 \cdot 2^{-\Delta(\tau,\tau')}.$$

If $\tau$ and $\tau'$ are not coupled, the events "$\tau$ is good" and "$\tau'$ is good" are conditionally independent, so $\mathrm{Covar}(\tau,\tau' \mid \tau,\tau' \text{ coupled})$ vanishes. Thus we have

$$\mathrm{Covar}(\tau,\tau') \leq \Pr(\tau \text{ good}, \tau' \text{ good} \mid \tau,\tau' \text{ coupled}) \Pr(\tau,\tau' \text{ coupled}).$$

Let $u_1,\ldots,u_h$ be the successive links of the subpath $\tau^*$ comprising the first $h$ links of $\tau$, and let $u'_1,\ldots,u'_h$ be the successive links of the subpath $\tau'^*$ comprising the first $h$ links of $\tau'$. Even if $\tau$ and $\tau'$ are coupled, a route can intersect the union of $\tau^*$ and $\tau'^*$ only in a segment of consecutive links. Thus we have

$$\begin{aligned}
\Pr(\tau \text{ good}, \tau' \text{ good} \mid \tau,\tau' \text{ coupled}) \leq \\
&\Pr(u_1 \text{ idle} \mid \tau,\tau' \text{ coupled}) \prod_{2\leq j\leq h} \Pr(u_j \text{ idle} \mid u_{j-1} \text{ idle}, \tau,\tau' \text{ coupled}) \\
&\Pr(u'_1 \text{ idle} \mid \tau,\tau' \text{ coupled}) \prod_{2\leq j\leq h} \Pr(u'_j \text{ idle} \mid u'_{j-1} \text{ idle}, \tau,\tau' \text{ coupled}).
\end{aligned}$$

Again we have $\Pr(u_1 \text{ idle} \mid \tau,\tau' \text{ coupled}) = \Pr(u'_1 \text{ idle} \mid \tau,\tau' \text{ coupled}) = q$, and the remaining $2h-2$ factors each equal $\big((1+q)/2\big)$. Thus we have

$$\mathrm{Covar}(\tau,\tau') \leq q^2\big((1+q)/2\big)^{2h-2}\, 8 \cdot 2^{-\Delta(\tau,\tau')}.$$

The proposition now follows by summing over $\tau$ and $\tau'$, with $a \neq a'$ and $b \neq b'$, and applying Proposition 8.13. $\triangle$

*Proof of Proposition 8.7:* The proof is immediate from Lemmas 8.8, 8.9 and 8.10, Corollary 8.11 and Proposition 8.12. $\triangle$

*Proof of Proposition 8.5:* We have (Chebyshev's inequality)

$$\Pr(A, B \text{ severed}) = \Pr(T = 0) \leq \mathrm{Var}(T)/\mathrm{Ex}(T)^2.$$

The proposition thus follows from Lemma 8.6 and Proposition 8.7. $\triangle$

*Proof of Proposition 8.1:* From Proposition 8.2 and Corollary 8.4 we have

$$\Pr(v \text{ ample}, w \text{ ample}) \to q^2$$

as $k \to \infty$, since the restriction of the extended to the initial zone is independent of the restriction to the final zone. If $v$ is ample, we can let $A$ be a set of $Z$ bi-accessible left interface links, and if $w$ is ample, we can let $B$ be a set of bi-accessible right interface links. We thus have

$$\Pr(v, w \text{ bi-linked}) \geq \Pr(v \text{ ample}, w \text{ ample}) - \max_{A,B} \Pr(A, B \text{ severed})$$

(where the maximum is over all sets $A$ of left interface links and sets $B$ of right interface links with $\#A = \#B = Z$), since the restriction of the extended to the central zone is independent of the restriction to the initial and final zones. The proposition now follows from Proposition 8.5. $\triangle$

## 9. Reconciliation

Our goal in this section will be to complete the proof of Proposition 7.2. To do this we shall need Proposition 9.1, Corollary 9.3, Lemmas 9.4 and 9.5, and Corollary 9.6 below.

We shall say that $v$ is "excessive" if there are more than $Y = y^h$ co-accessible left interface links; otherwise we shall say that $v$ is "moderate".

*Proposition 9.1:* We have

$$\Pr(v \text{ excessive}) \to 0$$

as $k \to \infty$.

For the proof of Proposition 9.1 we shall need the following lemma.

*Lemma 9.2:* The generating function for the number of co-accessible left interface links is $g^{(h)}(\eta)$, the $h$-th iterate of $g(\eta) = \eta(a + b\eta)$.

*Proof:* The proof is analogous to that of Lemma 8.3, with "co-" concepts replacing "bi-" concepts. $\triangle$

*Proof of Proposition 9.1:* Let the random variable $T$ denote the number of co-accessible left interface links. By Lemma 9.2 $g^{(h)}(\eta)$ is the generating function for $T$. For any $\eta > 1$ we have (Bernstein's inequality)

$$\Pr(v \text{ excessive}) = \Pr(T > Y) \leq g^{(h)}(\eta)/\eta^Y.$$

The function $g(\eta) = \eta(a + b\eta)$ has a repulsive fixed point at $\eta = 1$ with deriviative $g'(1) = 1 + b$. Since $1 + b < y$, we can choose $x$ such that $1 + b < x < y$. We shall take $\eta = 1 + 1/x^h$. Since $x > 1 + b$, the iterate $g^{(h)}(1 + 1/x^h)$ remains bounded (indeed, tends to unity) as $h \to \infty$. On the other hand, since $x < y$, $1/(1 + 1/x^h)^Y$ tends to zero in this limit. Thus

$$g^{(h)}(\eta)/\eta^Y \to 0$$

for this choice of $\eta > 1$, and the proposition follows. $\triangle$

We shall say that $w$ is "excessive" if there are more than $Y = y^h$ co-accessible right interface links; otherwise we shall say that $w$ is "moderate".

*Corollary 9.3:* We have

$$\Pr(w \text{ excessive}) \to 0$$

as $k \to \infty$.

*Proof:* The proof is the mirror image of that of Proposition 9.1. $\triangle$

If $K$ is a set of left interface links, and $L$ is a set of right interface links, we shall say that $K$ and $L$ "conflict" if there exists a central subroute from some link in $K$ to some link in $L$.

*Lemma 9.4:* If $\#K \leq Y$ and $\#L \leq Y$, we have

$$\Pr(K, L \text{ conflict}) \to 0$$

as $k \to \infty$.

*Proof:* By the banyan property of the central zone, there is a unique path $\tau$ between any link $a \in K$ and any link $b \in L$. There is a central subroute from $a$ to $b$ only if the $2h+1$ crossbars on $\tau$ all have appropriate orientations, which occurs with probability $1/2^{2h+1}$. Since there are at most $y$ ways to choose $a \in K$ and at most $Y$ ways to choose $b \in L$, we have

$$\Pr(K, L \text{ conflict}) \leq Y^2/2^{2h+1} = (y/2)^{2h}/2.$$

Since $y < 2$, the lemma follows. $\triangle$

For the purposes of the following lemma and corollary, we shall break each central subroute into two sub-subroutes: a "left central" sub-subroute and a "right central" sub-subroute. (The break occurs at the middle stage.) If $M$ is a set of left interface links and $D$ is a right central block, we shall say that $M$ and $D$ "clash" if there is a left central sub-subroute from some link in $M$ to a middle crossbar that shares a link with $D$.

*Lemma 9.5:* If $\#M \leq Y$, we have

$$\Pr(M, D \text{ clash}) \to 0$$

as $k \to \infty$.

*Proof:* Let $a$ be a link in $M$ and let $C$ be the left central block having $a$ as an inlink. By the banyan property in the central zone, there is unique outlink $b$ of $C$ through which any path from $a$ into $D$ must pass, and there is a unique path $\varrho$ from $a$ to $b$ in $C$. There is a left central sub-subroute from $a$ to $b$ only if the $h$ crossbars on $\varrho$ all have appropriate orientations, and this occurs with probability $1/2^h$. Since there are at most $Y$ ways to choose $a \in M$, we have

$$\Pr(M, D \text{ clash}) \leq Y/2^h = (y/2)^h.$$

Since $y < 2$, the lemma follows. $\triangle$

If $N$ is a set of right interface links and $C$ is a left central block, we shall say that $N$ and $C$ "clash" if there is a right central sub-subroute from a middle crossbar that shares a link with $C$ to some link in $M$.

*Corollary 9.6:* If $\#N \leq Y$, we have

$$\Pr(N, C \text{ clash}) \to 0$$

as $k \to \infty$.

*Proof:* The proof is the mirror image of that of Lemma 9.5. $\triangle$

*Proof of Proposition 7.2:* Suppose that we are given an extended state and we wish to find an idle path from $v$ to $w$. We shall do this in four steps as follows.

In the first step, we shall verify that $v$ and $w$ are both moderate. By Proposition 9.1 and Corollary 9.3, we shall succeed with probability approaching unity. We shall let $K$, with $\#K \leq Y$, be the set of co-accessible left interface links, and let $L$, with $\#L \leq Y$ be the set of co-accessible right interface links. In verifying these conditions and determining these sets, only the restriction of the extended state to the initial and final zones need be examined; indeed, only the orientations of crossbars and the co-dispositions of subroutes in these zones need be examined.

In the second step, we shall verify that $K$ and $L$ do not conflict. In verifying this condition, we need only examine the restriction of the extended state to the central zone, which is hitherto unexamined. Thus Lemma 9.4 applies to show that we shall again succeed with probability approaching unity.

Furthermore, if we now pretend that we have not examined the restriction of the extended state to the central zone, the conditional probabilities of further events will be affected only by factors approaching unity, since the condition we have just verified had probability approaching unity. (It is worth observing that we cannot use the same argument to pretend that we have not examined the restriction of the extended state to the initial and final zones, since we observed the values of of the sets $K$ and $M$, and these values are not assumed with probabilty approaching unity.)

In the third step, we shall verify that $v$ and $w$ are bi-linked, with the idle path $\pi$ comprising the bi-idle initial subpath $\alpha$, the idle central subpath $\tau$, and the bi-idle final subpath $\beta$. Since we have already examined the crossbar orientations and subroute co-dispositions in the initial and final zones, we now need only examine the subroute dispositions in the initial and final zones (intersecting the sets of accessible interface links with the sets $K$ and $L$ to obtain sets $A$ and $B$ of bi-accessible interface links, with $\#A = \#B = Z$), and the extended state restricted to those central blocks that have a bi-accessible interface link, and those middle crossbars that share a link with such blocks. By Proposition 8.1, we shall succeed with probabilty approaching $q^2$.

We shall say that an interface link is "uni-accessible" if it is co-accessible but not bi-accessible. At this point, we have still not examined those central blocks that have a uni-accessible interface link.

In the fourth step, we shall verify that the set $M \subseteq K$ of uni-accessible left interface links does not clash with the right central block $D$ through which $\tau$ passes, and that set $N \subseteq L$ of uni-accessible right interface links does not clash with the left central block $C$ through which $\tau$ passes. In verifying this condition, we need only examine the crossbar orientations in central blocks that have a uni-accessible link, which are hitherto unexamined. Thus Lemma 9.5 and Corollary 9.6 apply to show that we shall again succeed with probability approaching unity.

If we complete these four steps, which we do with probability approaching $q^2$, we have a path $\pi$ establishing that $v$ and $w$ are bi-linked in the extended state. We claim that $\pi$ also established that $v$ and $w$ are linked in the coherent state obtained by reconciliation.

First, the subpath $\alpha$ must remain idle after reconciliation. Indeed, the initial subroutes through its vertices are bi-rejected, and therefore terminate at bi-accessible left interface links. Routes through such links take their dispositions from their initial subroutes, so these routes are rejected, and the vertices of $\alpha$ remain idle.

Second, the subpath $\beta$ must remain idle after reconciliation. Indeed, the final subroutes through its vertices are bi-rejected, and therefore originate at bi-accessible right interface links. Routes through such links take their dispositions from their final subroutes, unless they also pass through a bi-accessible left interface link, which they cannot do since $K$ and $L$ do not conflict. Thus these routes are rejected, and the vertices of $\beta$ remain idle.

Third, the subpath $\tau$ must remain idle after reconciliation. Indeed, central subroutes through these links are rejected. Routes take their disposition from their central subroute unless they pass through a co-accessible interface link. Thus a route including rejected central subroute can be accepted only if it passes through a uni-accessible interface link, which it cannot since $M$ does not clash with the right central block $D$ through which $\tau$ passes and $N$ does not clash with the left central block $C$ through which $\tau$ passes. Thus these routes are rejected, and the links of $\tau$ remain idle.

Thus, if we complete the four steps described above, which we do with probability approaching $q^2$, $v$ and $w$ are linked. $\triangle$

## 10. Conclusion

There is a striking agreement between the analytic result of this paper and the empirical observations of Neiman and Vvedenskaya [NV]. It should be kept in mind, however, that the assumptions, as well as the methodologies, of these two studies differ. We assume a probability distribution on the states; their assumption concern the traffic offered by the subscribers and policy by which the network is operated. It would be interesting, though presumably also difficult, to reformulate and reprove the result of this paper in a setting closer to those explored by simulations.

## 11. References


[BNV] L. A. Bassalygo, V. I. Neiman and N. D. Vvedenskaya, "Blocking Probability of Some Binary Connecting Networks", *Systemy Raspredeleniya Inforatsiĭ*, 79–93, Nauka, Moscow, 1972.

[B] V. E. Beneš, "Optimal Rearrangeable Multistage Connecting Networks", *Bell Sys. Tech. J.*, 43 (1964) 1641–1656.

[CH] F. R. K. Chung and F. K. Hwang, "The Connection Patterns of Two Complete Binary Trees", *SIAM J. Algeb. and Discr. Meth.*, 1 (1980) 322–335.

[D] W. Doeblin, "Exposé de la Théorie de Chaines Simple Constantes de Markov à un Nombre Fini d'Etats", *Rev. Math. Union Interbalkanique*, 2 (1938) 77–105.

[I] N. Ikeno, "A Limit on Crosspoint Number", *IRE Trans. on Info. Theory (Spec. Suppl.)*, 5 (1959) 187–196.

[K] I. V. Koverninskiĭ, "Estimation of the Blocking Probability for Switching Circuits by Means of Probability Graphs", *Problems of Info. Transm.*, 11 (1975) 63–71.

[L] C. Y. Lee, "Analysis of Switching Networks", *Bell Sys. Tech. J.*, 34 (1955) 1287–1315.

[L1] P. Le Gall, "Étude du Blocage dans les Systèmes de Commutation Téléphonique Automatique Utilisant des Commutateurs Électroniques du Type Crossbar", *Ann. des Télécomm.*, 11 (1956) 159–171, 180–194, 197.

[L2] P. Le Gall, "Méthode de Calcul de l'Encombrement dans les Systèmes Téléphoniques Automatiques a Marquage", *Ann. des Télécomm.*, 12 (1957) 374–386.

[N] V. I. Neiman, "Some Binary Connecting Networks and Their Traffic Characteristics", *Proc. Internat. Teletraffic Congr.*, 7 (1973) 314/1–10.

[NV] V. I. Neiman and N. D. Vvedenskaya, "A Study of One Class of Binary Connecting Networks", *Brooklyn Polytechnic Inst. Symp. on Computer-Communiations Networks and Teletraffic*, (1972) 215–225.

[P1] N. Pippenger, "On Crossbar Switching Networks", *IEEE Trans. on Communications*, 23 (1975) 646–659.

[P2] N. Pippenger, "The Asymptotic Optimality of Spider-Web Networks", *Discr. Appl. Math.*, to appear.

[RJS] S. V. Ramanan, H. F. Jordan and J. R. Sauer, "A New Time-Domain, Multistage Permutation Algorithm", *IEEE Trans. on Info. Theory*, 36 (1990) 171–173.

[T] K. Takagi, "Design of Multi-Stage Link Systems by Means of Optimal Channel Graphs", *Electr. and Comm. in Japan*, 51–A (1968) 37–46.

2.1 A $2 \times 2$ crossbar has seven "modes": one (a) has no established connections, four (b–e) have one connection, and two (f–g) have two connections. The last two are the "configurations": "straight" (f) and "crossed" (g).

2.2 The "shuffle" connection pattern ($k = 3$).

2.3 The "inverse shuffle" connection pattern ($k = 3$).

2.4 A spider-web network with $2k - 1$ stages can be partitioned into two 'left sectors", which are banyan networks with $k - 1$ stages, $2^{k-1}$ crossbars in the "middle" stage, and two "right sectors", which are banyan networks with $k - 1$ stages.

2.5 A banyan network with $2h$ stages can be partitioned into $2^h$ "left blocks" and $2^h$ "right blocks", all of which are banyan networks with $h$ stages.

4.1 The limiting value of the blocking probability (plotted against occupancy probability) for series-parallel networks in the independent model. Below the threshold $p = (2 - \sqrt{2})/2$, the blocking probability is $P = (2p - p^2)^2/(1 - p)^4$. Above the threshold it is unity.

4.2 The limiting value of the blocking probability (plotted against occupancy probability) for spider-web networks in the independent model. Below the threshold $p = (2 - \sqrt{2})/2$, the blocking probability is $P = (2p^2(1-p)^2 - p^4)/(1-p)^4$, reaching the value $P = 8\sqrt{2} - 11$ at the threshold. Above the threshold it is unity.

4.3 The limiting value of the blocking probability (plotted against occupancy probability) for series-parallel networks in the coherent model. Below the threshold $p = 2 - \sqrt{2}$, the blocking probability is $P = p^2/2(1-p)^2$. Above the threshold it is unity.

4.4 The limiting value of the blocking probability (plotted against occupancy probability) for spider-web networks in the coherent model. Below the threshold $p = 2 - \sqrt{2}$, the blocking probability is zero. Above the threshold it is unity.

FIGURE CAPTIONS

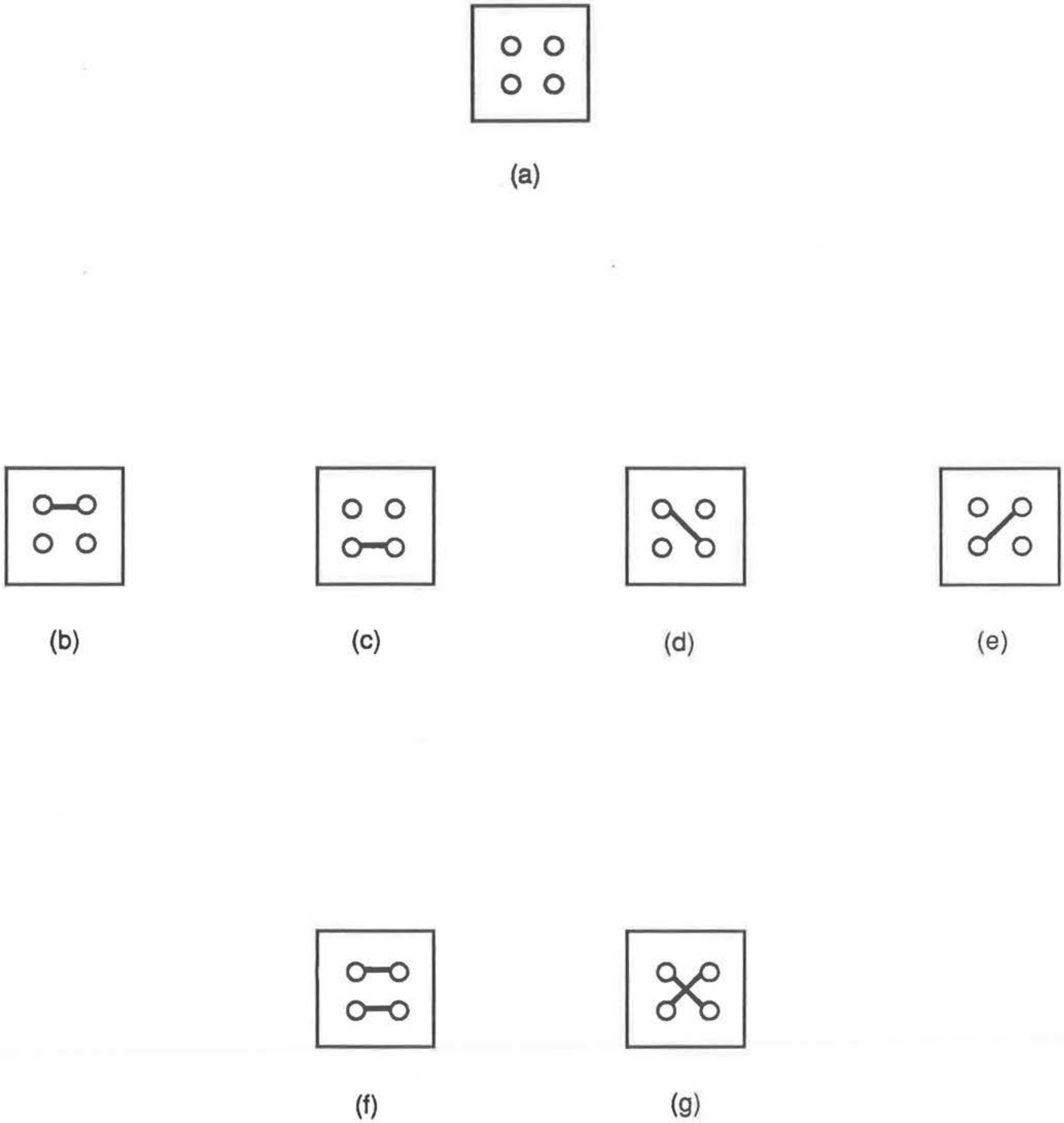


Figure 2.1

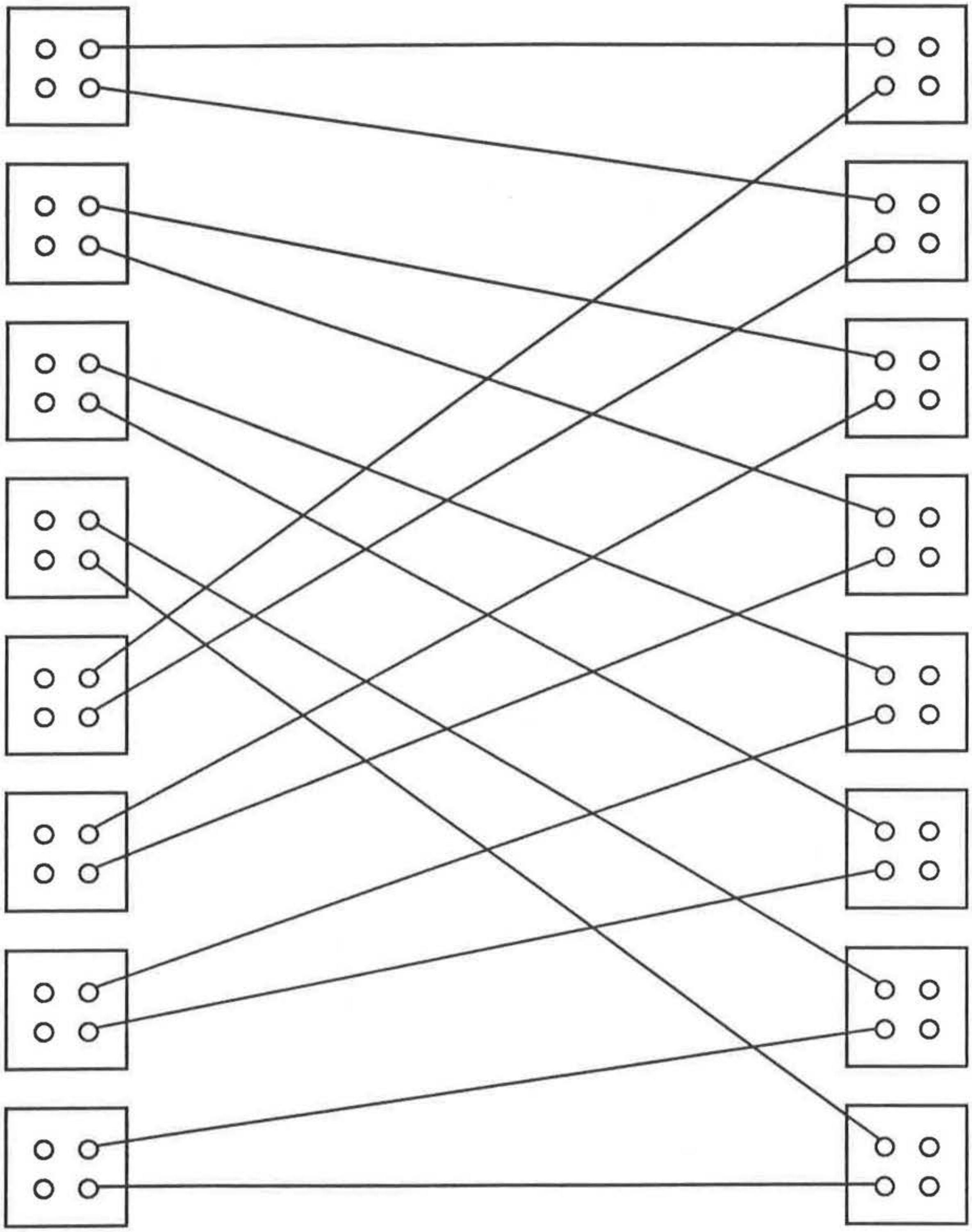

Figure 2.2

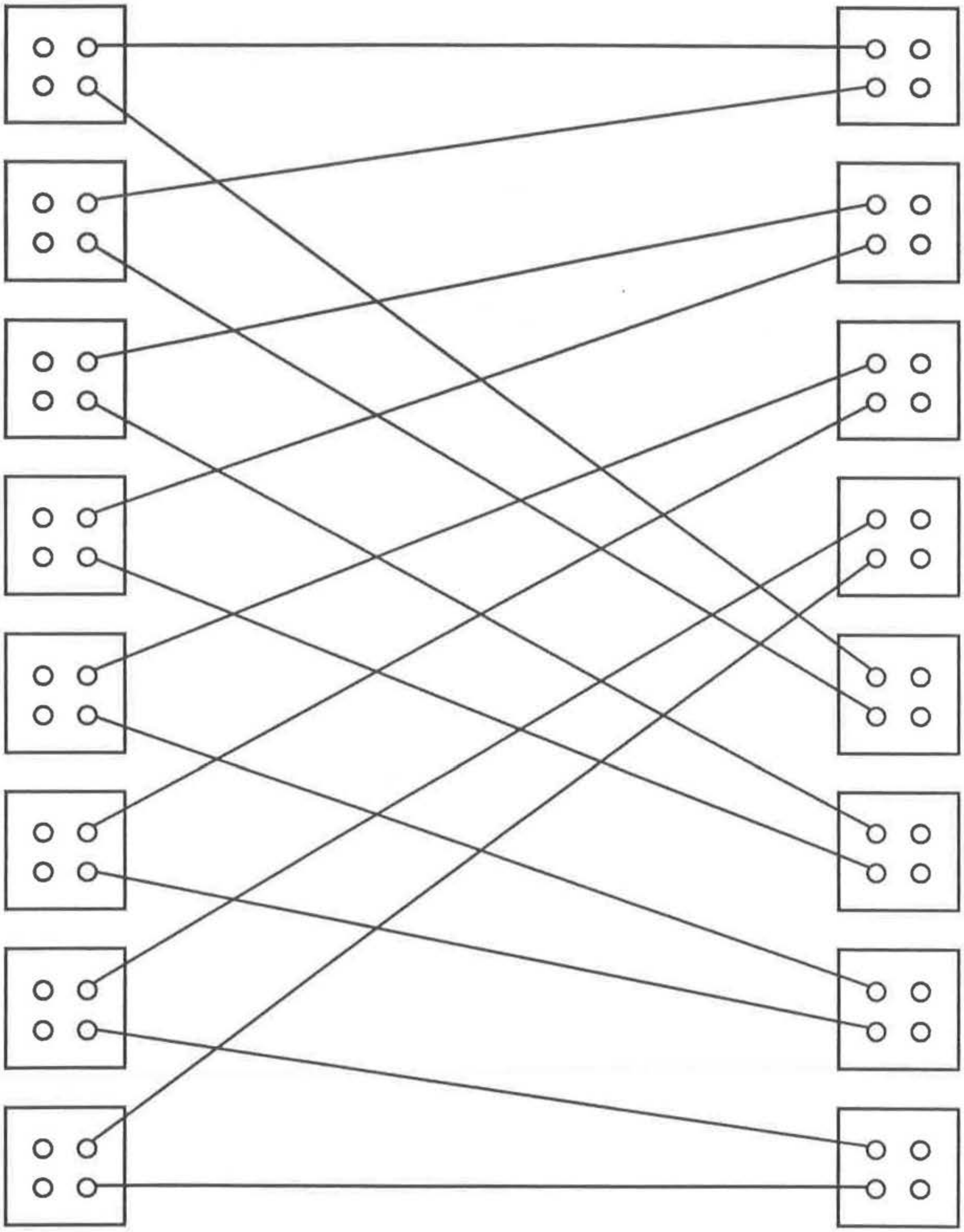

Figure 2.3

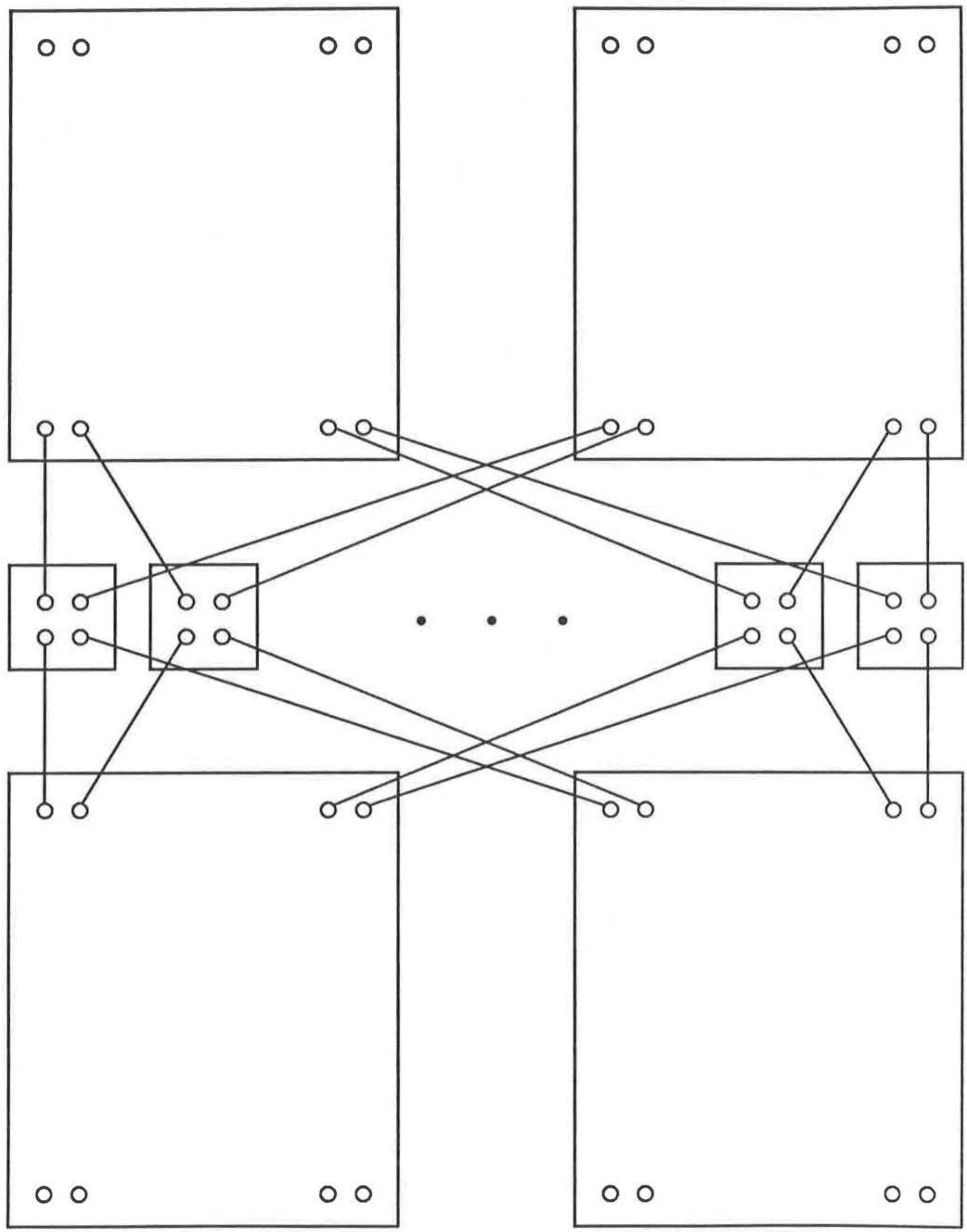

Figure 2.4 (rotated 90°)

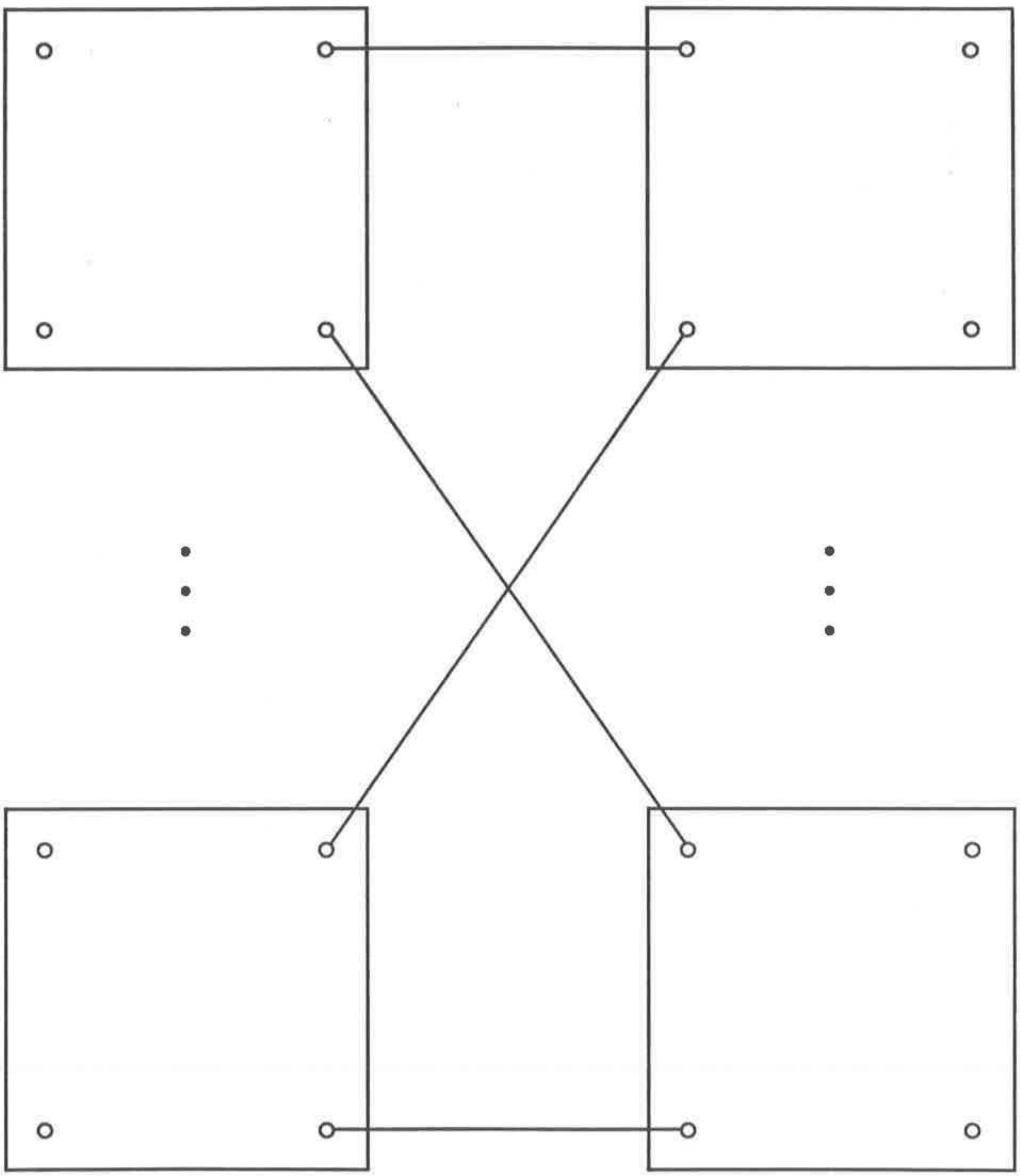

Figure 2.5

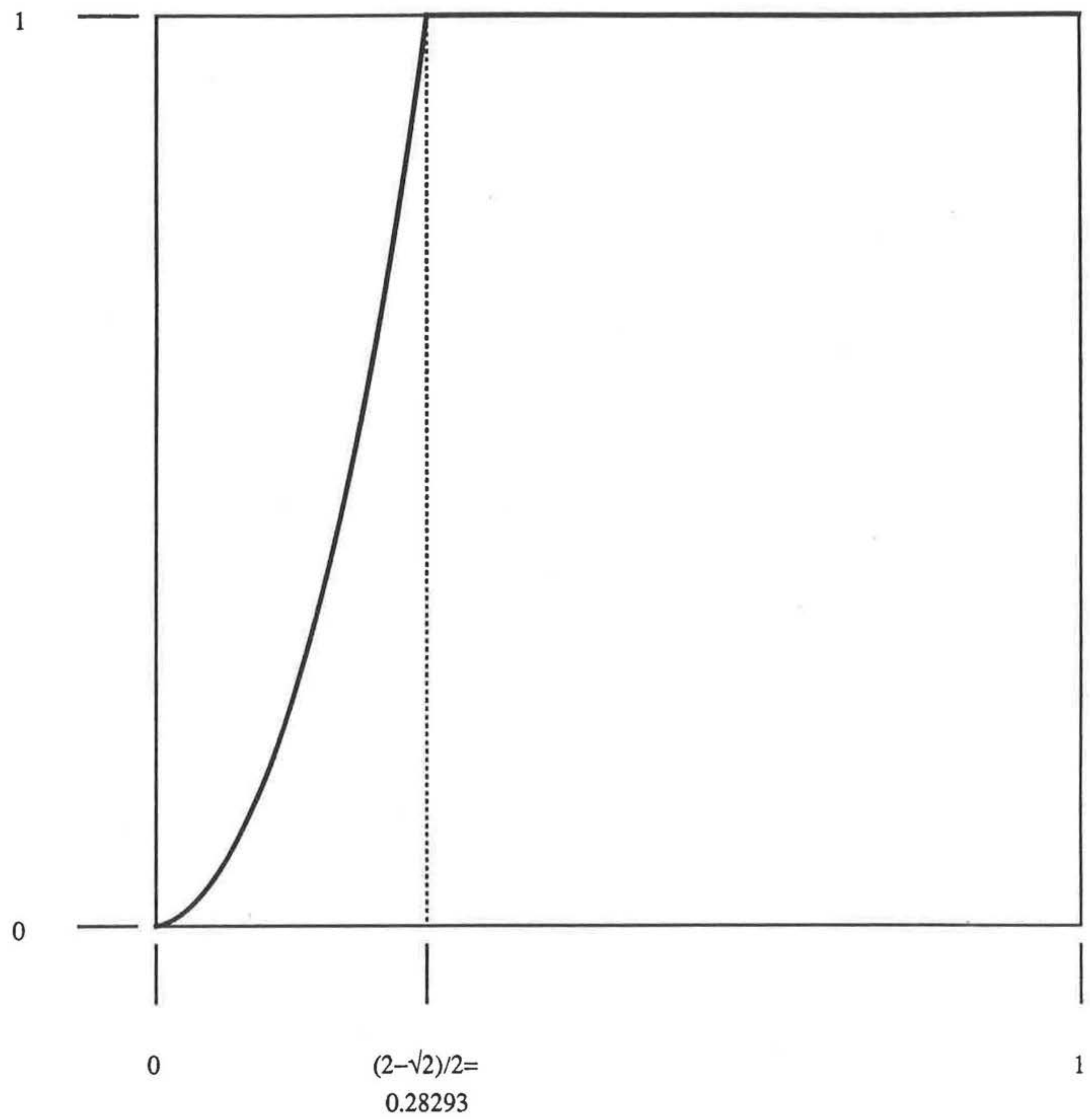


Figure 4.1

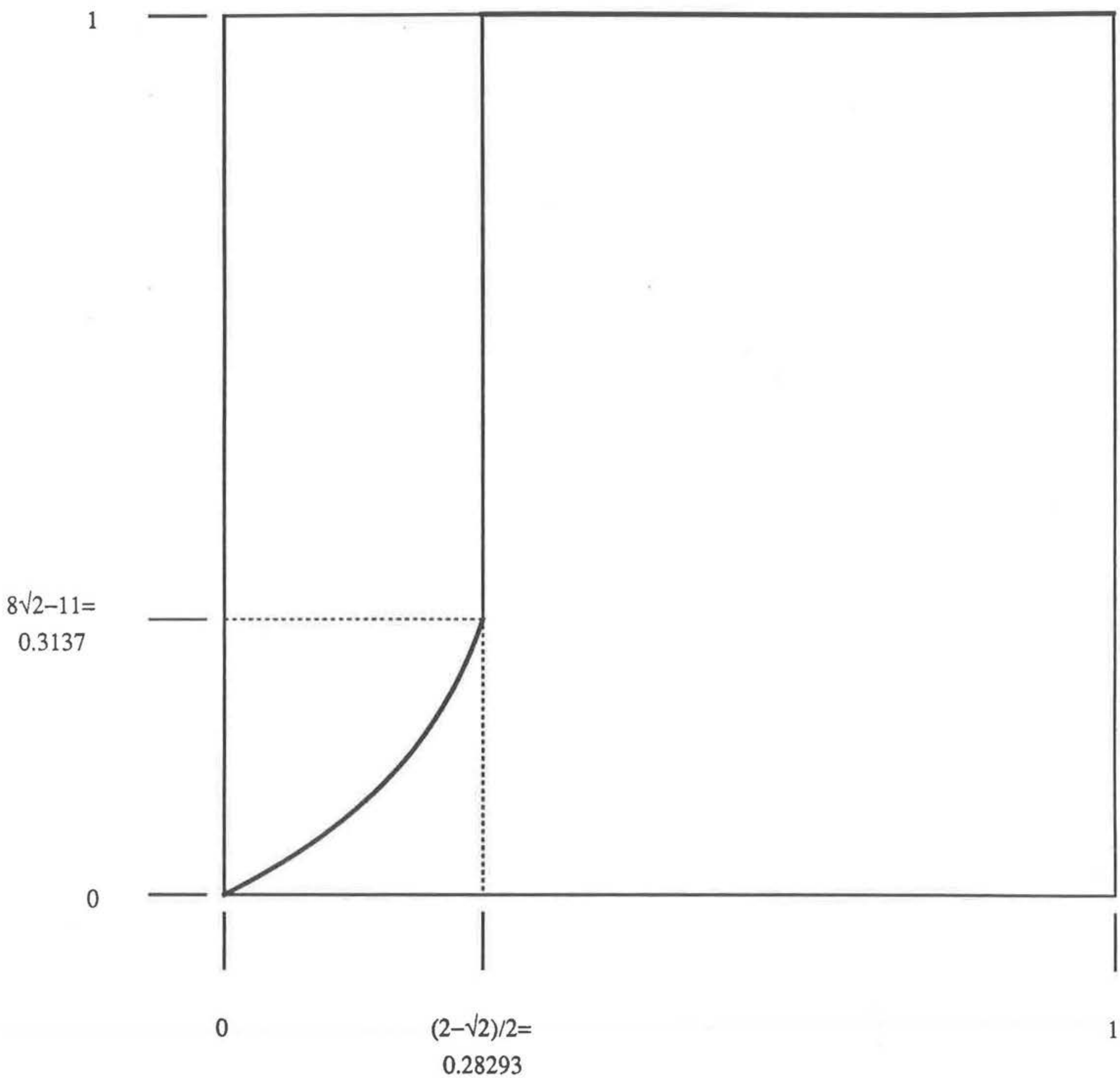


Figure 4.2

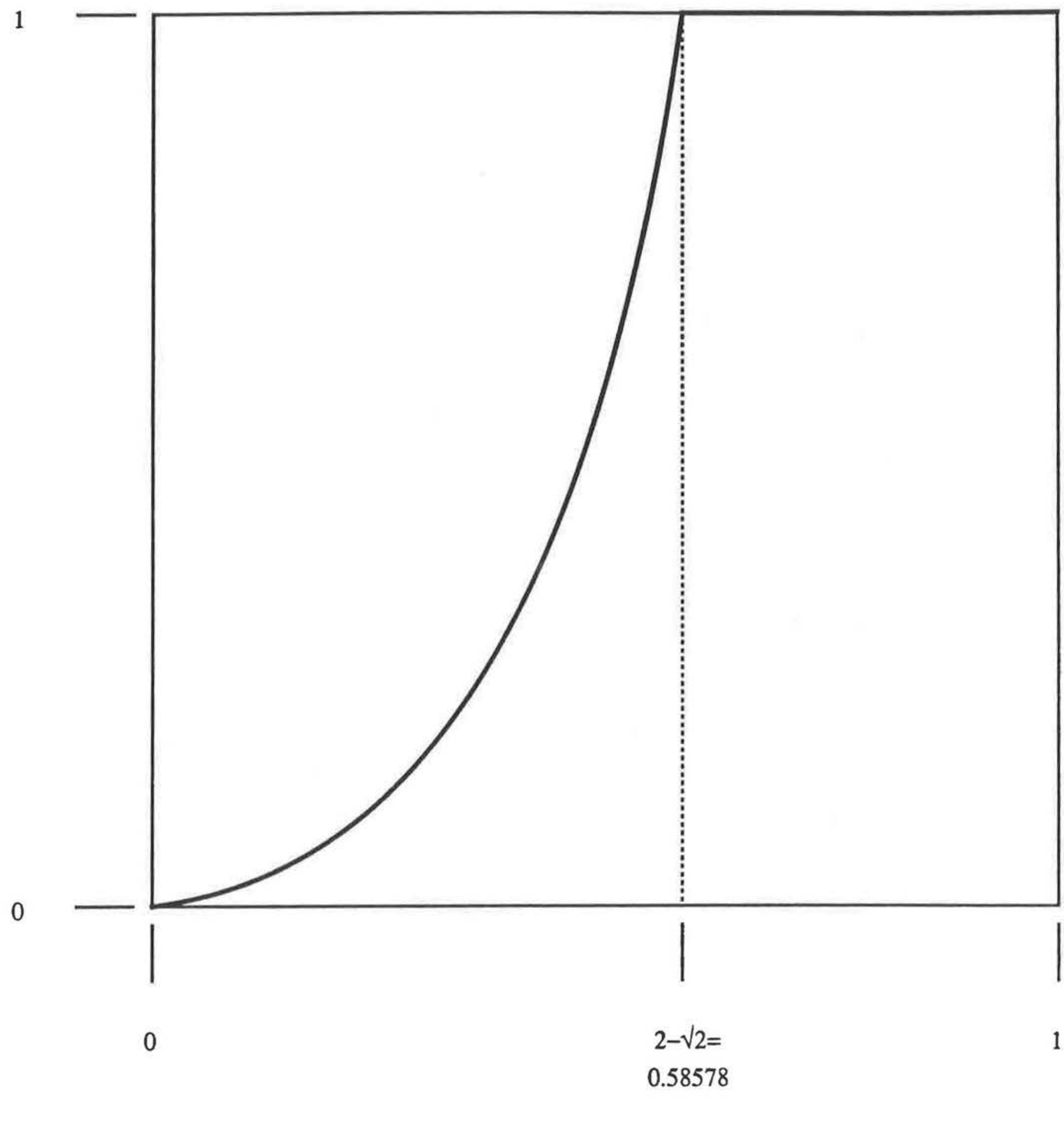


Figure 4.3

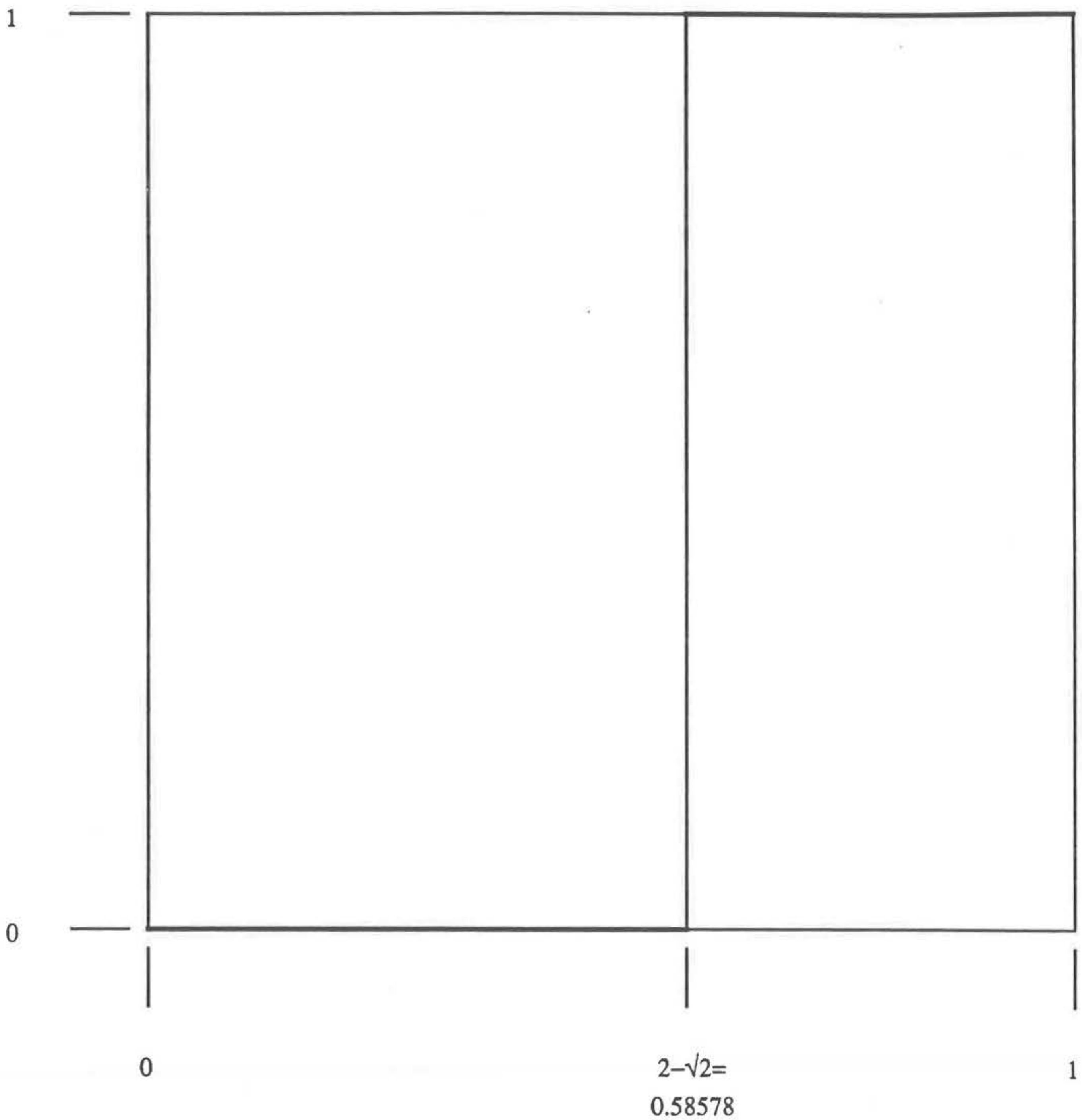


Figure 4.4