e-mail addresses: <gilmore@cs.ubc.ca>, <tsiknis@cs.ubc.ca>

A LOGIC for CATEGORY THEORY

by

Paul C. Gilmore* and George K. Tsiknis

Technical Report TR 90-2

May, 1990

ABSTRACT

Category theory provides an abstract and uniform treatment for many mathematical structures, and increasingly has found applications in computer science. Nevertheless, no suitable logic within which the theory can be developed has been provided. The classical set theories of Zermelo-Fraenkel and Gödel-Bernays, for example, are not suitable because of the use category theory makes of self-referencing abstractions, such as in the theorem that the set of categories forms a category. That a logic for the theory must be developed, Feferman has argued, follows from the use in the theory of concepts fundamental to logic, namely, propositional logic, quantification and the abstractions of set theory.

In this paper a demonstration that the logic and set theory NaDSet is suitable for category theory is provided. Specifically, a proof of the cited theorem of category theory is provided within NaDSet.

NaDSet succeeds as a logic for category theory because the resolution of the paradoxes provided for it is based on a reductionist semantics similar to the classical semantics of Tarski for first and second order logic. Self-membership and self-reference is not explicitly excluded. The reductionist semantics is most simply presented as a natural deduction logic. In this paper a sketch of the elementary and logical syntax or proof theory of the logic is described.

Formalizations for most of the fundamental concepts and constructs in category theory are presented. NaDSet definitions for natural transformations and functor categories are given and an equivalence relation on categories is defined. Additional definitions and discussions on products, comma categories, universals limits and adjoints are presented. They provide enough evidence to support the claim that any construct, not only in categories, but also in toposes, sheaves, triples and similar theories can be formalised within NaDSet.

---

*The author gratefully acknowledges financial support from the Natural Science and Engineering Research Council of Canada

1. INTRODUCTION

Section A1 of [Feferman84] reinforces the argument presented in [Feferman77] that category theory cannot by itself provide a foundation for mathematics since it makes use of prior notions of logic and set abstraction. At the same time the first paper provides motivation for constructing set theories other than the traditional Zermelo-Fraenkel and Gödel-Bernays set theories. An example of a common argument in modern algebra is presented using structures $<A, \otimes, =_A>$ consisting of a set A, a commutative and associative binary operation $\otimes$ and an identity relation $=_A$ over A. If B is the set of all such structures, PR is the Cartesian product on B and ISO isomorphism between the elements of B, then the structure $<B, PR, ISO>$ is itself a member of B. However, a proof of this fact cannot be formalized within the traditional set theories because of the prohibition against self-membership or self-reference.

In [Gilmore89] a natural deduction based set theory NaDSet was described and a proof that $<B, PR, ISO>$ is a member of B was provided within NaDSet. This encouraged the conjecture that NaDSet could provide a logic within which category theory could be formalized. This paper substantiates this conjecture by providing a proof within NaDSet that the set of all categories is itself a category.

Category theory, of course, involves many more primitive concepts than the theory of B-structures. Section 3 presents a definition of a category within NaDSet that is more general in two respects than the definition given in [Barr&Wells85] or in [Mac Lane71]. First, a category is defined in terms of its arrows only with no reference to objects, as suggested in [Lawvere66]. Secondly, the identity relation of a category is an explicit part of its structure. While the first simplification is not fundamental, the second generalization has important repercussions. It allows each category to assume its own identity relation that generally may be different than the extensional identity implied by the traditional definitions.

The definition of category theory in section 3 is typical for definitions of an axiomatic theory within NaDSet. The axioms of the theory are used only to define the set of structures satisfying the axioms, and in no way imply the existence of a structure satisfying the axioms. Therefore, the formalization of the theory within NaDSet has no existential implications for NaDSet. This fact may help to provide an answer to the question posed in [Blass84]: Does category theory necessarily involve existential principles that go beyond those of other mathematical disciplines? When a traditional set theory is used as a foundation for category theory, it is necessary to distinguish between small and large categories [Mac Lane71]. That is not necessary when category theory is formalized within NaDSet. Of course this does not provide an answer to the question: Does the proof of the existence of some categories involve existential principles that go beyond those of other mathematical disciplines?

In section 4 the notion of a functor on categories is formalized. In section 5, which constitutes the larger part of the paper, the necessary definitions for the category of categories and the detailed NaDSet proof that the category of categories is itself a category, is provided. This proof is of necessity greatly abbreviated, but nevertheless remains long and tedious. Since NaDSet is a logic novel to most readers, and since formal derivations are generally foreign to category theory, the paper possibly errs on the side of providing too much detail, rather too little. However, by examining only parts of the derivations provided, readers may gain confidence in the principal result and in the capability of NadSet to provide logical foundations for category theory.

The ubiquitous notions of natural transformations and functor categories are formalized in section 6, while in section 7, definitions and theorems for a variety of basic constructions including comma categories, universals, limits and adjoints are provided. These two sections further demonstrate that NaDSet may be used as the logic for category theory and suggest that any construct in category theory, as well as in the theories of toposes, sheaves, triples etc. can be formalized within NaDSet in a similar way.

In sections 2 and 3 a definition of NaDSet is given and motivations for its form are given that are illustrated in part by the proofs of later sections. The logic differs from a conventional presentation of set theory in four respects:

(1) To provide a transparent formalization of the traditional reductionist semantics of Tarski, NaDSet is formalized as a natural deduction based set theory. Since in a reductionist semantics the meaning of a complex sentence is reduced to that of simpler sentences, the meaning given to the irreducible atomic sentences is critical.

(2) A nominalist interpretation of atomic sentences is used: Only the name of a set, not the set itself, can be a member of another set. To avoid confusions of use and mention, it is necessary that NaDSet be a second order logic, but no higher order form of NaDSet is necessary or consistent.

(3) Although NaDSet is second order, both first and second order quantification is expressed by the same quantifier. It is only necessary that NaDSet have two distinct kinds of parameters (free variables) one first order and the other second order.

(4) A generalized set abstraction term {**ta**|**F**} is admitted in which **ta** may be a term, not just a single variable, and **F** may be any formula.

These points are elaborated upon in section 2.

2. NaDSet

In this section the logic is described, and its use motivated. The elementary syntax, that is the definitions relating to well-formed formulas and terms, is described in section 2.1, while the logical syntax, that is the definitions relating to well-formed proofs, is described in section 2.2. In section 2.3 some motivation for the choice of NaDSet as a logic is provided.

2.1.Elementary Syntax

The elementary syntax of this version of NaDSet differs from the second order form of NaDSet presented in [Gilmore86] in three respects:

(1) The conventional use of epsilon to denote membership predicate is replaced in this version by ':'. This use of ':' is similar to the use made of it in category theory and some programming languages.

(2) The elementary syntax requires only one kind of variable, rather than first and second order variables, although, as with the earlier version, both first and second order parameters are required. The latter are used as variables unbound by abstraction or quantification; that is, they are free variables. Providing distinct notations for free and bound variables removes from the logical syntax of the theory, the complications of substitution of terms with free variables. First and second order constants are also admitted. The forms chosen for these syntactic objects are unimportant; the only requirements are that there are denumerably many of each form, and objects of distinct forms are distinct.

(3) In the earlier version, a formula **ta:tb** is well formed only if **ta** is a first order term and **tb** is a second order. This implicitly restricts the abstraction introduction rules of the earlier version, since formulas introduced by the rules must satisfy the restriction. The removal of this restriction in the present version is necessary for the development of category theory.

Another way in which the extended theory differs from the earlier is in the existence of a consistency proof; although one was provided for the earlier version of NaDSet in [Gilmore86], none has yet been provided for the extended theory.

Definition of Elementary Syntax

1.1. A variable is a term. The single occurrence of the variable in the term is a free occurrence in the term.

1.2. Any parameter or constant is a term. No variable has a free occurrence in the term.

2.1. If **ta** and **tb** are any terms, then **ta:tb** is a formula. A free occurrence of a variable in **ta** or in **tb**, is a free occurrence of the variable in the formula.

2.2. If **F** and **G** are formulas then **(F↓G)** is a formula. A free occurrence of a variable in **F** or in **G** is a free occurrence in **(F↓G)**.

2.3. If **F** is a formula and **v** a variable, then ∀**vF** is a formula. A free occurrence of a variable other than **v** in **F**, is a free occurrence in ∀**vF**; no occurrence of **v** is free in ∀**vF**.

3. Let **ta** be any term in which there is at least one free occurrence of a variable and no occurrence of a parameter. Let **F** be any formula. Then {**ta**|**F**} is an abstraction term. A free occurrence of a variable in **F** which does not also have a free occurrence in **ta**, is a free occurrence in {**ta**|**F**}. A variable with a free occurrence in **ta** has no free occurrence in {**ta**|**F**}.

4. A term is first order if no second order parameter occurs in it. A formula **t:T** is atomic if **t** is first order, and **T** is a second order parameter or constant. A term or formula in which no variable has a free occurrence is said to be closed.

Clause 3 of this definition introduces the syntax for set abstraction. It generalizes the conventional syntax in which **ta** may only be a single variable. The more general form of the abstraction term is a genuine extension of the logic that is essential for many of its applications, including its use for category theory.

2.2. Logical Syntax

The extended NaDSet, like the original, is presented as a Gentzen Sequent Calculus [Genzen31-32], although it may be presented in any system of natural deduction. A sequent in NaDSet takes the form

$\Gamma \rightarrow \Theta$,

where Γ and Θ are finite, possibly empty, sequences of closed formulas. The formulas Γ form the **antecedent** of the sequent, and the formulas of Θ the **succedent**. A sequent can be interpreted as asserting that one of the formulas of its antecedent is false, or one of the formulas of its succedent is true.

By the logical syntax of NaDSet is to be understood the description of the axioms and the rules of deduction for sequents.

Definition of Logical Syntax

Axioms

$$\mathbf{G} \rightarrow \mathbf{G},$$

where **G** is a closed atomic formula

Propositional Rules

$$\frac{\Gamma, \mathbf{G} \rightarrow \Theta \qquad \Delta, \mathbf{H} \rightarrow \Lambda}{\Gamma, \Delta \rightarrow (\mathbf{G}\downarrow\mathbf{H}), \Theta, \Lambda} \qquad\qquad \frac{\Gamma \rightarrow \mathbf{G}, \mathbf{H}, \Theta}{\Gamma, (\mathbf{G}\downarrow\mathbf{H}) \rightarrow \Theta}$$

Quantification Rules

$$\frac{\Gamma \rightarrow [\mathbf{p}/\mathbf{u}]\mathbf{F}, \Theta}{\Gamma \rightarrow \forall\mathbf{u}\mathbf{F}, \Theta} \qquad\qquad \frac{\Gamma, [\mathbf{t}/\mathbf{u}]\mathbf{F} \rightarrow \Theta}{\Gamma, \forall\mathbf{u}\mathbf{F} \rightarrow \Theta}$$

In the first rule, **p** is a parameter that does not occur in **F**, or in any formula of Γ or Θ. In the second rule, **t** is any closed term.

Abstraction Rules

$$\frac{\Gamma \rightarrow [\underline{\mathbf{t}}/\underline{\mathbf{u}}]\mathbf{F}, \Theta}{\Gamma \rightarrow [\underline{\mathbf{t}}/\underline{\mathbf{u}}]\mathbf{ta}{:}\{\mathbf{ta}|\mathbf{F}\}, \Theta} \qquad\qquad \frac{\Gamma, [\underline{\mathbf{t}}/\underline{\mathbf{u}}]\mathbf{F} \rightarrow \Theta}{\Gamma, [\underline{\mathbf{t}}/\underline{\mathbf{u}}]\mathbf{ta}{:}\{\mathbf{ta}|\mathbf{F}\} \rightarrow \Theta}$$

$\underline{\mathbf{u}}$ is a sequence of the distinct variables with free occurrences in the term **ta**.

$\mathbf{F}$ is a formula in which only the variables $\underline{\mathbf{u}}$ have free occurrences.

$\underline{\mathbf{t}}$ is a sequence of closed terms, one for each variable in $\underline{\mathbf{u}}$.

Structural Rules

The structural rules of [Gentzen31-32] as described in [Kleene52] consist of contraction rules, interchange rules and thinning rules. The contraction rules permit the removal of a duplicate formula in the antecedent or in the succedent of a sequent, the interchange rules permit changing the order of the formulas in the antecedent or succedent of a sequent, while the thinning rules permit the introduction of a new formula into the antedent or succedent of a sequent. The effect of the contraction and interchange rules is to treat the antecedent and succedent of a sequent as finite sets of formulas. For this reason these rules will be ignored in this paper.

Thinning rules

$$\frac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow \mathbf{F}, \Theta} \qquad\qquad \frac{\Gamma \rightarrow \Theta}{\Gamma, \mathbf{F} \rightarrow \Theta}$$

where $\mathbf{F}$ is any closed formula.

Cut Rule

$$\frac{\Gamma, \mathbf{G} \rightarrow \Theta \qquad \mathbf{G}, \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda}$$

<u>End of definition</u>

Because the axioms are restricted to being sequents of closed formulas and the thinning rules may only introduce closed formulas, only sequents of closed formulas are derivable in NaDSet.

The propositional, quantification and abstraction rules will be denoted respectively by:

$\rightarrow\downarrow$, $\downarrow\rightarrow$, $\rightarrow\forall$, $\forall\rightarrow$, $\rightarrow\{\}$ and $\{\}\rightarrow$.

The thinning and cut rules will be referred to by name.

All the usual logical connectives, $\wedge$, $\vee$ and $\supset$ and the existential quantifier $\exists$ can be defined using $\downarrow$ and $\forall$. Corresponding rules of deduction can be derived and when necessary will be denoted respectively by: $\rightarrow\wedge$, $\wedge\rightarrow$, $\rightarrow\vee$, $\vee\rightarrow$, $\rightarrow\supset$, $\supset\rightarrow$, $\rightarrow\exists$ and $\exists\rightarrow$.

The abstraction rules differ in one important respect from those of [Gilmore86]: In an application of an abstraction rule, the term $\underline{\mathbf{t}}$ may be one in which second order parameters occur. In the original NaDSet, such applications of the rule were prohibited because the resulting formula $[\underline{\mathbf{t}}/\underline{\mathbf{a}}]\mathbf{ta}{:}\{\mathbf{ta}|\mathbf{Fa}\}$ would not be well-formed.

The quantification rules require only one kind of universal quantifier, not the two of second order logic. The parameter appearing in the premiss of the $\to\forall$ rule, but not in its conclusion, will be either a first or a second order parameter, with the order of the parameter implicitly determining the order of the quantifier. There is not a similar restriction on the $\forall\to$ rule; however, should a second order parameter occur in the term **t** appearing in the premiss of the rule, the quantifier can be understood to be a second order quantifier.

Bounded quantification is frequently used in the derivations for category theory. The definition of bounded quantifiers will be provided in section 2.3.4.

2.3 Why NaDSet?

The success of the axiomatization of category theory within NaDSet is dependent upon three features of the logic that distinguish it from classical set theories:

1) NaDSet is formalized as a natural deduction logic.
2) The atomic sentences of the logic receive a nominalistic interpretation.
3) Generalized abstraction terms are introduced into arguments through deduction rules, not through the use of comprehension axioms.

Each of these features will be discussed in a separate subsection of this section, following the next section which describes the form and interpretation of definitions provided in later sections of the paper.

2.3.1 Definitions

Essential to an understanding of this paper is the proper interpretation of definitions such as

Cat **for** {<Ar, $=_a$, Sr, Tg, Cp> | Category[Ar, $=_a$, Sr, Tg, Cp] }

Category[**Ar**, $=_{\mathbf{a}}$, **Sr**, **Tg**, **Cp**] **for** <u>axioms</u>

In the first of these definitions, 'Cat' is provided as an abbreviation for the abstraction term

$\{ \langle Ar, =_a, Sr, Tg, Cp \rangle \mid Category[Ar, =_a, Sr, Tg, Cp] \}$

This means that any term or formula in which 'Cat' is used as a term, should be understood as the term or formula in which 'Cat' is replaced by the abstraction term.

The second of these definitions is a definition scheme of individual definitions of the first kind. In the second definition, **Ar**, $\mathbf{=_a}$, **Sr**, **Tg** and **Cp** are used as metavariables ranging over the terms of NaDSet. When they are replaced with particular terms, as they are in the formula

$Category[Ar, =_a, Sr, Tg, Cp]$

by variables 'Ar', '$=_a$', 'Sr', 'Tg' and 'Cp', the resulting formula

$Category[Ar, =_a, Sr, Tg, Cp]$

is an abbreviation for the conjunction of all the axioms for categories in which the terms **Ar**, $\mathbf{=_a}$, **Sr**, **Tg** and **Cp** are replaced by the variables 'Ar', '$=_a$', 'Sr', 'Tg' and 'Cp'.

This second use of the **for** definitions is similar to the quasi-quotation corner notation of [Quine51].

2.3.2 Why a Natural Deduction Presentation?

Natural deduction presentations of first order logic are intended to formalize in a transparent fashion the standard reductionist semantics for the logic generally attributed to [Tarski36]. That semantics, for a given interpretation, provides a simultaneous recursive definition for the true closed formulas and the false closed formulas of the logic. The definition is reductionist in the sense that the truth or falsehood of a formula is reduced to the truth or falsefhood of simpler formulas. Extending Tarski semantics in the natural way to a logic such as NaDSet provides a means for avoiding the paradoxes of set theory as described in [Gilmore71,80,86]. In this respect NaDSet is similar to logics introduced by [Schütte60], although it must be emphasized the NaDSet, unlike some of the logics described in [Schütte60], is a fully formal logic with rules of deduction with finitely many premisses.

Because Tarski semantics reduces the truth or falsehood of complex formulas to that of atomic formulas, the interpretation of atomic formulas is critical. A closed atomic formula in NaDSet, as described in section 2.1, takes the form:

t:T

where **t** is a closed first order term without free variables, and **T** is either a second order constant or a second order parameter. The term **t** can only be a first order constant, a first order parameter, a second order constant, or a closed abstraction term {**ta:F**} without any occurrences of a second order parameter. However, closed atomic formulas appearing in the axioms of NaDSet used in the derivations for category theory take only one of the following forms:

p:P or {**ta:F**}:**P**

where **p** is a first order parameter, **P** a second order parameter. For example, an axiom of the first form, used in the derivation of lemma 4.1, is

$pd{:}Ar_C \rightarrow pd{:}Ar_C$

where 'pd' is a first order parameter and 'Ar_C' a second order parameter.

An example of the second form used in the same derivation is

$<pc,pc>{:}Sr_C \rightarrow <pc,pc>{:}Sr_C$

where 'pc' is a first order parameter, 'Sr_C' is a second order parameter, and the ordered pair '<pc,pc>', can be taken to be an abbreviation for

$\{x \mid (pc{:}C \downarrow pc{:}C)\}$

for a selected second order constant 'C', as defined in [Gilmore89], or can be taken to be a primitive of the logic incorporated into the elementary syntax. As noted in [Gilmore89], many different abstraction terms may be used as the definition of order pair, and the particular form of the one used here is not relevant for the purposes of this paper. The only question that must be answered here is how the atomic sentence

$\{x \mid (pc{:}C \downarrow pc{:}C)\}{:}Sr_C$

and the atomic sentence

$pd{:}Ar_C$

are to be interpreted?

The latter atomic formula is interpretated as it normally is in a second order logic. Namely, the first order parameter 'pd' is assigned an element in the domain of discourse $\mathbb{D}$ of the logic, while the second order parameter'Ar_C' is assigned a subset of $\mathbb{D}$. Then '$pd{:}Ar_C$' is "true" if the element assigned to 'pd' is a member of the set assigned to 'Ar_C', and is "false" otherwise.

The first of the two atomic formulas, however, is not a formula of conventional second order logic because the term '$\{x \mid (pc{:}C \downarrow pc{:}C)\}$', being a set abstraction term, would denote a set in the conventional interpretation, while no set is a member of $\mathbb{D}$. The interpretation of this atomic sentence in NaDSet requires that $\mathbb{D}$ consist of all the closed terms of NaDSet in which no parameter, first or second order, occurs. Then, as in the conventional interpretation, 'pd' may be assigned any term from $\mathbb{D}$. The term '{x | (pc:C ↓ pc:C)}', on the other hand, is assigned itself; that is, an occurrence of an abstraction term to the left of ':' in an atomic sentence is regarded as a name for itself, rather than as a name for a set.

This nominalistic interpretation of atomic sentences has as a consequence that NaDSet is an intensional theory rather than an extensional. The intensional identity of two abstraction terms does not follow from their extensional identity. Indeed, as shown in [Gilmore89], the assumption of an axiom or rule of extensionality within NaDSet, results in an inconsistent theory. This is, therefore, another way in which NaDSet differs from set theories such as Zermelo-Fraenkel and Gödel-Bernays.

2.3.3 Generalized Abstraction

The definition in section 3 of the set Cat of categories provides an excellant example of the generalized abstraction of NaDSet:

Cat **for** { <Ar, $=_a$, Sr, Tg, Cp> | Category[Ar, $=_a$, Sr, Tg, Cp] }

In this definition, 'Ar', '$=_a$', 'Sr', 'Tg' and 'Cp' are variables that are bound in the abstraction term

{ <Ar, $=_a$, Sr, Tg, Cp> | Category[Ar, $=_a$, Sr, Tg, Cp] }

while '<Ar, $=_a$, Sr, Tg, Cp>' is the term **ta** for the abstraction term {**ta**|**F**}, as defined in section 2. The formula 'Category[Ar, $=_a$, Sr, Tg, Cp]' is an abbreviation of the conjunction of the axioms (c1)-(c20) for categories listed in section 4.

The definition of Cat has the same form as that usually given for finite automata; namely, a category is a quintuple of predicates satisfying given axioms. In a classical set theory the definition of Cat would take the following form:

CCat **for** { x | [∃ Ar, $=_a$, Sr, Tg, Cp] (x = <Ar, $=_a$, Sr, Tg, Cp> ∧
Category[Ar, $=_a$, Sr, Tg, Cp]) },

where 'x' is a variable bound in the abstraction term, and now 'Ar', '$=_a$', 'Sr', 'Tg' and 'Cp' are variables bound by the multiple quantifier

[∃ Ar, $=_a$, Sr, Tg, Cp].

This definition of CCat can't be used in place of Cat. For example, the derivation of lemma 4.1 in section 4 would fail because second order parameters replace the abstracted variables' Ar', '$=_a$', 'Sr', 'Tg' and 'Cp' of Cat in the derivation.

The abstraction rules $\rightarrow\{\}$ and $\{\}\rightarrow$ take the place of the comprehension axiom scheme of a classical set theory. Although no restrictions are placed on the terms {**ta|F**} that may be introduced in the conclusion of the rules, the paradoxes are avoided because paradoxical conclusions like

$\rightarrow \{x \mid \sim x{:}x\}{:}\{x \mid \sim x{:}x\}$

have no premiss from which they can be derived. Thus the presentation of NaDSet must be as a natural deduction logic, and in the logic, abstraction is given equal treatment with the logical connectives and quantifiers in the sense that each of these fundamental logical concepts is formalized in a pair of rules of deduction.

2.3.4 Bounded Quantifiers

Bounded quantifiers are frequently used throughout the paper. For example, each of the axioms (c1) - (c20) for categories given in section 3 uses a single or multiple bounded universal quantifier. Consider (c2):

$[\forall f,g{:}\mathbf{Ar}](\, f{=_{\mathbf{a}}}g \supset g{=_{\mathbf{a}}}f)$

This expression is an abbreviation for the expression:

$[\forall f][\forall g](\, f{:}\mathbf{Ar} \wedge g{:}\mathbf{Ar} \supset (\, f{=_{\mathbf{a}}}g \supset g{=_{\mathbf{a}}}f))$

Thus $[\forall f,g{:}\mathbf{Ar}]$ is a conventional bounded quantifier.

But a more general form of bounded quantifier is also used: A single variable may be bounded by an abstraction term. For example, lemma 4.1 of section 4 below takes the form:

$\rightarrow [\forall x,y{:}\mathrm{Cat}]\, P[x,y]$

where P[x,y] is a formula in which the variables 'x' and 'y' occur free. 'Cat', as described in section 2.3.1 above is an abbreviation for the abstraction term

$\{\langle Ar, =_a, Sr, Tg, Cp\rangle \mid \mathrm{Category}[Ar, =_a, Sr, Tg, Cp]\}$

where Ar, $=_a$, Sr, Tg and Cp are variables that are bound in the abstraction term.

A single bounded quantifier of the form

$[\forall x{:}\mathrm{Cat}]\, P[x]$

is an abbreviation for the formula

$$[\forall Ar'][\forall =_a'][\forall Sr'][\forall Tg'][\forall Cp'](\\ \langle Ar',=_a',Sr',Tg',Cp'\rangle:\{\langle Ar,=_a,Sr,Tg,Cp\rangle \mid Category[Ar,=_a,Sr,Tg,Cp]\} \supset \\ P[\langle Ar',=_a',Sr',Tg',Cp'\rangle]),$$

where Ar', $=_a'$, Sr', Tg' and Cp' are distinct variables free to replace respectively the variable x in the formula P[x].

The formula

$$[\forall x,y{:}Cat]\ P[x,y]$$

is then an abbreviation for the formula

$$[\forall Ar'][\forall =_a'][\forall Sr'][\forall Tg'][\forall Cp'][\forall Ar''][\forall =_a''][\forall Sr''][\forall Tg''][\forall Cp''](\\ \langle Ar',=_a',Sr',Tg',Cp'\rangle:\{\langle Ar,=_a,Sr,Tg,Cp\rangle \mid Category[Ar,=_a,Sr,Tg,Cp]\} \wedge \\ \langle Ar'',=_a'',Sr'',Tg'',Cp''\rangle:\{\langle Ar,=_a,Sr,Tg,Cp\rangle \mid Category[Ar,=_a,Sr,Tg,Cp]\} \\ \supset P[\langle Ar',=_a',Sr',Tg',Cp'\rangle, \langle Ar'',=_a'',Sr'',Tg'',Cp''\rangle]),$$

where the variables Ar', $=_a'$, Sr', Tg', Cp', Ar'', $=_a''$, Sr'', Tg'' and Cp'' are suitably chosen.

The following rules of deduction for a single bounded quantifier can be derived:

$$\frac{\Gamma,\ [\mathbf{p}/\mathbf{u}]ta{:}\{ta \mid F\} \rightarrow [[\mathbf{p}/\mathbf{u}]ta/v]G,\ \Theta}{\Gamma \rightarrow [\forall v{:}\{\mathbf{ta} \mid \mathbf{F}\}]\ \mathbf{G},\ \Theta}$$

where

$\mathbf{u}$ is a sequence of the distinct variables with free occurrences in **ta**,

$\mathbf{p}$ is a sequence of the same length as $\mathbf{u}$ of distinct parameters, none of which occur in the conclusion of the rule.

$$\frac{\Gamma \rightarrow \Theta,\ [\mathbf{t}/\mathbf{u}]ta{:}\{ta \mid F\} \qquad \Delta,\ [[\mathbf{t}/\mathbf{u}]ta/v]G \rightarrow \Lambda}{\Gamma,\ \Delta,\ [\forall v{:}\{\mathbf{ta} \mid \mathbf{F}\}]\ \mathbf{G} \rightarrow \Theta,\ \Lambda}$$

where

$\mathbf{u}$ is the same as in the first rule and

$\mathbf{t}$ is a sequence of the same length as $\mathbf{u}$ of closed terms.

A derivation of these rules will be left to the reader.

3. CATEGORIES

Category theory has been generally viewed as a means of providing an abstract and uniform treatment of many mathematical structures. Therefore, categories are a complex enough algebraic structure that can be used as a benchmark for the capability of a formalism that aspires to provide a foundation for mathematics. In this section, a NaDSet definition of the set of categories will be given analogous to the definition of the structure B in section 8 of [Gilmore 89]. The terminology provided in the introduction of [Barr&Wells85] will be used with one exception: Instead of using *objects* and *arrows* in defining a category, by following Lawvere's definition [Lawvere66], objects can be dispensed with altogether, and only arrows used. Nevertheless, for the readers who are accustomed to the more traditonal definition of categories, a definition of the objects for a category in terms of its arrows is provided.

The formalization of category theory within NaDSet is typical of the formalization of any axiomatic theory within the logic: The set of structures satisfying the axioms of category theory is defined. The theorems of category theory are then the sentences that can be proven to be true in any member of the set of categories.

In this section, and throughout the remainder of the paper, the notation for abstraction variables and for parameters is greatly expanded to include more conventional algebraic notations. These notations will be explained as they are introduced. Additionally, metavariables ranging over terms of NaDSet that are intended to represent algebraic concepts, are used. They will always be printed in bold type. For example, the variables of this kind used in this section, together with their intended interpretation are:

Ar the set of arrows or morphisms

$=_{\mathbf{a}}$ identity of arrows

Sr a binary term with first argument an arrow and second argument its source object

Tg a binary term with first argument an arrow and second argument its target object

Cp a ternary term the third argument of which is the composite of the arrows that are its first two terms.

The first use of these metavariables is in the following definition:

Category[**Ar**, $=_{\mathbf{a}}$, **Sr**, **Tg**, **Cp**] **for** <u>axioms</u>

<u>Axioms</u> is the conjunction of the following sentences, where the usual infix notation for $=_{\mathbf{a}}$ is

used instead of the postfix notation of NaDSet:

Identity Axioms

$$[\forall f{:}\mathbf{Ar}]\ f{=_{\mathbf{a}}}f \tag{c1}$$

$$[\forall f,g{:}\mathbf{Ar}](\ f{=_{\mathbf{a}}}g \supset g{=_{\mathbf{a}}}f) \tag{c2}$$

$$[\forall f,g,h{:}\mathbf{Ar}](\ f{=_{\mathbf{a}}}g \wedge g{=_{\mathbf{a}}}h \supset f{=_{\mathbf{a}}}h\ ) \tag{c3}$$

$$[\forall f,g,a{:}\mathbf{Ar}](\ f{=_{\mathbf{a}}}g \wedge \langle f,a\rangle{:}\mathbf{Sr} \supset \langle g,a\rangle{:}\mathbf{Sr}\ ) \tag{c4}$$

$$[\forall f,a,b{:}\mathbf{Ar}](\ a{=_{\mathbf{a}}}b \wedge \langle f,a\rangle{:}\mathbf{Sr} \supset \langle f,b\rangle{:}\mathbf{Sr}\ ) \tag{c5}$$

$$[\forall f,g,a{:}\mathbf{Ar}](\ f{=_{\mathbf{a}}}g \wedge \langle f,a\rangle{:}\mathbf{Tg} \supset \langle g,a\rangle{:}\mathbf{Tg}\ ) \tag{c6}$$

$$[\forall f,a,b{:}\mathbf{Ar}](\ a{=_{\mathbf{a}}}b \wedge \langle f,a\rangle{:}\mathbf{Tg} \supset \langle f,b\rangle{:}\mathbf{Tg}\ ) \tag{c7}$$

$$[\forall f,g,h,k{:}\mathbf{Ar}]\ (\ f{=_{\mathbf{a}}}k \wedge \langle f,g,h\rangle{:}\mathbf{Cp} \supset \langle k,g,h\rangle{:}\mathbf{Cp}\ ) \tag{c8}$$

$$[\forall f,g,h,k{:}\mathbf{Ar}]\ (\ g{=_{\mathbf{a}}}k \wedge \langle f,g,h\rangle{:}\mathbf{Cp} \supset \langle f,k,h\rangle{:}\mathbf{Cp}\ ) \tag{c9}$$

$$[\forall f,g,h,k{:}\mathbf{Ar}]\ (\ h{=_{\mathbf{a}}}k \wedge \langle f,g,h\rangle{:}\mathbf{Cp} \supset \langle f,g,k\rangle{:}\mathbf{Cp}\ ) \tag{c10}$$

Sr, Tg and Cp are functions

$$[\forall f{:}\mathbf{Ar}][\ \exists a{:}\mathbf{Ar}]\ \langle f,a\rangle{:}\mathbf{Sr} \tag{c11}$$

$$[\forall f,a,b{:}\mathbf{Ar}](\ \langle f,a\rangle{:}\mathbf{Sr} \wedge \langle f,b\rangle{:}\mathbf{Sr} \supset a{=_{\mathbf{a}}}b) \tag{c12}$$

$$[\forall f{:}\mathbf{Ar}][\ \exists a{:}\mathbf{Ar}]\ \langle f,a\rangle{:}\mathbf{Tg} \tag{c13}$$

$$[\forall f,a,b{:}\mathbf{Ar}](\ \langle f,a\rangle{:}\mathbf{Tg} \wedge \langle f,b\rangle{:}\mathbf{Tg} \supset a{=_{\mathbf{a}}}b) \tag{c14}$$

$$[\forall f,g,b{:}\mathbf{Ar}](\ \langle f,b\rangle{:}\mathbf{Tg} \wedge \langle g,b\rangle{:}\mathbf{Sr}\ \supset [\ \exists h{:}\mathbf{Ar}]\langle f,g,h\rangle{:}\mathbf{Cp}\ ) \tag{c15}$$

$$\begin{aligned}&[\forall f,g,h,a,b,c{:}\mathbf{Ar}](\ \langle f,g,h\rangle{:}\mathbf{Cp} \supset ((\ \langle f,a\rangle{:}\mathbf{Sr}\ \supset \langle h,a\rangle{:}\mathbf{Sr}) \wedge \\ &\qquad (\langle g,b\rangle{:}\mathbf{Tg} \supset \langle h,b\rangle{:}\mathbf{Tg}) \wedge (\langle f,c\rangle{:}\mathbf{Tg}\ \equiv \langle g,c\rangle{:}\mathbf{Sr}\ )))\end{aligned} \tag{c16}$$

$$[\forall f,g,h,k{:}\mathbf{Ar}]\ (\ \langle f,g,h\rangle{:}\mathbf{Cp} \wedge \langle f,g,k\rangle{:}\mathbf{Cp} \supset h{=_{\mathbf{a}}}k\ ) \tag{c17}$$

Note that compositions are written in the order of the arrows from left to right. Therefore, $\langle f,g,h\rangle{:}\mathbf{Cp}$ if and only if h is $g\circ f$ where $\circ$ denotes morphism composition.

Identity Arrows Exist

$$[\forall f,a{:}\mathbf{Ar}](\ \langle f,a\rangle{:}\mathbf{Sr}\ \supset \langle a,a\rangle{:}\mathbf{Sr} \wedge \langle a,a\rangle{:}\mathbf{Tg} \wedge \langle a,f,f\rangle{:}\mathbf{Cp}\ ) \tag{c18}$$

$$[\forall f,a{:}\mathbf{Ar}](\ \langle f,a\rangle{:}\mathbf{Tg}\ \supset \langle a,a\rangle{:}\mathbf{Sr} \wedge \langle a,a\rangle{:}\mathbf{Tg} \wedge \langle f,a,f\rangle{:}\mathbf{Cp}\ ) \tag{c19}$$

Composition is Associative

$$[\forall f,g,h,fg,gh,fg1h,f1gh{:}\mathbf{Ar}](\ \langle f,g,fg\rangle{:}\mathbf{Cp} \wedge \langle g,h,gh\rangle{:}\mathbf{Cp} \wedge \langle fg,h,fg1h\rangle{:}\mathbf{Cp} \wedge \langle f,gh,f1gh\rangle{:}\mathbf{Cp} \supset fg1h{=_\mathbf{a}}f1gh\ ) \tag{c20}$$

The set of categories is now defined:

Cat for { <Ar, $=_a$, Sr, Tg, Cp> | Category[Ar, $=_a$, Sr, Tg, Cp]}

where Ar, $=_a$, Cp, Sr and Tg are all used as variables that are bound in the abstraction term.

Finally, the projections on a tuple that represents a category can be given by the following definitions.

Ar[<**Ar**, $=_\mathbf{a}$, **Sr**, **Tg**, **Cp**>] **for** { u | u:**Ar** }

$=_\mathbf{a}$[<**Ar**, $=_\mathbf{a}$, **Sr**, **Tg**, **Cp**>] **for** { <u,v> | <u,v>: $=_\mathbf{a}$ }

Sr[<**Ar**, $=_\mathbf{a}$, **Sr**, **Tg**, **Cp**>] **for** { <u,v> | <u,v>: **Sr** }

Tg[<**Ar**, $=_\mathbf{a}$, **Sr**, **Tg**, **Cp**>] **for** { <u,v> | <u,v>: **Tg** }

Cp[<**Ar**, $=_\mathbf{a}$, **Sr**, **Tg**, **Cp**>] **for** { <u,v,w> | <u,v,w>: **Cp** }

3.1 Objects, Hom-Sets and Commutative Diagrams

The axiomatization of category theory presented here does not require the specification of a set of objects, since the objects of a category correspond exactly to its identity arrows. Therefore the set of objects Ob[<**Ar**, $=_\mathbf{a}$, **Sr**, **Tg**, **Cp**>] of a category <**Ar**, $=_\mathbf{a}$, **Sr**, **Tg**, **Cp**> may be defined to be any one of the following extensionally identical terms.

(i) $\{\ x \mid x{:}\mathbf{Ar} \wedge \langle x, x\rangle{:}\mathbf{Sr} \wedge \langle x, x\rangle{:}\mathbf{Tg}\ \}$

(ii) $\{\ x \mid x{:}\mathbf{Ar} \wedge ([\exists f{:}\mathbf{Ar}]\langle f, x\rangle{:}\mathbf{Sr} \vee [\exists f{:}\mathbf{Ar}]\langle f, x\rangle{:}\mathbf{Tg})\ \}$

(iii) $\{\ x \mid x{:}\mathbf{Ar} \wedge [\forall f,g{:}\mathbf{Ar}](\langle f, x, g\rangle{:}\mathbf{Cp} \supset f{=_\mathbf{a}}g)$
$\wedge [\forall f,g{:}\mathbf{Ar}](\langle x, f, g\rangle{:}\mathbf{Cp} \supset f{=_\mathbf{a}}g)\ \}$

The hom-set for objects **ob1** and **ob2** can be defined:

Hom[**ob1**,**ob2**] **for** $\{\ x \mid x{:}\mathbf{Ar} \wedge \langle x,\mathbf{ob1}\rangle{:}\mathbf{Sr} \wedge \langle x,\mathbf{ob2}\rangle{:}\mathbf{Tg}\ \}$

Finally, that the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "h"'] & b \arrow[d, "g"] \\ & c \end{tikzcd}$$

commutes means

$\langle f,a\rangle{:}\mathbf{Sr} \wedge \langle f,b\rangle{:}\mathbf{Tg} \wedge \langle g,b\rangle{:}\mathbf{Sr} \wedge \langle g,c\rangle{:}\mathbf{Tg} \wedge \langle f,g,h\rangle{:}\mathbf{Cp}$,

while that the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "k"'] & b \arrow[d, "g"] \\ d \arrow[r, "m"'] & c \end{tikzcd}$$

commutes means

$\langle f,a\rangle{:}\mathbf{Sr} \wedge \langle f,b\rangle{:}\mathbf{Tg} \wedge \langle g,b\rangle{:}\mathbf{Sr} \wedge \langle g,c\rangle{:}\mathbf{Tg} \wedge \langle k,a\rangle{:}\mathbf{Sr} \wedge \langle k,d\rangle{:}\mathbf{Tg} \wedge$
$\langle m,d\rangle{:}\mathbf{Sr} \wedge \langle m,c\rangle{:}\mathbf{Tg} \wedge [\exists h{:}\mathbf{Ar}](\ \langle f,g,h\rangle{:}\mathbf{Cp}\ \wedge \langle k,m,h\rangle{:}\mathbf{Cp}\ )$,

that is, that both the following diagrams commute:

$$\begin{tikzcd} a \arrow[d, "k"'] \arrow[dr, "h"] & \\ d \arrow[r, "m"'] & c \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "h"'] & b \arrow[d, "g"] \\ & c \end{tikzcd}$$

4. FUNCTORS

To define the category of categories the notion of *functor* from one category to another is needed. Its definition is given in the typical NaDSet style with the symbols $\mathbf{F}$, $\mathbf{Ar_C}$, $\mathbf{=_{aC}}$, $\mathbf{Sr_C}$, $\mathbf{Tg_C}$, $\mathbf{Cp_C}$, $\mathbf{Ar_D}$, $\mathbf{=_{aD}}$, $\mathbf{Sr_D}$, $\mathbf{Tg_D}$ and $\mathbf{Cp_D}$ used as metavariables ranging over second order terms.

Functor$[\mathbf{F},\langle \mathbf{Ar_C}, \mathbf{=_{aC}}, \mathbf{Sr_C}, \mathbf{Tg_C}, \mathbf{Cp_C}\rangle,\langle \mathbf{Ar_D}, \mathbf{=_{aD}}, \mathbf{Sr_D}, \mathbf{Tg_D}, \mathbf{Cp_D}\rangle]$
for <u>axioms</u>

where <u>axioms</u> consists of the conjunction of the following sentences:

<u>F is a map for categories</u>

$$< \mathbf{Ar_C}, \mathbf{=_{aC}}, \mathbf{Sr_C}, \mathbf{Tg_C}, \mathbf{Cp_C} >:\mathrm{Cat} \qquad \text{(f1)}$$

$$< \mathbf{Ar_D}, \mathbf{=_{aD}}, \mathbf{Sr_D}, \mathbf{Tg_D}, \mathbf{Cp_D} >:\mathrm{Cat} \qquad \text{(f2)}$$

F maps arrows to arrows, preserving arrow identity

$$[\forall fc:\mathbf{Ar_C}][\exists fd:\mathbf{Ar_D}] <fc,fd>:\mathbf{F} \qquad \text{(f3)}$$

$$[\forall fc,gc:\mathbf{Ar_C}][\forall fd,gd:\mathbf{Ar_D}](fc \mathbf{=_{aC}} gc \wedge <fc,fd>:\mathbf{F} \wedge <gc,gd>:\mathbf{F} \supset fd \mathbf{=_{aD}} gd) \qquad \text{(f4)}$$

$$[\forall fc,gc:\mathbf{Ar_C}][\forall fd:\mathbf{Ar_D}](fc \mathbf{=_{aC}} gc \wedge <fc,fd>:\mathbf{F} \supset <gc,fd>:\mathbf{F}) \qquad \text{(f5)}$$

$$[\forall fc:\mathbf{Ar_C}][\forall fd,gd:\mathbf{Ar_D}](fd \mathbf{=_{aD}} gd \wedge <fc,fd>:\mathbf{F} \supset <fc,gd>:\mathbf{F}) \qquad \text{(f6)}$$

F preserves source, target and composition

$$[\forall fc,c:\mathbf{Ar_C}][\forall fd,d:\mathbf{Ar_D}] (<fc,c>:\mathbf{Sr_C} \wedge <fc,fd>:\mathbf{F} \wedge <c,d>:\mathbf{F} \supset <fd,d>:\mathbf{Sr_D}) \qquad \text{(f7)}$$

$$[\forall fc,c:\mathbf{Ar_C}][\forall fd,d:\mathbf{Ar_D}] (<fc,c>: \mathbf{Tg_C} \wedge <fc,fd>:\mathbf{F} \wedge <c,d>:\mathbf{F} \supset <fd,d>: \mathbf{Tg_D}) \qquad \text{(f8)}$$

$$[\forall fc1,fc2,fc3:\mathbf{Ar_C}] [\forall fd1,fd2,fd3:\mathbf{Ar_D}](< fc1,fc2,fc3>:\mathbf{Cp_C} \wedge$$
$$<fc1,fd1>:\mathbf{F} \wedge <fc2,fd2>:\mathbf{F} \wedge <fc3,fd3>:\mathbf{F} \supset < fd1,fd2,fd3>:\mathbf{Cp_D}) \qquad \text{(f9)}$$

Functors, following a suggestion of [Lawvere66], are defined as triples that include the source and target categories. The set of functors is defined:

$$\mathrm{Func}\ \mathbf{for}\ \{ < F, <Ar_C, =_{aC}, Sr_C, Tg_C, Cp_C>, <Ar_D, =_{aD}, Sr_D, Tg_D, Cp_D> > \mid$$
$$\mathrm{Functor}[F,<Ar_C, =_{aC}, Sr_C,Tg_C, Cp_C>,<Ar_D, =_{aD}, Sr_D,Tg_D, Cp_D>]\}$$

The set of functors from a category $<\mathbf{Ar_C}, \mathbf{=_{aC}}, \mathbf{Sr_C}, \mathbf{Tg_C}, \mathbf{Cp_C}>$ to a category $<\mathbf{Ar_D}, \mathbf{=_{aD}}, \mathbf{Sr_D}, \mathbf{Tg_D}, \mathbf{Cp_D}>$ is defined as

$$\mathrm{Func}[<\mathbf{Ar_C}, \mathbf{=_{aC}}, \mathbf{Sr_C}, \mathbf{Tg_C}, \mathbf{Cp_C}>, <\mathbf{Ar_D}, \mathbf{=_{aD}}, \mathbf{Sr_D}, \mathbf{Tg_D}, \mathbf{Cp_D}>]$$
$$\mathbf{for}$$
$$\{x \mid < x, <\mathbf{Ar_C}, \mathbf{=_{aC}}, \mathbf{Sr_C}, \mathbf{Tg_C}, \mathbf{Cp_C}>, <\mathbf{Ar_D}, \mathbf{=_{aD}}, \mathbf{Sr_D}, \mathbf{Tg_D}, \mathbf{Cp_D}>>:\mathrm{Func} \}.$$

In [Mac Lane71] and [Barr&Wells85] an additional axiom is included in the definition of functors; the axiom states that a functor must map identity arrows to identity arrows. But that axiom is not independent of the seven axioms given here. Since the identity arrows of a category are its objects, they can be defined by one of the three equivalent definitions given in section 3.1.

The first of these definitions will be used here:

$$\mathrm{Id}[<Ar_C, =a_C, Sr_C, Tg_C, Cp_C>] \quad \textbf{for} \ \{x \mid x{:}Ar_C \wedge <x,x>{:}Sr_C \wedge <x,x>{:}Tg_C\}$$

The sequent asserted to be derivable in the following lemma, expresses the additional axiom.

4.1 Lemma The sequent

$$\rightarrow [\forall x,y{:}Cat][\forall f{:}Func[x,y]][\forall c{:}Ar[x]][\forall d{:}Ar[y]](\ c{:}Id[x] \wedge <c,d>{:}f \supset d{:}Id[y]\ )$$

is derivable.

Proof:

A derivation of the sequent follows. This derivation, like the others presented later, is condensed. Several applications of the rules of deduction may be represented as one application. To assist in identifying the rule being applied, the principal sentence in the conclusion of the rule is identified with a prefixed *; here "principal sentence" means the explicitly displayed sentence in the conclusion of the rule. More than one sentence may be so prefixed when a single step represents applications of more than one rule. In addition, the prefix # is used to identify the cut formula in an application of the cut rule. Finally, when a step involves a single premiss rule, the line between the premiss and the conclusion is omitted. However, in a step with many premisses, the line is retained and the premisses are numbered using Roman numerals. In this case, all premisses except the first are either axioms or have been previously derived. Axioms of NaDSet are referenced simply as '(axiom)', while other axioms are referenced by their number.

In the following derivation, pc and pd are first order parameters, F, Ar_C, $=_{aC}$, Sr_C, Tg_C, Cp_C and Ar_D, $=_{aD}$, Sr_D, Tg_D, Cp_D are second order parameters and C and D are abbreviations for the tuples $<Ar_C, =_{aC}, Sr_C, Tg_C, Cp_C>$ and $<Ar_D, =_{aD}, Sr_D, Tg_D, Cp_D>$ respectively.

(i) $<pc,pc>{:}Sr_C \rightarrow <pc,pc>{:}Sr_C$ (axiom)

(ii) $<pc,pd>{:}F \rightarrow <pc,pd>{:}F, *<pc,pd>{:}F$ (axiom+thin)

(i) $<pc,pc>{:}Sr_C, <pc,pd>{:}F \rightarrow *(<pc,pc>{:}Sr_C \wedge <pc,pd>{:}F \wedge <pc,pd>{:}F)$

(ii) $<pd,pd>{:}Sr_D \rightarrow <pd,pd>{:}Sr_D$ (axiom)

(i) $*(<pc,pc>{:}Sr_C \wedge <pc,pd>{:}F \wedge <pc,pd>{:}F \supset <pd,pd>{:}Sr_D)$,

$<pc,pc>{:}Sr_C, <pc,pd>{:}F \rightarrow <pd,pd>{:}Sr_D$

(ii) $pd{:}Ar_D \rightarrow pd{:}Ar_D$ (axiom)

(iii) $pd:Ar_D \rightarrow pd:Ar_D$ (axiom)

(iv) $pc:Ar_C \rightarrow pc:Ar_C$ (axiom)

(v) $pc:Ar_C \rightarrow pc:Ar_C$ (axiom)

$*[\forall fc,c:Ar_C]*[\forall fd,d:Ar_D]\ (<fc,c>:Sr_C \wedge <fc,fd>:F \wedge <c,d>:F \supset <fd,d>:Sr_D),$

$pc:Ar_C,\ pd:Ar_D,\ <pc,pc>:Sr_C,\ <pc,pd>:F \ \rightarrow\ <pd,pd>:Sr_D$

(i) $*Functor[F,C,D],\ pc:Ar_C,\ pd:Ar_D,\ <pc,pc>:Sr_C,\ <pc,pd>:F$

$\rightarrow\ <pd,pd>:Sr_D$ (f7,thin)

(ii) $Functor[F,C,D],\ pc:Ar_C,\ pd:Ar_D,\ <pc,pc>:Tg_C,\ <pc,pd>:F$

$\rightarrow\ <pd,pd>:Tg_D$ (similar to i)

(iii) $pd:Ar_D\ \rightarrow\ pd:Ar_D$ (axiom)

$Functor[F,C,D],\ pc:Ar_C,\ pd:Ar_D,\ <pc,pc>:Sr_C,\ <pc,pc>:Tg_C,\ <pc,pd>:F$

$\rightarrow\ *(pd:Ar_D \wedge <pd,pd>:Sr_D \wedge <pd,pd>:Tg_D)$

$Functor[F,C,D],\ pc:Ar_C,\ pd:Ar_D,\ *pc:Id[C],\ <pc,pd>:F\ \rightarrow\ *pd:Id[D]$

$Functor[F,C,D],\ pc:Ar_C,\ pd:Ar_D\ \rightarrow\ *(\ pc:Id[C] \wedge <pc,pd>:F \supset\ pd:Id[D]\ )$

$Functor[F,C,D]\ \rightarrow\ *[\forall c:Ar[C]]*[\forall d:Ar[D]](\ c:Id[C] \wedge <c,d>F \supset d:Id[D]\ )$

$*F:Func[C,D] \rightarrow [\forall c:Ar[C]][\forall d:Ar[D]](\ c:Id[C] \wedge <c,d>:F \supset d:Id[D]\ )$

$*C:Cat,\ *D:Cat,\ F:Func[C,D] \rightarrow [\forall c:Ar[C]][\forall d:Ar[D]](\ c:Id[C] \wedge <c,d>:F \supset d:Id[D]\ )$ (thin)

$\rightarrow *[\forall x,y:Cat]*[\forall f:Func[x,y][\forall c:Ar[x]][\forall d:Ar[y]](\ c:Id[x] \wedge <c,d>:f \supset d:Id[y]\ )$

End of proof of lemma 4.1

5. THE CATEGORY OF CATEGORIES

5.1 Definitions and Preliminaries

The category of categories is defined as the tuple **<Ar, $=_a$, Sr, Tg, Cp>** of the second order terms **Ar, $=_a$, Sr, Tg, Cp** whose definitions are given in this section. Because of the great number of variables used in this section, some abbreviations similar to those used in the derivation of lemma 1, are again used here, and later: The capital letters A, B, C, D, E with or without subscripts, are used to abbreviate the tuples $<Ar_A, =_{aA}, Sr_A, Tg_A, Cp_A>$, ... , $<Ar_E, =_{aE}, Sr_E, Tg_E, Cp_E>$ of the terms Ar_A, $=_{aA}$, Sr_A, Tg_A and Cp_A, ... , Ar_E, $=_{aE}$, Sr_E, Tg_E and Cp_E respectively. At different occasions these terms can be second order parameters, abstraction variables or metavariables that range over the second order terms. However, what the terms are to be in a particular context will be described prior to their use.

In the following definitions, the letters C and D, with or without subscripts, are abbreviations for the previously mentioned tuples of abstraction variables, while the letters F, G, H possibly subscripted, are regular abstraction variables.

A definition of the set **Ar** of arrows for the category of categories will be provided first; it is just the set of functors, as defined in section 4:

Ar for Func

The identity $=_a$ for members of **Ar** is defined in terms of extensional identity.

$$=_a \textbf{ for } \{<<F1, C1, D1>, <F2, C2, D2>> \mid C1 =_e C2 \wedge D1 =_e D2 \wedge F1 =_e F2 \ \}$$

In this definition $=_e$ is the coordinate-wise extensional identity among tuples of terms defined by

$$=_e \textbf{ for } \{<<Ar1, =_{a1}, Sr1, Tg1, Cp1>, <Ar2, =_{a2}, Sr2, Tg2, Cp2>> \mid$$
$$Ar1 =_e Ar2 \wedge =_{a1} =_e =_{a2} \wedge Sr1 =_e Sr2$$
$$\wedge Tg1 =_e Tg2 \wedge Cp1 =_e Cp2 \ \}$$

where Ar1, ..., Cp2 are all being used as abstraction variables. The definition of extensional identity $=_e$ is depends upon the context:

$$Ar1 =_e Ar2 \textbf{ for } [\forall f{:}Ar1]\, f{:}Ar2 \wedge [\forall f{:}Ar2]\, f{:}Ar1$$

$=_{a1} =_e =_{a2}$ **for** $[\forall f,g:Ar1](f =_{a1} g \supset f =_{a2} g) \wedge [\forall f,g:Ar2](f =_{a2} g \supset f =_{a1} g)$

$Sr1 =_e Sr2$ **for** $[\forall f,g:Ar1](\langle f, g\rangle:Sr1 \supset \langle f, g\rangle:Sr2) \wedge$
$[\forall f,g:Ar2](\langle f, g\rangle:Sr2 \supset \langle f, g\rangle:Sr1)$

$Tg1 =_e Tg2$ **for** $[\forall f,g:Ar1](\langle f, g\rangle:Tg1 \supset \langle f, g\rangle:Tg2) \wedge$
$[\forall f,g:Ar2](\langle f, g\rangle:Tg2 \supset \langle f, g\rangle:Tg1)$

$Cp1 =_e Cp2$ **for** $[\forall f,g,h:Ar1](\langle f, g,h\rangle:Cp1 \supset \langle f, g,h\rangle:Cp2) \wedge$
$[\forall f,g,h:Ar2](\langle f, g,h\rangle:Cp2 \supset \langle f, g,h\rangle:Cp1)$

$F1 =_e F2$ **for** $[\forall f:Ar_{C1}][\forall g:Ar_{D1}](\langle f,g\rangle:F1 \supset \langle f,g\rangle:F2) \wedge$
$[\forall f:Ar_{C2}][\forall g:Ar_{D2}](\langle f,g\rangle:F2 \supset \langle f,g\rangle:F1)$

Clearly, the source and target of an arrow has to coincide with the identity functor of the source and target category, respectively. Their definitions follow, in a style similar to that of $\mathbb{A}r$.

$\mathbb{S}r$ **for** $\{\langle\langle F1, C1, D1\rangle, \langle F2, C2, D2\rangle\rangle \mid$

$C2 =_e C1 \wedge D2 =_e C1 \wedge [\forall f,g:Ar_{C1}](\langle f,g\rangle:F2 \equiv f =_{aC1} g)\ \}$

Similarly,

$\mathbb{T}g$ **for** $\{\langle\langle F1, C1, D1\rangle, \langle F2, C2, D2\rangle\rangle \mid$

$C2 =_e D1 \wedge D2 =_e D1 \wedge [\forall f,g:Ar_{D1}](\langle f,g\rangle:F2 \equiv f =_{aD1} g)\ \}$

The final definition needed is of $\mathbb{C}p$, composition of the arrows for the category of categories.

$\mathbb{C}p$ **for** $\{\ \langle\langle F1, C1, D1\rangle, \langle F2, C2, D2\rangle, \langle F3, C3, D3\rangle\rangle \mid$

$C1 =_e C3 \wedge D1 =_e C2 \wedge D2 =_e D3 \wedge$

$[\forall f:Ar_{C1}][\forall g:Ar_{D2}](\langle f,g\rangle:F3 \equiv [\exists h:Ar_{D1}](\langle f,h\rangle:F1 \wedge \langle h,g\rangle:F2))\ \}$.

The main goal of the paper is to show that the set Cat with the defined constructs is itself a category. The proof of this result provided later makes use of some preliminary results that are discussed next.

Some trivial consequences of the definitions of $=_e$ and $=_e$ are listed in the following lemma.

5.1.1 Lemma

For any second order parameters P, Q and R, the following sequents are derivable:

(1) $P=_eQ \rightarrow P=_eQ$

(2) $\rightarrow P=_eP$

(3) $P=_eQ \rightarrow Q=_eP$

(4) $P=_eQ,\ Q=_eR \rightarrow P=_eR$

For any tuples C, D and E of second order parameters, as defined in this section, and any first order parameters a, b, c, the following sequents are derivable.

(5) $C =_{\mathbf{e}} D \rightarrow C =_{\mathbf{e}} D$

(6) $C =_{\mathbf{e}} C$

(7) $C =_{\mathbf{e}} D \rightarrow D =_{\mathbf{e}} C$

(8) $C =_{\mathbf{e}} D,\ D =_{\mathbf{e}} E \rightarrow C =_{\mathbf{e}} E$

(9) $a{:}Ar_C,\ C=_{\mathbf{e}}D \rightarrow a{:}Ar_D$

(10) $a =_{aC} b,\ C=_{\mathbf{e}}D \rightarrow a =_{aD} b$

(11) $\langle a,b\rangle{:}Sr_C,\ C=_{\mathbf{e}}D \rightarrow \langle a,b\rangle{:}Sr_D$

(12) $\langle a,b\rangle{:}Tg_C,\ C=_{\mathbf{e}}D \rightarrow \langle a,b\rangle{:}Tg_D$

(13) $\langle a,b,c\rangle{:}Cp_C,\ C=_{\mathbf{e}}D \rightarrow \langle a,b,c\rangle{:}Cp_D$.

The sequents 1-13 are simple consequences of the definitions of $=_e$ and $=_{\mathbf{e}}$. Their derivations are elementary and are therefore omitted.

5.2 Identity Functors

The next lemma insures that for any category (an element of Cat) there exists an identity functor from the category to itself.

5.2.1 Lemma.

Let

$\mathbb{I}\mathrm{d}[C]$ **for** $=_{aC}$.

If C is any tuple $\langle Ar_C, =_{aC}, Sr_C, Tg_C, Cp_C\rangle$ of second order parameters, the sequent

$$C{:}Cat \rightarrow \langle \mathbb{I}d[C], C, C\rangle{:}Ar$$

is derivable.

Proof of Lemma 5.2.1

Let Ax[G,A,B] be the result of replacing **F** by G, $\langle \mathbf{Ar_C}, \mathbf{=_{aC}}, \mathbf{Sr_C}, \mathbf{Tg_C}, \mathbf{Cp_C}\rangle$ by A, and $\langle \mathbf{Ar_D}, \mathbf{=_{aD}}, \mathbf{Sr_D}, \mathbf{Tg_D}, \mathbf{Cp_D}\rangle$ by B in an axiom of (f1) to (f9). From the definition of **Ar**, it is obvious that a proof of the lemma can be obtained from a derivation of the sequent

$$C{:}Cat \rightarrow Functor[\ \mathbb{I}d[C], C, C\ ]$$

by a single application of $\rightarrow\{\}$. The latter derivation can in turn be obtained if for each axiom (f1) to (f10) a derivation for the sequent

$$C{:}Cat \rightarrow Ax[\ \mathbb{I}d[C], C, C\ ] \qquad \text{(L1)}$$

is provided. Derivations of the last sequent will be given for axioms (f3), (f4), (f7) and (f9) only, the rest being either similar or trivial.

5.2.1.1 Sequent (L1) is derivable when Ax is (f3).

A derivation in which f is a first order parameter follows:

(i) $f =_{aC} f \rightarrow f =_{aC} f$ (axiom)

(ii) $f{:}Ar_C \rightarrow f{:}Ar_C$ (axiom)

$*[\forall g{:}Ar_C]\ g =_{aC} g,\ f{:}Ar_C \rightarrow f =_{aC} f$

--- thinning

$*C{:}Cat,\ f{:}Ar_C \rightarrow f =_{aC} f$

(i) $C{:}Cat, f{:}Ar_C \rightarrow *\langle f,f\rangle{:}\mathbb{I}d[C]$

(ii) $f{:}Ar_C \rightarrow f{:}Ar_C$ (axiom)

$C{:}Cat,\ f{:}Ar_C \rightarrow *[\exists fd{:}Ar_C]\ \langle f,fd\rangle{:}\mathbb{I}d[C]$

$C{:}Cat \rightarrow *[\forall fc{:}Ar_C][\ \exists fd{:}Ar_C]\ \langle fc,fd\rangle{:}\mathbb{I}d[C]$

5.2.1.2 Sequent (L1) is derivable when Ax is (f4).

In the following derivation f1, f2, f3 and f4 are first order parameters:

(i) $f3 =_{aC} f1 \rightarrow f3 =_{aC} f1$ (axiom)

(ii) $f1 =_{aC} f4 \rightarrow f1 =_{aC} f4$ (axiom)

(iii) $f3 =_{aC} f4 \rightarrow f3 =_{aC} f4$ (axiom)

(i) $*(f3 =_{aC} f1 \wedge f1 =_{aC} f4 \supset f3 =_{aC} f4),$

$$f3 =_{aC} f1,\ f1 =_{aC} f4 \rightarrow f3 =_{aC} f4$$

(ii) $f1{:}Ar_C \rightarrow f1{:}Ar_C$ (axiom)

(iii) $f3{:}Ar_C \rightarrow f3{:}Ar_C$ (axiom)

(iv) $f4{:}Ar_C \rightarrow f4{:}Ar_C$ (axiom)

(i) $*[\forall f,g,h{:}Ar_C](f =_{aC} g \wedge g =_{aC} h \supset f =_{aC} h),$

$f1{:}Ar_C,\ f3{:}Ar_C,\ f4{:}Ar_C,\ f3 =_{aC} f1,\ f1 =_{aC} f4 \rightarrow f3 =_{aC} f4$

(ii) $f1 =_{aC} f3 \rightarrow f1 =_{aC} f3$ (axiom)

(i) $[\forall f,g,h{:}Ar_C](f =_{aC} g \wedge g =_{aC} h \supset f =_{aC} h),$

$*(f1 =_{aC} f3 \supset f3 =_{aC} f1),\ f1{:}Ar_C,\ f3{:}Ar_C,\ f4{:}Ar_C,$

$$f1 =_{aC} f3,\ f1 =_{aC} f4 \rightarrow f3 =_{aC} f4$$

(ii) $f1{:}Ar_C \rightarrow f1{:}Ar_C$ (axiom)

(iii) $f3{:}Ar_C \rightarrow f3{:}Ar_C$ (axiom)

(i) $[\forall f,g,h{:}Ar_C](f =_{aC} g \wedge g =_{aC} h \supset f =_{aC} h),$

$*[\forall f,g{:}Ar_C](f =_{aC} g \supset g =_{aC} f),$

$f1{:}Ar_C,\ f3{:}Ar_C,\ f4{:}Ar_C,$

$f1 =_{aC} f4,\ f1 =_{aC} f3 \rightarrow f3 =_{aC} f4$

(ii) $f1 =_{aC} f2 \rightarrow f1 =_{aC} f2$ (axiom)

(iii) $f2 =_{aC} f4 \rightarrow f2 =_{aC} f4$ (axiom)

(i) $[\forall f,g,h{:}Ar_C](f =_{aC} g \wedge g =_{aC} h \supset f =_{aC} h),$

$*(f1 =_{aC} f2 \wedge f2 =_{aC} f4 \supset f1 =_{aC} f4),$

$[\forall f,g{:}Ar_C](\ f =_{aC} g \supset g =_{aC} f)$,

$f1{:}Ar_C,\ f3{:}Ar_C,\ f4{:}Ar_C$,

$f1 =_{aC} f2,\ f1 =_{aC} f3,\ f2 =_{aC} f4 \ \rightarrow\ f3 =_{aC} f4$

(ii) $f1{:}Ar_C \rightarrow f1{:}Ar_C$ (axiom)

(iii) $f2{:}Ar_C \rightarrow f2{:}Ar_C$ (axiom)

(iv) $f4{:}Ar_C \rightarrow f4{:}Ar_C$ (axiom)

$*[\forall f,g,h{:}Ar_C](\ f =_{aC} g \wedge g =_{aC} h \supset f =_{aC} h)$,

$[\forall f,g{:}Ar_C](\ f =_{aC} g \supset g =_{aC} f)$,

$f1{:}Ar_C,\ f2{:}Ar_C,\ f3{:}Ar_C,\ f4{:}Ar_C$,

$f1 =_{aC} f2,\ f1 =_{aC} f3,\ f2 =_{aC} f4 \ \rightarrow\ f3 =_{aC} f4$

---------thinning

$*C{:}Cat,\ f1{:}Ar_C,\ f2{:}Ar_C,\ f3{:}Ar_C,\ f4{:}Ar_C$,

$f1 =_{aC} f2,\ f1 =_{aC} f3,\ f2 =_{aC} f4 \ \rightarrow\ f3 =_{aC} f4$

$C{:}Cat,\ f1{:}Ar_C,\ f2{:}Ar_C,\ f3{:}Ar_C,\ f4{:}Ar_C$,

$f1 =_{aC} f2,\ *\langle f1,f3\rangle{:}\mathbb{I}\mathrm{d}[C],\ *\langle f2,f4\rangle{:}\mathbb{I}\mathrm{d}[C] \ \rightarrow\ f3 =_{aC} f4$

$C{:}Cat,\ f1{:}Ar_C,\ f2{:}Ar_C,\ f3{:}Ar_C,\ f4{:}Ar_C \ \rightarrow$

$*(f1 =_{aC} f2 \wedge \langle f1,f3\rangle{:}\mathbb{I}\mathrm{d}[C] \wedge \langle f2,f4\rangle{:}\mathbb{I}\mathrm{d}[C] \supset f3 =_{aC} f4\ )$

$C{:}Cat \ \rightarrow\ *[\forall fc,gc,fd,\ gd{:}Ar_C]($

$fc =_{aC} gc \wedge \langle fc,fd\rangle{:}\mathbb{I}\mathrm{d}[C] \wedge \langle gc,gd\rangle{:}\mathbb{I}\mathrm{d}[C] \supset fd =_{aC} gd\ )$

5.2.1.3 Sequent (L1) is derivable when Ax is (f7).

In the following derivation f1, f2, a1 and a2 are first order parameters:

(i) $a1 =_{aC} a2 \ \rightarrow\ a1 =_{aC} a2$ (axiom)

(ii) $\langle f2,\ a1\rangle{:}Sr_C \rightarrow \langle f2,\ a1\rangle{:}Sr_C$ (axiom)

(iii) $\langle f2,\ a2\rangle{:}Sr_C \rightarrow \langle f2,\ a2\rangle{:}Sr_C$ (axiom)

(i) $*(a1 =_{aC} a2 \wedge \langle f2, a1\rangle{:}Sr_C \supset \langle f2, a2\rangle{:}Sr_C)$,

$\langle f2,a1\rangle{:}Sr_C,\ a1 =_{aC} a2 \rightarrow \langle f2,a2\rangle{:}Sr_C$

(ii) $f2{:}Ar_C \rightarrow f2{:}Ar_C$ (axiom)

(iii) $a1{:}Ar_C \rightarrow a1{:}Ar_C$ (axiom)

(iv) $a2{:}Ar_C \rightarrow a2{:}Ar_C$ (axiom)

(i) $*[\forall f,g,h{:}Ar_C](g =_{aC} h \wedge \langle f, g\rangle{:}Sr_C \supset \langle f, h\rangle{:}Sr_C)$,

$a1{:}Ar_C, f2{:}Ar_C, a2{:}Ar_C,$

$\langle f2,a1\rangle{:}Sr_C, a1 =_{aC} a2 \rightarrow \langle f2,a2\rangle{:}Sr_C$

(ii) $f1 =_{aC} f2 \rightarrow f1 =_{aC} f2$ (axiom)

(iii) $\langle f1, a1\rangle{:}Sr_C \rightarrow \langle f1, a1\rangle{:}Sr_C$ (axiom)

(i) $*(f1 =_{aC} f2 \wedge \langle f1, a1\rangle{:}Sr_C \supset \langle f2, a1\rangle{:}Sr_C)$,

$[\forall f,g,h{:}Ar_C](g =_{aC} h \wedge \langle f, g\rangle{:}Sr_C \supset \langle f, h\rangle{:}Sr_C)$,

$a1{:}Ar_C,\ f2{:}Ar_C,\ a2{:}Ar_C,$

$\langle f1,a1\rangle{:}Sr_C,\ f1 =_{aC} f2,\ a1 =_{aC} a2 \rightarrow \langle f2,a2\rangle{:}Sr_C$

(ii) $f1{:}Ar_C \rightarrow f1{:}Ar_C$ (axiom)

(iii) $f2{:}Ar_C \rightarrow f2{:}Ar_C$ (axiom)

(iv) $a1{:}Ar_C \rightarrow a1{:}Ar_C$ (axiom)

$*[\forall f,g,h{:}Ar_C](f =_{aC} g \wedge \langle f, h\rangle{:}Sr_C \supset \langle g, h\rangle{:}Sr_C)$,

$[\forall f,g,h{:}Ar_C](g =_{aC} h \wedge \langle f, g\rangle{:}Sr_C \supset \langle f, h\rangle{:}Sr_C)$,

$f1{:}Ar_C,\ a1{:}Ar_C,\ f2{:}Ar_C,\ a2{:}Ar_C,$

$\langle f1,a1\rangle{:}Sr_C,\ f1 =_{aC} f2,\ a1 =_{aC} a2 \rightarrow \langle f2,a2\rangle{:}Sr_C$

-----------thinning

$*C{:}Cat,\ f1{:}Ar_C,\ a1{:}Ar_C,\ f2{:}Ar_C,\ a2{:}Ar_C,$

$\langle f1,a1\rangle{:}Sr_C,\ f1 =_{aC} f2,\ a1 =_{aC} a2 \rightarrow \langle f2,a2\rangle{:}Sr_C$

$C{:}Cat,\ f1{:}Ar_C,\ a1{:}Ar_C,\ f2{:}Ar_C,\ a2{:}Ar_C,$

$\langle f1,a1\rangle{:}Sr_C,\ *\langle f1,f2\rangle{:}\mathbb{I}\mathrm{d}[C],\ *\langle a1,a2\rangle{:}\mathbb{I}\mathrm{d}[C] \rightarrow \langle f2,a2\rangle{:}Sr_C$

$C{:}Cat,\ f1{:}Ar_C,\ a1{:}Ar_C,\ f2{:}Ar_C,\ a2{:}Ar_C \rightarrow$

$$*(\ \langle f1,a1\rangle{:}Sr_C \wedge \langle f1,f2\rangle{:}\mathbb{I}d[C]\ \wedge \langle a1,a2\rangle{:}\mathbb{I}d[C]\ \supset \langle f2,a2\rangle{:}Sr_C\ )$$

$C{:}Cat \rightarrow *[\forall fc,c,fd,d{:}Ar_C]\ ($

$$\langle fc,c\rangle{:}Sr_C \wedge \langle fc,fd\rangle{:}\mathbb{I}d[C]\ \wedge \langle c,d\rangle{:}\mathbb{I}d[C]\ \supset \langle fd,d\rangle{:}Sr_C\ )$$

5.2.1.4 Sequent (L1) is derivable when Ax is (f9).

A derivation of this sequent is provided in which f1, f2, f3, g1, g2 and g3 are first order parameters:

(i) $\langle g1,g2,g3\rangle{:}Cp_C \rightarrow \langle g1,g2,g3\rangle{:}Cp_C$ (axiom)

(ii) $g1{:}Ar_C \rightarrow g1{:}Ar_C$ (axiom)

(iii) $g2{:}Ar_C \rightarrow g2{:}Ar_C$ (axiom)

(iv) $f3{:}Ar_C \rightarrow f3{:}Ar_C$ (axiom)

(v) $g3{:}Ar_C \rightarrow g3{:}Ar_C$ (axiom)

(vi) $f3 =_{aC} g3 \rightarrow f3 =_{aC} g3$ (axiom)

(vii) $\langle g1,g2,f3\rangle{:}Cp_C \rightarrow \langle g1,g2,f3\rangle{:}Cp_C$ (axiom)

(i) $*[\forall f,g,h,k{:}Ar_C]*(h =_{aC} k\ \wedge\ \langle f,g,h\rangle{:}Cp_C \supset \langle f,g,k\rangle{:}Cp_C\ ),$

$f3{:}Ar_C,\ g1{:}Ar_C,\ g2{:}Ar_C,\ g3{:}Ar_C,\ \langle g1,g2,f3\rangle{:}Cp_C,$

$f3 =_{aC} g3 \rightarrow \langle g1,g2,g3\rangle{:}Cp_C$

(ii) $g1{:}Ar_C \rightarrow g1{:}Ar_C$ (axiom)

(iii) $f2{:}Ar_C \rightarrow f2{:}Ar_C$ (axiom)

(iv) $f3{:}Ar_C \rightarrow f3{:}Ar_C$ (axiom)

(v) $g2{:}Ar_C \rightarrow g2{:}Ar_C$ (axiom)

(vi) $f2 =_{aC} g2 \rightarrow f2 =_{aC} g2$ (axiom)

(vii) $\langle g1,f2,f3\rangle{:}Cp_C \rightarrow \langle g1,f2,f3\rangle{:}Cp_C$ (axiom)

(i) $*[\forall f,g,h,k{:}Ar_C]*(\ g =_{aC} k\ \wedge\ \langle f,g,h\rangle{:}Cp_C \supset \langle f,k,h\rangle{:}Cp_C\ ),$

$[\forall f,g,h,k{:}Ar_C](h =_{aC} k\ \wedge\ \langle f,g,h\rangle{:}Cp_C \supset \langle f,g,k\rangle{:}Cp_C\ ),$

$f2{:}Ar_C,\ f3{:}Ar_C,\ g1{:}Ar_C,\ g2{:}Ar_C,\ g3{:}Ar_C,\ \langle g1,f2,f3\rangle{:}Cp_C,$

$f2 =_{aC} g2,\ f3 =_{aC} g3\ \rightarrow\ \langle g1,g2,g3\rangle{:}Cp_C$

(ii) $f1{:}Ar_C \rightarrow f1{:}Ar_C$ (axiom)

(iii) $f2{:}Ar_C \rightarrow f2{:}Ar_C$ (axiom)

(iv) $f3{:}Ar_C \rightarrow f3{:}Ar_C$ (axiom)

(v) $g1{:}Ar_C \rightarrow g1{:}Ar_C$ (axiom)

(vi) $f1 =_{aC} g1 \rightarrow f1 =_{aC} g1$ (axiom)

(vii) $\langle f1,f2,f3\rangle{:}Cp_C \rightarrow \langle f1,f2,f3\rangle{:}Cp_C$ (axiom)

$*[\forall f,g,h,k{:}Ar_C]*(f =_{aC} k \wedge \langle f,g,h\rangle{:}Cp_C \supset \langle k,g,h\rangle{:}Cp_C),$

$[\forall f,g,h,k{:}Ar_C](g =_{aC} k \wedge \langle f,g,h\rangle{:}Cp_C \supset \langle f,k,h\rangle{:}Cp_C),$

$[\forall f,g,h,k{:}Ar_C](h =_{aC} k \wedge \langle f,g,h\rangle{:}Cp_C \supset \langle f,g,k\rangle{:}Cp_C),$

$f1{:}Ar_C,\ f2{:}Ar_C,\ f3{:}Ar_C,\ g1{:}Ar_C,\ g2{:}Ar_C,\ g3{:}Ar_C,\ \langle f1,f2,f3\rangle{:}Cp_C,$

$f1 =_{aC} g1,\ f2 =_{aC} g2,\ f3 =_{aC} g3\ \rightarrow\ \langle g1,g2,g3\rangle{:}Cp_C$

-----------thinning

$*C{:}Cat,\ f1{:}Ar_C,\ f2{:}Ar_C,\ f3{:}Ar_C,\ g1{:}Ar_C,\ g2{:}Ar_C,\ g3{:}Ar_C,\ \langle f1,f2,f3\rangle{:}Cp_C,$

$f1 =_{aC} g1,\ f2 =_{aC} g2,\ f3 =_{aC} g3\ \rightarrow\ \langle g1,g2,g3\rangle{:}Cp_C$

$C{:}Cat,\ f1{:}Ar_C,\ f2{:}Ar_C,\ f3{:}Ar_C,\ g1{:}Ar_C,\ g2{:}Ar_C,\ g3{:}Ar_C,\ \langle f1,f2,f3\rangle{:}Cp_C,$

$*\langle f1,g1\rangle{:}\mathbb{I}\mathrm{d}[C],\ *\langle f2,g2\rangle{:}\mathbb{I}\mathrm{d}[C],\ *\langle f3,g3\rangle{:}\mathbb{I}\mathrm{d}[C]\ \rightarrow\ \langle g1,g2,g3\rangle{:}Cp_C$

$C{:}Cat,\ f1{:}Ar_C,\ f2{:}Ar_C,\ f3{:}Ar_C,\ g1{:}Ar_C,\ g2{:}Ar_C,\ g3{:}Ar_C,\ *(\langle f1,f2,f3\rangle{:}Cp_C \wedge$

$\langle f1,g1\rangle{:}\mathbb{I}\mathrm{d}[C] \wedge \langle f2,g2\rangle{:}\mathbb{I}\mathrm{d}[C] \wedge \langle f3,g3\rangle{:}\mathbb{I}\mathrm{d}[C] \supset \langle g1,g2,g3\rangle{:}Cp_C)$

$C{:}Cat\ \rightarrow\ *[\forall fc1,fc2,fc3,fd1,fd2,fd3{:}Ar_C](\langle fc1,fc2,fc3\rangle{:}Cp_C \wedge$

$\langle fc1,fd1\rangle{:}\mathbb{I}\mathrm{d}[C] \wedge \langle fc2,fd2\rangle{:}\mathbb{I}\mathrm{d}[C] \wedge \langle fc3,fd3\rangle{:}\mathbb{I}\mathrm{d}[C] \supset \langle fd1,fd2,fd3\rangle{:}Cp_C)$

End of proof of lemma 5.2.1

5.3 Composition Functors

The next lemma states that if two functors are composable, their composite is also a functor.

5.3.1 Lemma.

Let

FC[F1,C1,D1,F2,C2,D2] for

{< x,g> | [∃f:Ar$_{\mathbf{D1}}$](< x,f>:F1 ∧ <f, g >:F2) } .

The sequent

<F1,C1,D1>:Ar, <F2,C2,D2>:Ar, D1=$_e$C2 →

< FC[F1,C1,D1,F2,C2,D2], C1, D2 >:Ar

where F1, F2 are second order parameters and C1, D1, C1, D1 are the usual tuples of second order parameters, is derivable.

Proof of Lemma 5.3.1.

Clearly, the sequent of the lemma can be derived from the sequent

Functor[F1,C1,D1], Functor[F2,C2,D2] D1=$_e$C2 →

Functor[FC[F1,C1,D1,F2,C2,D2], C1, D2]

by two applications of { }→ and one of →{ }.

Let Ax[G,A,B] be as in the proof of lemma 5.2.1. From the functor definition it is obvious that a proof of the latter sequent can be obtained if for each axiom (f1) to (f9) a derivation for the sequent

Ax[F1,C1,D1], Ax[F2,C2,D2] D1=$_e$C2 →

Ax[FC[F1,C1,D1,F2,C2,D2], C1, D2] (L2)

is provided. Derivations for this sequent will be given for the axioms (f3), (f4), (f7) and (f9) only, the rest being either similar or trivial.

5.3.1.1 Sequent (L2) is derivable when Ax is (f3).

In the following derivation pc, pd and pe are first order parameters:

(i) <pc,pd>:F1 → <pc,pd>:F1 (axiom)
(ii) <pd,pe>:F2 → <pd,pe>:F2 (axiom)
(iii) pd:Ar$_{D1}$ → pd:Ar$_{D1}$ (axiom)

$pd{:}Ar_{D1}, \langle pc,pd\rangle{:}F1, \langle pd,pe\rangle{:}F2 \rightarrow *[\exists d{:}Ar_{D1}]*(\langle pc,d\rangle{:}F1 \wedge \langle d,pe\rangle{:}F2)$

(i) $pd{:}Ar_{D1}, \langle pc,pd\rangle{:}F1, \langle pd,pe\rangle{:}F2 \rightarrow *\langle pc,pe\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2]$

(ii) $pe{:}Ar_{D2} \rightarrow pe{:}Ar_{D2}$ (axiom)

$pd{:}Ar_{D1}, \langle pc,pd\rangle{:}F1, pe{:}Ar_{D2}, \langle pd,pe\rangle{:}F2$

$\rightarrow *[\exists d{:}Ar_{D2}]\ \langle pc,d\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2]$

(i) $pd{:}Ar_{D1}, \langle pc,pd\rangle{:}F1,\ *[\exists d{:}Ar_{D2}]\ \langle pd,d\rangle{:}F2$

$\rightarrow [\exists d{:}Ar_{D2}]\ \langle pc,d\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2]$

(ii) $pd{:}Ar_{D1},\ D1{=_e}C2 \rightarrow pd{:}Ar_{C2}$ (lemma 5.1.1)

$pd{:}Ar_{D1}, \langle pc,pd\rangle{:}F1,\ *[\forall c{:}Ar_{C2}][\exists d{:}Ar_{D2}]\ \langle c,d\rangle{:}F2,\ D1{=_e}C2$

$\rightarrow [\exists d{:}Ar_{D2}]\ \langle pc,d\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2]$

(i) $*[\exists d{:}Ar_{D1}]\ \langle pc,d\rangle{:}F1, [\forall c{:}Ar_{C2}][\exists d{:}Ar_{D2}]\ \langle c,d\rangle{:}F2,\ D1{=_e}C2$

$\rightarrow [\exists d{:}Ar_{D2}]\ \langle pc,d\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2]$

(ii) $pc{:}Ar_{C1} \rightarrow pc{:}Ar_{C1}$ (axiom)

$*[\forall c{:}Ar_{C1}][\exists d{:}Ar_{D1}]\ \langle c,d\rangle{:}F1, [\forall c{:}Ar_{C2}][\exists d{:}Ar_{D2}]\ \langle c,d\rangle{:}F2,\ D1{=_e}C2$

$pc{:}Ar_{C1} \rightarrow [\exists d{:}Ar_{D2}]\ \langle pc,d\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2]$

$[\forall c{:}Ar_{C1}][\exists fd{:}Ar_{D1}]\ \langle c,d\rangle{:}F1, [\forall c{:}Ar_{C2}][\exists d{:}Ar_{D2}]\ \langle c,d\rangle{:}F2,\ D1{=_e}C2$

$\rightarrow *[\forall c{:}Ar_{C1}][\exists d{:}Ar_{D2}]\ \langle c,d\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2]$

5.3.1.2 Sequent (L2) is derivable when Ax is (f4).

In the following derivation pc1, pc2, pd1, pd2, pdd1 and pdd2 are first order parameters:

(i) $pc1{=_{aC1}}pc2 \rightarrow pc1{=_{aC1}}pc2$ (axiom)

(ii) $\langle pc1,pdd1\rangle{:}F1 \rightarrow \langle pc1,pdd1\rangle{:}F1$ (axiom)

(iii) $\langle pc2,pdd2\rangle{:}F1 \rightarrow \langle pc2,pdd2\rangle{:}F1$ (axiom)

(i) $pc1{=_{aC1}}pc2, \langle pc1,pdd1\rangle{:}F1, \langle pc2,pdd2\rangle{:}F1,$

$\rightarrow *(pc1 =_{aC1} pc2 \wedge \langle pc1, pdd1\rangle : F1 \wedge \langle pc2, pdd2\rangle : F1)$

(ii) $pdd1 =_{aD1} pdd2,\ =_{aD1} =_e =_{aC2}\ \rightarrow pdd1 =_{aC2} pdd2$ (lemma 5.1.1)

(i) $*(pc1 =_{aC1} pc2 \wedge \langle pc1, pdd1\rangle : F1 \wedge \langle pc2, pdd2\rangle : F1 \supset pdd1 =_{aD1} pdd2),$

$=_{aD1} =_e =_{aC2},\ pc1 =_{aC1} pc2,$

$\langle pc1, pdd1\rangle : F1,\ \langle pc2, pdd2\rangle : F1,$

$\rightarrow pdd1 =_{aC2} pdd2$

(ii) $\langle pdd1, pd1\rangle : F2 \rightarrow \langle pdd1, pd1\rangle : F2$ (axiom)

(iii) $\langle pdd2, pd2\rangle : F2 \rightarrow \langle pdd2, pd2\rangle : F2$ (axiom)

(i) $(pc1 =_{aC1} pc2 \wedge \langle pc1, pdd1\rangle : F1 \wedge \langle pc2, pdd2\rangle : F1 \supset pdd1 =_{aD1} pdd2),$

$=_{aD1} =_e =_{aC2},\ pc1 =_{aC1} pc2,$

$\langle pc1, pdd1\rangle : F1,\ \langle pdd1, pd1\rangle : F2,\ \langle pc2, pdd2\rangle : F1,\ \langle pdd2, pd2\rangle : F2$

$\rightarrow *(pdd1 =_{aC2} pdd2 \wedge \langle pdd1, pd1\rangle : F2 \wedge \langle pdd2, pd2\rangle : F2)$

(ii) $pd1 =_{aD2} pd2 \rightarrow pd1 =_{aD2} pd2$ (axiom)

(i) $(pc1 =_{aC1} pc2 \wedge \langle pc1, pdd1\rangle : F1 \wedge \langle pc2, pdd2\rangle : F1 \supset pdd1 =_{aD1} pdd2),$

$*(pdd1 =_{aC2} pdd2 \wedge \langle pdd1, pd1\rangle : F2 \wedge \langle pdd2, pd2\rangle : F2 \supset pd1 =_{aD2} pd2),$

$=_{aD1} =_e =_{aC2},\ pc1 =_{aC1} pc2,$

$\langle pc1, pdd1\rangle : F1,\ \langle pdd1, pd1\rangle : F2,\ \langle pc2, pdd2\rangle : F1,\ \langle pdd2, pd2\rangle : F2$

$\rightarrow pd1 =_{aD2} pd2$

(ii) $pdd1 : Ar_{D1},\ Ar_{D1} =_e Ar_{C2} \rightarrow pdd1 : Ar_{C2}$ (lemma 5.1.1)

(iii) $pdd2 : Ar_{D1},\ Ar_{D1} =_e Ar_{C2} \rightarrow pdd2 : Ar_{C2}$ (lemma 5.1.1)

(iv) $pd1 : Ar_{D2} \rightarrow pd1 : Ar_{D2}$ (axiom)

(v) $pd2 : Ar_{D2} \rightarrow pd2 : Ar_{D2}$ (axiom)

---

thinning

(i) $(pc1 =_{aC1} pc2 \wedge \langle pc1, pdd1\rangle : F1 \wedge \langle pc2, pdd2\rangle : F1 \supset pdd1 =_{aD1} pdd2),$

$*[\forall c1, c2 : Ar_{C2}]\ *[\forall d1, d2 : Ar_{D2}]($

$c1 =_{aC2} c2 \wedge \langle c1, d1\rangle : F2 \wedge \langle c2, d2\rangle : F2 \supset d1 =_{aD2} d2),\ *D1 =_e C2,$

$pc1 : Ar_{C1},\ pc2 : Ar_{C1},\ pd1 : Ar_{D2},\ pd2 : Ar_{D2},\ pc1 =_{aC1} pc2,$

$pdd1 : Ar_{D1},\ \langle pc1, pdd1\rangle : F1,\ \langle pdd1, pd1\rangle : F2,$

$pdd2{:}Ar_{D1},\ \langle pc2,pdd2\rangle{:}F1,\ \langle pdd2, pd2\rangle{:}F2$

$\to\ pd1 =_{aD2} pd2$

(ii) $pdd1{:}Ar_{D1} \to pdd1{:}Ar_{D1}$ (axiom)

(iii) $pdd2{:}Ar_{D1} \to pdd2{:}Ar_{D1}$ (axiom)

---

*$[\forall d1,d2{:}Ar_{D1}]($

$pc1 =_{aC1} pc2 \wedge \langle pc1,d1\rangle{:}F1 \wedge \langle pc2,d2\rangle{:}F1 \supset d1 =_{aD1} d2\ ),$

$[\forall c1,c2{:}Ar_{C2}]\ [\forall d1,d2{:}Ar_{D2}]($

$c1 =_{aC2} c2 \wedge \langle c1,d1\rangle{:}F2 \wedge \langle c2,d2\rangle{:}F2 \supset d1 =_{aD2} d2\ ),\ D1 =_{e} C2,$

$pc1{:}Ar_{C1},\ pc2{:}Ar_{C1},\ pd1{:}\ Ar_{D2},\ pd2{:}Ar_{D2},\ pc1 =_{aC1} pc2,$

$pdd1{:}Ar_{D1},\ \langle pc1,pdd1\rangle{:}F1,\ \langle pdd1, pd1\rangle{:}F2,$

$pdd2{:}Ar_{D1},\ \langle pc2,pdd2\rangle{:}F1,\ \langle pdd2, pd2\rangle{:}F2$

$\to\ pd1 =_{aD2} pd2$

$[\forall d1,d2{:}Ar_{D1}]($

$pc1 =_{aC1} pc2 \wedge \langle pc1,d1\rangle{:}F1 \wedge \langle pc2,d2\rangle{:}F1 \supset d1 =_{aD1} d2\ ),$

$[\forall c1,c2{:}Ar_{C2}]\ [\forall d1,d2{:}Ar_{D2}]($

$c1 =_{aC2} c2 \wedge \langle c1,d1\rangle{:}F2 \wedge \langle c2,d2\rangle{:}F2 \supset d1 =_{aD2} d2\ ),\ D1 =_{e} C2,$

$pc1{:}Ar_{C1},\ pc2{:}Ar_{C1},\ pd1{:}\ Ar_{D2},\ pd2{:}Ar_{D2},\ pc1 =_{aC1} pc2,$

*$[\ \exists d{:}Ar_{D1}](\langle pc1,d\rangle{:}F1 \wedge \langle d, pd1\rangle{:}F2),$

*$[\ \exists d{:}Ar_{D1}](\langle pc2,d\rangle{:}F1 \wedge \langle d, pd2\rangle{:}F2)$

$\to\ pd1 =_{aD2} pd2$

(i) $[\forall d1,d2{:}Ar_{D1}]($

$pc1 =_{aC1} pc2 \wedge \langle pc1,d1\rangle{:}F1 \wedge \langle pc2,d2\rangle{:}F1 \supset d1 =_{aD1} d2\ ),$

$[\forall c1,c2{:}Ar_{C2}]\ [\forall d1,d2{:}Ar_{D2}]($

$c1 =_{aC2} c2 \wedge \langle c1,d1\rangle{:}F2 \wedge \langle c2,d2\rangle{:}F2 \supset d1 =_{aD2} d2\ ),\ D1 =_{e} C2,$

$pc1{:}Ar_{C1},\ pc2{:}Ar_{C1},\ pd1{:}\ Ar_{D2},\ pd2{:}Ar_{D2},\ pc1 =_{aC1} pc2,$

*$\langle pc1,pd1\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2]$, *$\langle pc2,pd2\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2]$

$\to\ pd1 =_{aD2} pd2$

(ii) $pc1{:}Ar_{C1} \rightarrow pc1{:}Ar_{C1}$ (axiom)

(iii) $pc2{:}Ar_{C1} \rightarrow pc2{:}Ar_{C1}$ (axiom)

$$*[\forall c1,c2{:}Ar_{C1}]\,[\forall d1,d2{:}Ar_{D1}](\; c1 =_{aC1} c2 \wedge \langle c1,d1\rangle{:}F1 \wedge \langle c2,d2\rangle{:}F1 \supset d1 =_{aD1} d2\,),$$

$$[\forall c1,c2{:}Ar_{C2}]\,[\forall d1,d2{:}Ar_{D2}](\; c1 =_{aC2} c2 \wedge \langle c1,d1\rangle{:}F2 \wedge \langle c2,d2\rangle{:}F2 \supset d1 =_{aD2} d2\,),\; D1 =_e C2,$$

$$pc1{:}Ar_{C1},\; pc2{:}Ar_{C1},\; pd1{:}\,Ar_{D2},\; pd2{:}Ar_{D2}$$

$$\rightarrow *(pc1 =_{aC1} pc2 \wedge \langle pc1,pd1\rangle{:}\, \mathbb{FC}[F1,C1,D1,F2,C2,D2] \wedge \langle pc2,pd2\rangle{:}\, \mathbb{FC}[F1,C1,D1,F2,C2,D2] \supset pd1 =_{aD2} pd2\,)$$

$$[\forall c1,c2{:}Ar_{C1}]\,[\forall d1,d2{:}Ar_{D1}](\; c1 =_{aC1} c2 \wedge \langle c1,d1\rangle{:}F1 \wedge \langle c2,d2\rangle{:}F1 \supset d1 =_{aD1} d2\,),$$

$$[\forall c1,c2{:}Ar_{C2}]\,[\forall d1,d2{:}Ar_{D2}](\; c1 =_{aC2} c2 \wedge \langle c1,d1\rangle{:}F2 \wedge \langle c2,d2\rangle{:}F2 \supset d1 =_{aD2} d2\,),\; D1 =_e C2$$

$$\rightarrow *[\forall c1,c2{:}Ar_{C1}]\; *[\forall d1,d2{:}Ar_{D2}](\; c1 =_{aC1} c2 \wedge \langle c1,d1\rangle{:}\, \mathbb{FC}[F1,C1,D1,F2,C2,D2] \wedge \langle c2,d2\rangle{:}\, \mathbb{FC}[F1,C1,D1,F2,C2,D2] \supset d1 =_{aD2} d2\,)$$

5.3.1.3 Sequent (L2) is derivable when Ax is (f7).
In the following derivation, the terms fd1, d1, fd2 and d2 are first order parameters:

(i) $\langle fd2,d2\rangle{:}Sr_{D2} \rightarrow \langle fd2,d2\rangle{:}Sr_{D2}$ (axiom)

(ii) $fd1{:}Ar_{D1},\; Ar_{D1} =_e Ar_{C2} \rightarrow fd1{:}\, Ar_{C2}$ (lemma 5.1.1)

(iii) $d1{:}Ar_{D1},\; Ar_{D1} =_e Ar_{C2} \rightarrow d1{:}\, Ar_{C2}$ (lemma 5.1.1)

(iv) $fd2{:}\, Ar_{D2} \rightarrow fd2{:}\, Ar_{D2}$ (axiom)

(v) $d2{:}\, Ar_{D2} \rightarrow d2{:}\, Ar_{D2}$ (axiom)

(vi) $\langle fd1,d1\rangle{:}Sr_{D1},\; Sr_{D1} =_e Sr_{C2} \rightarrow \langle fd1,d1\rangle{:}Sr_{C2}$ (lemma 5.1.1)

(vii) $\langle fd1,fd2\rangle{:}F2 \rightarrow \langle fd1,fd2\rangle{:}F2$ (axiom)

(viii) $\langle d1,d2\rangle{:}F2 \rightarrow \langle d1,d2\rangle{:}F2$ (axiom) thinning

(i) $*[\forall fc,c{:}Ar_{C2}]*[\forall fd,d{:}Ar_{D2}]*($

$\langle fc,c\rangle{:}Sr_{C2} \wedge \langle fc,fd\rangle{:}F2 \wedge \langle c,d\rangle{:}F2 \supset \langle fd,d\rangle{:}Sr_{D2}),\ *D1{=_e}C2,$

$\langle fd1,d1\rangle{:}Sr_{D1},$

$fd2{:}\ Ar_{D2},\ d2{:}Ar_{D2},\ d1{:}Ar_{D1},\ fd1{:}Ar_{D1},\ \langle fd1,fd2\rangle{:}F2,\ \langle d1,d2\rangle{:}F2$

$\rightarrow \langle fd2,d2\rangle{:}Sr_{D2}$

(ii) $fc1{:}\ Ar_{C1} \rightarrow fc1{:}\ Ar_{C1}$ (axiom)

(iii) $c1{:}\ Ar_{C1} \rightarrow c1{:}\ Ar_{C1}$ (axiom)

(iv) $fd1{:}\ Ar_{D1} \rightarrow fd1{:}\ Ar_{D1}$ (axiom)

(v) $d1{:}\ Ar_{D1} \rightarrow d1{:}\ Ar_{D1}$ (axiom)

(vi) $\langle fc1,c1\rangle{:}Sr_{C1} \rightarrow \langle fc1,c1\rangle{:}Sr_{C1}$ (axiom)

(vii) $\langle fc1,fd1\rangle{:}F1 \rightarrow \langle fc1,fd1\rangle{:}F1$ (axiom)

(viii) $\langle c1,d1\rangle{:}F1 \rightarrow \langle c1,d1\rangle{:}F1$ (axiom)

$*[\forall fc,c{:}Ar_{C1}]*[\forall fd,d{:}Ar_{D1}]*($

$\langle fc,c\rangle{:}Sr_{C1} \wedge \langle fc,fd\rangle{:}F1 \wedge \langle c,d\rangle{:}F1 \supset \langle fd,d\rangle{:}Sr_{D1}),$

$[\forall fc,c{:}Ar_{C2}][\forall fd,d{:}Ar_{D2}]\ ($

$\langle fc,c\rangle{:}Sr_{C2} \wedge \langle fc,fd\rangle{:}F2 \wedge \langle c,d\rangle{:}F2 \supset \langle fd,d\rangle{:}Sr_{D2}),\ D1{=_e}C2,$

$fc1{:}\ Ar_{C1},\ c1{:}Ar_{C1},\ fd2{:}\ Ar_{D2},\ d2{:}Ar_{D2},\ \langle fc1,c1\rangle{:}Sr_{C1},$

$fd1{:}Ar_{D1},\ \langle fc1,fd1\rangle{:}F1,\ \langle fd1,fd2\rangle{:}F2,\ d1{:}Ar_{D1},\ \langle c1,d1\rangle{:}F1,\ \langle d1,d2\rangle{:}F2$

$\rightarrow \langle fd2,d2\rangle{:}Sr_{D2})$

$[\forall fc,c{:}Ar_{C1}][\forall fd,d{:}Ar_{D1}]\ ($

$\langle fc,c\rangle{:}Sr_{C1} \wedge \langle fc,fd\rangle{:}F1 \wedge \langle c,d\rangle{:}F1 \supset \langle fd,d\rangle{:}Sr_{D1}),$

$[\forall fc,c{:}Ar_{C2}][\forall fd,d{:}Ar_{D2}]\ ($

$\langle fc,c\rangle{:}Sr_{C2} \wedge \langle fc,fd\rangle{:}F2 \wedge \langle c,d\rangle{:}F2 \supset \langle fd,d\rangle{:}Sr_{D2}),\ D1{=_e}C2,$

$fc1{:}\ Ar_{C1},\ c1{:}Ar_{C1},\ fd2{:}\ Ar_{D2},\ d2{:}Ar_{D2},\ \langle fc1,c1\rangle{:}Sr_{C1},$

$*[\exists f{:}Ar_{D1}]*(\langle fc1,f\rangle{:}F1 \wedge \langle f,fd2\rangle{:}F2),\ *[\exists f{:}Ar_{D1}]*(\langle c1,f\rangle{:}F1 \wedge \langle f,d2\rangle{:}F2)$

$\rightarrow \langle fd2,d2\rangle{:}Sr_{D2})$

$[\forall fc,c{:}Ar_{C1}][\forall fd,d{:}Ar_{D1}]\ ($

$\langle fc,c\rangle{:}Sr_{C1} \wedge \langle fc,fd\rangle{:}F1 \wedge \langle c,d\rangle{:}F1 \supset \langle fd,d\rangle{:}Sr_{D1}),$

$[\forall fc,c{:}Ar_{C2}][\forall fd,d{:}Ar_{D2}]\ ($

$\langle fc,c\rangle{:}Sr_{C2} \wedge \langle fc,fd\rangle{:}F2 \wedge \langle c,d\rangle{:}F2 \supset \langle fd,d\rangle{:}Sr_{D2}),\ D1 =_e C2,$

$fc1{:}\ Ar_{C1},\ c1{:}Ar_{C1},\ fd2{:}\ Ar_{D2},\ d2{:}Ar_{D2},\ \langle fc1,c1\rangle{:}Sr_{C1},$

$*\langle fc1,fd2\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2],\ *\langle c1,d2\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2]$

$\rightarrow\ \langle fd2,d2\rangle{:}Sr_{D2})$

$[\forall fc,c{:}Ar_{C1}][\forall fd,d{:}Ar_{D1}]\ ($

$\langle fc,c\rangle{:}Sr_{C1} \wedge \langle fc,fd\rangle{:}F1 \wedge \langle c,d\rangle{:}F1 \supset \langle fd,d\rangle{:}Sr_{D1}),$

$[\forall fc,c{:}Ar_{C2}][\forall fd,d{:}Ar_{D2}]\ ($

$\langle fc,c\rangle{:}Sr_{C2} \wedge \langle fc,fd\rangle{:}F2 \wedge \langle c,d\rangle{:}F2 \supset \langle fd,d\rangle{:}Sr_{D2}),\ D1 =_e C2$

$\rightarrow\ *[\forall fc,c{:}Ar_{C1}]*[\forall fd,d{:}Ar_{D2}]\ *(\langle fc,c\rangle{:}Sr_{C1} \wedge$

$\langle fc,fd\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2] \wedge \langle c,d\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2]$

$\supset \langle fd,d\rangle{:}Sr_{D2})$

5.3.1.4 Sequent (L2) is derivable when Ax is (f9).

In the following derivation the first order parameters pc1, pc2, pc3, pd1, pd2, pd3, pe1, pe2 and pe3 are used:

(i) $pe1{:}Ar_{D1},\ Ar_D =_e Ar_{C2} \rightarrow pe1{:}\ Ar_{C2}$ (lemma 5.1.1)

(ii) $pe2{:}Ar_{D1},\ Ar_{D1} =_e Ar_{C2} \rightarrow pe2{:}\ Ar_{C2}$ (lemma 5.1.1)

(iii) $pe3{:}Ar_{D1},\ Ar_{D1} =_e Ar_{C2} \rightarrow pe3{:}\ Ar_{C2}$ (lemma 5.1.1)

(iv) $pd1{:}\ Ar_{D2} \rightarrow pd1{:}\ Ar_{D2}$ (axiom)

(v) $pd2{:}\ Ar_{D2} \rightarrow pd2{:}\ Ar_{D2}$ (axiom)

(vi) $pd3{:}\ Ar_{D2} \rightarrow pd3{:}\ Ar_{D2}$ (axiom)

(vii) $\langle pe1,pe2,pe3\rangle{:}Cp_{D1},\ Ar_{D1} =_e Ar_{C2} \rightarrow \langle pe1,pe2,pe3\rangle{:}Cp_{C2}$ (lemma 5.1.1)

(viii) $\langle pe1,pd1\rangle{:}F2 \rightarrow \langle pe1,pd1\rangle{:}F2$ (axiom)

(ix) $\langle pe2,pd2\rangle{:}F2 \rightarrow \langle pe2,pd2\rangle{:}F2$ (axiom)

(x) $\langle pe3,pd3\rangle{:}F2 \rightarrow \langle pe3,pd3\rangle{:}F2$ (axiom)

(xi) $\langle pd1,pd2,pd3\rangle{:}Cp_{D2} \rightarrow \langle pd1,pd2,pd3\rangle{:}Cp_{D2}$ (axiom) thinning

(i) $*[\forall fc1,fc2,fc3:Ar_{C2}]*[\forall fd1,fd2,fd3:Ar_{D2}]*(\langle fc1,fc2,fc3\rangle:Cp_{C2} \wedge$

$\langle fc1,fd1\rangle:F2 \wedge \langle fc2,fd2\rangle:F2 \wedge \langle fc3,fd3\rangle:F2 \supset \langle fd1,fd2,fd3\rangle:Cp_{D2}),\ *D1 =_{e} C2,$

$pd1: Ar_{D2},\ pd2: Ar_{D2},\ pd3: Ar_{D2},\ pe1:Ar_{D1},\ pe2:Ar_{D1},\ pe3:Ar_{D1},$

$\langle pe1,pe2,pe3\rangle:Cp_{D1},\ \langle pe1,pd1\rangle:F2,\ \langle pe2,pd2\rangle:F2,\ \langle pe3,pd3\rangle:F2$

$\rightarrow \langle pd1,pd2,pd3\rangle:Cp_{D2}$

(ii) $pc1: Ar_{C1} \rightarrow pc1: Ar_{C1}$ (axiom)

(iii) $pc2: Ar_{C1} \rightarrow pc2: Ar_{C1}$ (axiom)

(iv) $pc3: Ar_{C1} \rightarrow pc3: Ar_{C1}$ (axiom)

(v) $pe1: Ar_{C1} \rightarrow pe1: Ar_{C1}$ (axiom)

(vi) $pe2: Ar_{C1} \rightarrow pe2: Ar_{C1}$ (axiom)

(vii) $pe3: Ar_{C1} \rightarrow pe3: Ar_{C1}$ (axiom)

(viii) $\langle pc1,pc2,pc3\rangle:Cp_{C1} \rightarrow \langle pc1,pc2,pc3\rangle:Cp_{C1}$ (axiom)

(ix) $\langle pc1,pe1\rangle:F1 \rightarrow \langle pc1,pe1\rangle:F1$ (axiom)

(x) $\langle pc2,pe2\rangle:F1 \rightarrow \langle pc2,pe2\rangle:F1$ (axiom)

(xi) $\langle pc3,pe3\rangle:F1 \rightarrow \langle pc3,pe3\rangle:F1$ (axiom)

$[\forall fc1,fc2,fc3:Ar_{C1}]\ [\forall fd1,fd2,fd3:Ar_{D1}](\langle fc1,fc2,fc3\rangle:Cp_{C1} \wedge$

$\langle fc1,fd1\rangle:F1 \wedge \langle fc2,fd2\rangle:F1 \wedge \langle fc3,fd3\rangle:F1 \supset \langle fd1,fd2,fd3\rangle:Cp_{D1}),$

$[\forall fc1,fc2,fc3:Ar_{C2}]\ [\forall fd1,fd2,fd3:Ar_{D2}](\langle fc1,fc2,fc3\rangle:Cp_{C2} \wedge$

$\langle fc1,fd1\rangle:F2 \wedge \langle fc2,fd2\rangle:F2 \wedge \langle fc3,fd3\rangle:F2 \supset \langle fd1,fd2,fd3\rangle:Cp_{D2}),\ D1 =_{e} C2,$

$pc1: Ar_{C1},\ pc2: Ar_{C1},\ pc3: Ar_{C1},\ pd1: Ar_{D2},\ pd2: Ar_{D2},\ pd3: Ar_{D2},$

$\langle pc1,pc2,pc3\rangle:Cp_{C1},\ *[\exists f:Ar_{D1}]*(\langle pc1,f\rangle:F1 \wedge \langle f,pd1\rangle:F2),$

$*[\exists f:Ar_{D1}]*(\langle pc2,f\rangle:F1 \wedge \langle f,pd2\rangle:F2),\ *[\exists f:Ar_{D1}]*(\langle pc3,f\rangle:F1 \wedge \langle f,pd3\rangle:F2)$

$\rightarrow \langle pd1,pd2,pd3\rangle:Cp_{D2}$

$[\forall fc1,fc2,fc3:Ar_{C1}]\ [\forall fd1,fd2,fd3:Ar_{D1}](\langle fc1,fc2,fc3\rangle:Cp_{C1} \wedge$

$\langle fc1,fd1\rangle:F1 \wedge \langle fc2,fd2\rangle:F1 \wedge \langle fc3,fd3\rangle:F1 \supset \langle fd1,fd2,fd3\rangle:Cp_{D1}),$

$[\forall fc1,fc2,fc3:Ar_{C2}]\ [\forall fd1,fd2,fd3:Ar_{D2}](\langle fc1,fc2,fc3\rangle:Cp_{C2} \wedge$

$\langle fc1,fd1\rangle:F2 \wedge \langle fc2,fd2\rangle:F2 \wedge \langle fc3,fd3\rangle:F2 \supset \langle fd1,fd2,fd3\rangle:Cp_{D2}),\ D1 =_{e} C2,$

$pc1{:}\ Ar_{C1},\ pc2{:}\ Ar_{C1},\ pc3{:}\ Ar_{C1},\ pd1{:}\ Ar_{D2},\ pd2{:}\ Ar_{D2},\ pd3{:}\ Ar_{D2},$

$\langle pc1,pc2,pc3\rangle{:}Cp_{C1},\ {*}\langle pc1,pd1\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2],$

${*}\langle pc2,pd2\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2],\ {*}\langle pc3,pd3\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2]$

$\rightarrow\ \langle pd1,pd2,pd3\rangle{:}Cp_{D2}$

$[\forall fc1,fc2,fc3{:}Ar_{C1}]\ [\forall fd1,fd2,fd3{:}Ar_{D1}](\langle fc1,fc2,fc3\rangle{:}Cp_{C1} \wedge$

$\langle fc1,fd1\rangle{:}F1 \wedge \langle fc2,fd2\rangle{:}F1 \wedge \langle fc3,fd3\rangle{:}F1 \supset \langle fd1,fd2,fd3\rangle{:}Cp_{D1}),$

$[\forall fc1,fc2,fc3{:}Ar_{C2}]\ [\forall fd1,fd2,fd3{:}Ar_{D2}](\langle fc1,fc2,fc3\rangle{:}Cp_{C2} \wedge$

$\langle fc1,fd1\rangle{:}F2 \wedge \langle fc2,fd2\rangle{:}F2 \wedge \langle fc3,fd3\rangle{:}F2 \supset \langle fd1,fd2,fd3\rangle{:}Cp_{D2}),\ D1 =_{e} C2$

$\rightarrow {*}[\forall fc1,fc2,fc3{:}Ar_{C1}]\ {*}[\forall fd1,fd2,fd3{:}Ar_{D2}]{*}(\langle fc1,fc2,fc3\rangle{:}Cp_{C1} \wedge$

$\langle fc1,fd1\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2] \wedge \langle fc2,fd2\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2] \wedge$

$\langle fc3,fd3\rangle{:}\ \mathbb{FC}[F1,C1,D1,F2,C2,D2] \supset \langle fd1,fd2,fd3\rangle{:}Cp_{D2})$

End of proof of lemma 5.3.1

5.4 Cat is a Category

The main theorem of the paper is proved in this section.

5.4.1 Theorem

The sequent

$\rightarrow$ < **Ar**, $=_a$, **Sr**, **Tg**, **Cp**>:Cat

is derivable in NaDSet.

Proof of Theorem 5.4.1

In the proof of the theorem we use a notation similar to the one used in the previous lemmas. Ar_X, $=a_X$, Sr_X, Tg_X, Cp_X, with X being A, B, C, D, E, posibly subscripted, are used as second order parameters while X alone will abbreviate the tuple $<Ar_X, =a_X, Sr_X, Tg_X, Cp>$.

A derivation of $\rightarrow$ <**Ar**,$=_a$,**Sr**,**Tg**,**Cp**>:Cat can be obtained from a derivation of $\rightarrow$ Category[**Ar**,$=_a$,**Sr**,**Tg**,**Cp**] by one application of $\rightarrow${ } rule and the definition of Cat. To derive the latter sequent it is necessary to provide a derivation of each sequent of the form

$\rightarrow$ Ax[**Ar**, $=_a$, **Sr**, **Tg**, **Cp**] (T1)

where Ax[**Ar**,$=_a$,**Sr**,**Tg**,**Cp**] is one of the axioms c1 to c20. Derivations will be provided for the complicated and "interesting" axioms only.

5.4.1.1 The sequent T1 is derivable when Ax is the axiom c3. A derivation for it follows.

(i) $F1=_eF2,\ F2=_eF3 \rightarrow F1=_eF3$ (lemma 5.1.1)

(ii) $C1=_eC2,\ C2=_eC3 \rightarrow C1=_eC3$ (lemma 5.1.1)

(iii) $D1=_eD2,\ D2=_eD3 \rightarrow D1=_eD3$ (lemma 5.1.1)

---

thinning

<F1,C1,D1>:**Ar**, <F2,C2,D2>:**Ar**, <F3,C3,D3>:**Ar**,

$F1=_eF2,\ C1=_eC2,\ D1=_eD2,\ F2=_eF3,\ C2=_eC3,\ D2=_eD3$

$\rightarrow F1=_eF3,\ C1=_eC3,\ D1=_eD3$

<F1,C1,D1>:**Ar**, <F2,C2,D2>:**Ar**, <F3,C3,D3>:**Ar**

$$\rightarrow {*}(\ {*}\langle F1,C1,D1\rangle =_a \langle F2,C2,D2\rangle \wedge {*}\langle F2,C2,D2\rangle =_a \langle F3,C3,D3\rangle \supset$$

$$\qquad {*}\langle F1,C1,D1\rangle =_a \langle F3,C3,D3\rangle\ )$$

$$\qquad\qquad \rightarrow {*}[\forall f,g,h{:}Ar](\ f =_a g \wedge g =_a h \supset f =_a h\ )$$

5.4.1.2 The sequent T1 is derivable when Ax is the axiom c6.
In the following derivation, a1, a2 are first order parameters:

(i) $a1 =_{aD1} a2,\ D1 =_e D2 \rightarrow a1 =_{aD2} a2$ (lemma 5.1.1)

(ii) $\langle a1,a2\rangle{:}F3 \rightarrow \langle a1,a2\rangle{:}F3$ (axiom)

$${*}(\langle a1,a2\rangle{:}F3 \supset a1 =_{aD1} a2),\ D1 =_e D2,\ \langle a1,a2\rangle{:}F3 \rightarrow a1 =_{aD2} a2$$

(i) $(\langle a1,a2\rangle{:}F3 \supset a1 =_{aD1} a2),\ D1 =_e D2 \rightarrow {*}(\langle a1,a2\rangle{:}F3 \supset a1 =_{aD2} a2)$

(ii) $(f =_{aD1} g \supset \langle a1,a2\rangle{:}F3\ ),\ D1 =_e D2 \rightarrow {*}(a1 =_{aD2} a2 \supset \langle a1,a2\rangle{:}F3\ )$ (similarly)

(i) $(\langle a1,a2\rangle{:}F3 \equiv a1 =_{aD1} a2),\ D1 =_e D2 \rightarrow {*}(\langle a1,a2\rangle{:}F3 \equiv a1 =_{aD2} a2)$

(ii) $a1{:}Ar_{D2} \rightarrow a1{:}Ar_{D2}$ (axiom)

(iii) $a2{:}Ar_{D2} \rightarrow a2{:}Ar_{D2}$ (axiom)

$${*}[\forall f,g{:}Ar_{D1}](\langle f,g\rangle{:}F3 \equiv f =_{aD1} g),\ D1 =_e D2,\ a1{:}Ar_{D2},\ a2{:}Ar_{D2}$$

$$\qquad \rightarrow (\langle a1,a2\rangle{:}F3 \equiv a1 =_{aD2} a2)$$

(i) $[\forall f,g{:}Ar_{D1}](\langle f,g\rangle{:}F3 \equiv f =_{aD1} g),\ D1 =_e D2$

$$\qquad \rightarrow {*}[\forall f,g{:}Ar_{D2}](\langle f,g\rangle{:}F3 \equiv f =_{aD2} g)$$

(ii) $C3 =_e D1,\ D1 =_e D2 \rightarrow C3 =_e D2$ (lemma 5.1.1)

(iii) $D3 =_e D1,\ D1 =_e D2 \rightarrow D3 =_e D2$ (lemma 5.1.1)

---

thinning

$$\langle F1,C1,D1\rangle{:}Ar,\ \langle F2,C2,D2\rangle{:}Ar,\ \langle F3,C3,D3\rangle{:}Ar,$$

$$\qquad F1 =_e F2,\ C1 =_e C2,\ D1 =_e D2,$$

$$\qquad C3 =_e D1,\ D3 =_e D1,\ [\forall f,g{:}Ar_{D1}](\langle f,g\rangle{:}F3 \equiv f =_{aD1} g),$$

$\to *(C3 =_e D2 \wedge D3 =_e D2 \wedge [\forall f,g{:}Ar_{D2}](\langle f,g\rangle{:}F3 \equiv f =_{aD2} g))$

$\langle F1,C1,D1\rangle{:}Ar,\ \langle F2,C2,D2\rangle{:}Ar,\ \langle F3,C3,D3\rangle{:}Ar,$

$*\langle F1,C1,D1\rangle =_a \langle F2,C2,D2\rangle,\ *\langle\langle F1,C1,D1\rangle,\langle F3,C3,D3\rangle\rangle{:}Tg$

$\to *\langle\langle F2,C2,D2\rangle,\langle F3,C3,D3\rangle\rangle{:}Tg$

$\to *[\forall f,g,a{:}Ar]*(f =_a g \wedge \langle f,a\rangle{:}Tg \supset \langle g,a\rangle{:}Tg)$

5.4.1.3 The sequent T1 is derivable when Ax is the axiom c11.
In the following derivation a1, a2 are first order parameters.

(i) $a1 =_{aC} a2 \to a1 =_{aC} a2$ (axiom, used twice)

$\to (a1 =_{aC} a2) \equiv (a1 =_{aC} a2)$

$a1{:}Ar_C,\ a2{:}Ar_C \to (a1 =_{aC} a2) \equiv (a1 =_{aC} a2)$ thinning

(i) $\to * [\forall f,g{:}Ar_C](*\langle f,g\rangle{:}\mathbb{I}d[C] \equiv (f =_{aC} g))$

(ii) $\to C =_e C$ (lemma 5.1.1)

$\to *(C =_e C \wedge C =_e C \wedge [\forall f,g{:}Ar_C](\langle f,g\rangle{:}\mathbb{I}d[C] \equiv f =_{aC} g)$

(i) $\to *\langle\langle F,C,D\rangle, \langle \mathbb{I}d[C],C,C\rangle\rangle{:}\mathbb{S}r$

(ii) $C{:}Cat \to \langle \mathbb{I}d[C],C,C\rangle{:}Ar$ (Lemma 5.2.1)

--------thinning

$Functor[F,C,D] \to *[\exists a{:}Ar]\ \langle\langle F,C,D\rangle,a\rangle{:}\mathbb{S}r$

$*\langle F,C,D\rangle{:}Ar \to [\exists a{:}Ar]\ \langle\langle F,C,D\rangle,a\rangle{:}\mathbb{S}r$

$\to *[\forall f{:}Ar][\exists a{:}Ar]\ \langle f,a\rangle{:}\mathbb{S}r$

5.4.1.4 The sequent T1 is derivable when Ax is the axiom c15.

T1 will follow from

(1) $\langle F1,C1,D1\rangle{:}Ar,\ \langle F2,C2,D2\rangle{:}Ar,\ D3 =_e D1,\ D3 =_e C2$

$\to \langle \mathbb{F}\mathbb{C}[F1,C1,D1,F2,C2,D2],C1,D2\rangle{:}Ar$

and

(2) $D3 =_e D1,\ D3 =_e C2 \to$

$<<F1,C1,D1>, <F2,C2,D2>, <\mathbb{FC}[F1,C1,D1,F2,C2,D2],C1,D2>>:\mathbb{C}p$.

5.4.1.4.1 The sequent (1) is derived next:

(i) $D3 =_e D1,\ D3 =_e C2 \rightarrow D1 =_e C2$ (consequence of lemma 5.1.1)

(ii) $<F1,C1,D1>:Ar,\ <F2,C2,D2>:Ar,\ D1 =_e C2$

$\rightarrow <\mathbb{FC}[F1,C1,D1,F2,C2,D2],C1,D2>:Ar$ (lemma 5.3.1)

---
cut

$<F1,C1,D1>:Ar,\ <F2,C2,D2>:Ar,\ D3 =_e D1,\ D3 =_e C2$

$\rightarrow <\mathbb{FC}[F1,C1,D1,F2,C2,D2],C1,D2>:Ar$

5.4.1.4.2 The above sequent (2) is derivable.
In the following derivation, a1, a2 and a3 are first order parameters:.

(i) $a3:Ar_{D1} \rightarrow a3:Ar_{D1}$ (axiom)

(ii) $<a1,a3>:F1 \rightarrow <a1,a3>:F1$ (axiom)

(iii) $<a3,a2>F2 \rightarrow <a3,a2>F2$ (axiom)

$a3:Ar_{D1},\ <a1,a3>:F1,\ <a3,a2>F2 \rightarrow *[\exists h:Ar_{D1}]*(\ <a1,h>:F1 \wedge <h,a2>F2\ )$

(i) $*[\exists h:Ar_{D1}]*(\ <a1,h>:F1 \wedge <h,a2>F2\ )$

$\rightarrow [\exists h:Ar_{D1}](\ <a1,h>:F1 \wedge <h,a2>F2\ )$ (used twice)

$\rightarrow [\exists h:Ar_{D1}](\ <a1,h>:F1 \wedge <h,a2>F2\ ) \equiv [\exists h:Ar_{D1}](\ <a1,h>:F1 \wedge <h,a2>F2\ )$

$\rightarrow *<a1,a2>:\mathbb{FC}[F1,C1,D1,F2,C2,D2] \equiv [\exists h:Ar_{D1}](\ <a1,h>:F1 \wedge <h,a2>F2\ )$

$a1:Ar_{C1},\ a2:Ar_{D2} \rightarrow$

$(\ <a1,a2>:\mathbb{FC}[F1,C1,D1,F2,C2,D2] \equiv [\exists h:Ar_{D1}](\ <a1,h>:F1 \wedge <h,a2>F2\ ))$ (thin)

(i) $\rightarrow *[\forall f:Ar_{C1}][\forall g:Ar_{D2}]($

$<f,g>:\mathbb{FC}[F1,C1,D1,F2,C2,D2] \equiv [\exists h:Ar_{D1}](\ <f,h>:F1 \wedge <h,g>F2\ ))$

(ii) $D3 =_e D1,\ D3 =_e C2 \rightarrow D1 =_e C2$ (consequence of lemma 5.1.1)

(iii) $\rightarrow C1 =_e C1$ (lemma 5.1.1)

$(iv) \to D2 =_e D2$ (lemma 5.1.1)

$D3 =_e D1,\ D3 =_e C2 \ \to$

$*(\ C1 =_e C1 \wedge D1 =_e C2 \wedge D2 =_e D2 \wedge [\forall f{:}Ar_{C1}][\forall g{:}Ar_{D2}]($

$\langle f,g\rangle{:}\mathbb{FC}[F1,C1,D1,F2,C2,D2] \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle F2\ )))$

$D3 =_e D1,\ D3 =_e C2 \ \to$

$*\langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle, \langle \mathbb{FC}[F1,C1,D1,F2,C2,D2],C1,D2\rangle\rangle{:}\mathbb{C}p$

5.4.1.4.3 Finally, a derivation of T1 from (1) and (2) is given.

(1) $\langle F1,C1,D1\rangle{:}Ar,\ \langle F2,C2,D2\rangle{:}Ar,\ D3 =_e D1,\ D3 =_e C2$

$\to \langle\ \mathbb{FC}[F1,C1,D1,F2,C2,D2],C1,D2\rangle{:}Ar$

(2) $D3 =_e D1,\ D3 =_e C2 \ \to$

$*\langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle, \langle \mathbb{FC}[F1,C1,D1,F2,C2,D2],C1,D2\rangle\rangle{:}\mathbb{C}p$

$\langle F1,C1,D1\rangle{:}Ar,\ \langle F2,C2,D2\rangle{:}Ar,\ D3 =_e D1,\ D3 =_e C2$

$\to *[\ \exists h{:}Ar]\ \langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle,h\rangle{:}\mathbb{C}p$

$\langle F1,C1,D1\rangle{:}Ar,\ \langle F2,C2,D2\rangle{:}Ar,\ \langle F3,C3,D3\rangle{:}Ar,$

$*\langle\langle F1,C1,D1\rangle, \langle F3,C3,D3\rangle\ \rangle{:}\mathbb{T}g,\ *\langle\langle F2,C2,D2\rangle, \langle F3,C3,D3\rangle\rangle{:}\mathbb{S}r$

$\to [\exists h{:}Ar]\ \langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle,h\rangle{:}\mathbb{C}p$ (thin)

$\langle F1,C1,D1\rangle{:}Ar,\ \langle F2,C2,D2\rangle{:}Ar,\ \langle F3,C3,D3\rangle{:}A$

$\to *(\ \langle\langle F1,C1,D1\rangle, \langle F3,C3,D3\rangle\ \rangle{:}\mathbb{T}g \wedge \langle\langle F2,C2,D2\rangle, \langle F3,C3,D3\rangle\rangle{:}\mathbb{S}r$

$\supset [\ \exists h{:}Ar]\ \langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle,h\rangle{:}\mathbb{C}p\ )$

$\to *[\forall f,g,a{:}Ar](\ \langle f,a\rangle{:}\mathbb{T}g \wedge \langle g,a\rangle{:}\mathbb{S}r \ \supset [\exists h{:}Ar]\ \langle f,g,h\rangle{:}\mathbb{C}p\ )$

5.4.1.5 The sequent T1 is derivable when Ax is the axiom c17.
A derivation with first order parameters a1, a2, a3, follows:

(i) $a3{:}Ar_{D1} \to a3{:}Ar_{D1}$ (axiom)

(ii) $\langle a1,a3\rangle{:}F1 \to \langle a1,a3\rangle{:}F1$ (axiom)

(iii) $\langle a3,a2\rangle{:}F2 \rightarrow \langle a3,a2\rangle{:}F2$ (axiom)

(i) $a3{:}Ar_{D1},\ \langle a1,a3\rangle{:}F1,\ \langle a3,a2\rangle{:}F2 \rightarrow {*}[\exists h{:}Ar_{D1}]{*}(\langle a1,h\rangle{:}F1 \wedge \langle h,a2\rangle{:}F2)$

(ii) $\langle a1,a2\rangle{:}F4 \rightarrow \langle a1,a2\rangle{:}F4$ (axiom)

(i) $a3{:}Ar_{D1},\ \langle a1,a3\rangle{:}F1,\ \langle a3,a2\rangle{:}F2,$

$\quad {*}([\exists h{:}Ar_{D1}](\langle a1,h\rangle{:}F1 \wedge \langle h,a2\rangle{:}F2) \supset \langle a1,a2\rangle{:}F4)$

$\qquad \rightarrow \langle a1,a2\rangle{:}F4$

(ii) $\langle a1,a2\rangle{:}F3 \rightarrow \langle a1,a2\rangle{:}F3$ (axiom)

${*}(\langle a1,a2\rangle{:}F3 \supset {*}[\exists h{:}Ar_{D1}]{*}(\langle a1,h\rangle{:}F1 \wedge \langle h,a2\rangle{:}F2)),$

$\quad ([\exists h{:}Ar_{D1}](\langle a1,h\rangle{:}F1 \wedge \langle h,a2\rangle{:}F2) \supset \langle a1,a2\rangle{:}F4),$

$\qquad \langle a1,a2\rangle{:}F3 \rightarrow \langle a1,a2\rangle{:}F4$

(i) ${*}(\langle a1,a2\rangle{:}F3 \equiv [\exists h{:}Ar_{D1}](\langle a1,h\rangle{:}F1 \wedge \langle h,a2\rangle{:}F2)),$

$\quad {*}(\langle a1,a2\rangle{:}F4 \equiv [\exists h{:}Ar_{D1}](\langle a1,h\rangle{:}F1 \wedge \langle h,a2\rangle{:}F2)),$

$\qquad \langle a1,a2\rangle{:}F3 \rightarrow \langle a1,a2\rangle{:}F4$

(ii) $a1{:}Ar_{C3},\ C1 =_e C3 \rightarrow a1{:}Ar_{C1}$ used twice (lemma 5.1.1)

(iii) $a2{:}Ar_{D3},\ D2 =_e D3 \rightarrow a2{:}Ar_{D2}$ used twice (lemma 5.1.1)

$C1 =_e C3,\ D2 =_e D3,$

$\quad {*}[\forall f{:}Ar_{C1}]{*}[\forall g{:}Ar_{D2}](\langle f,g\rangle{:}F3 \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2)),$

$\quad {*}[\forall f{:}Ar_{C1}]{*}[\forall g{:}Ar_{D2}](\langle f,g\rangle{:}F4 \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2)),$

$\qquad a1{:}Ar_{C3},\ a2{:}Ar_{D3},\ \langle a1,a2\rangle{:}F3 \rightarrow \langle a1,a2\rangle{:}F4$

(i) $C1 =_e C3,\ D2 =_e D3,$

$\quad [\forall f{:}Ar_{C1}][\forall g{:}Ar_{D2}](\langle f,g\rangle{:}F3 \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2)),$

$\quad [\forall f{:}Ar_{C1}][\forall g{:}Ar_{D2}](\langle f,g\rangle{:}F4 \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2))$

$\qquad \rightarrow {*}[\forall f{:}Ar_{C3}]{*}[\forall g{:}Ar_{D3}]{*}(\langle f,g\rangle{:}F3 \supset \langle f,g\rangle{:}F4)$

(ii) $C1 =_e C4,\ D2 =_e D4,$

$\quad [\forall f{:}Ar_{C1}][\forall g{:}Ar_{D2}](\langle f,g\rangle{:}F3 \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2)),$

$\quad [\forall f{:}Ar_{C1}][\forall g{:}Ar_{D2}](\langle f,g\rangle{:}F4 \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2))$

$\rightarrow [\forall f{:}Ar_{C4}][\forall g{:}Ar_{D4}](\langle f,g\rangle{:}F4 \supset \langle f,g\rangle{:}F3)$ (similar to (i)

(iii) $C1 =_e C3,\ C1 =_e C4 \rightarrow C3 =_e C4$ (lemma 5.1.1)

(iv) $D2 =_e D3,\ D2 =_e D4 \rightarrow D3 =_e D4$ (lemma 5.1.1)

$C1 =_e C3,\ D2 =_e D3,$

$[\forall f{:}Ar_{C1}][\forall g{:}Ar_{D2}](\langle f,g\rangle{:}F3 \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2)),$

$C1 =_e C4,\ D2 =_e D4,$

$[\forall f{:}Ar_{C1}][\forall g{:}Ar_{D2}](\langle f,g\rangle{:}F4 \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2))$

$\rightarrow (C3 =_e C4 \wedge D3 =_e D4 \wedge$

$[\forall f{:}Ar_{C3}][\forall g{:}Ar_{D3}](\langle f,g\rangle{:}F3 \supset \langle f,g\rangle{:}F4) \wedge$

$[\forall f{:}Ar_{C4}][\forall g{:}Ar_{D4}](\langle f,g\rangle{:}F4 \supset \langle f,g\rangle{:}F3)\ )$

$\langle F1,C1,D1\rangle{:}Ar,\ \langle F2,C2,D2\rangle{:}Ar,\ \langle F3,C3,D3\rangle{:}Ar,\ \langle F4,C4,D4\rangle{:}Ar,$

$*\langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle, \langle F3,C3,D3\rangle\rangle{:}\mathbb{C}p,$

$*\langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle, \langle F4,C4,D4\rangle\rangle{:}\mathbb{C}p$

$\rightarrow *\langle F3,C3,D3\rangle =_a \langle F4,C4,D4\rangle$ (thin)

$\rightarrow *[\forall f,g,h,k{:}Ar]\ *(\ \langle f,g,h\rangle{:}\mathbb{C}p \wedge \langle f,g,k\rangle{:}\mathbb{C}p \supset h =_a k\ )$

5.4.1.6 The sequent T1 is derivable when Ax is the axiom c18.

First we derive the sequents:

(1) $\langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle\rangle{:}\mathbb{S}r \rightarrow \langle\langle F2,C2,D2\rangle, \langle F2,C2,D2\rangle\rangle{:}\mathbb{S}r$

(2) $\langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle\rangle{:}\mathbb{S}r \rightarrow \langle\langle F2,C2,D2\rangle, \langle F2,C2,D2\rangle\rangle{:}\mathbb{T}g$

(3) $\langle F1,C1,D1\rangle{:}Ar,\ \langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle\rangle{:}\mathbb{S}r \rightarrow$

$\langle\langle F2,C2,D2\rangle, \langle F1,C1,D1\rangle, \langle F1,C1,D1\rangle\rangle{:}\mathbb{C}p$

5.4.1.6.1 A derivation of (1) with first order parameters a1 and a2 follows:

$\langle a1,a2\rangle{:}F2 \rightarrow \langle a1,a2\rangle{:}F2$ (axiom)

$a1 =_{aC1} a2,\ C2 =_e C1 \rightarrow a1 =_{aC2} a2$ (lemma 5.1.1)

(i) $*(\langle a1,a2\rangle{:}F2 \supset a1 =_{aC1} a2),\ C2 =_e C1 \rightarrow *(\langle a1,a2\rangle{:}F2 \supset a1 =_{aC2} a2)$

(ii) $(a1 =_{aC1} a2 \supset \langle a1,a2\rangle{:}F2),\ C2 =_e C1 \rightarrow (a1 =_{aC2} a2 \supset \langle a1,a2\rangle{:}F2)$ (similar to i)

(i) $*(\langle a1,a2\rangle{:}F2 \equiv a1 =_{aC1} a2),\ C2 =_e C1 \rightarrow *(\langle a1,a2\rangle{:}F2 \equiv a1 =_{aC2} a2)$

(ii) $a1{:}Ar_{C2},\ C2 =_e C1 \rightarrow a1{:}Ar_{C1}$ (lemma 5.1.1)

(iii) $a2{:}Ar_{C2},\ C2 =_e C1 \rightarrow a2{:}Ar_{C1}$ (lemma 5.1.1)

$C2 =_e C1,\ *[\forall f,g{:}Ar_{C1}](\langle f,g\rangle{:}F2 \equiv f =_{aC1} g),$

$a1{:}Ar_{C2},\ a2{:}Ar_{C2} \rightarrow (\langle a1,a2\rangle{:}F2 \equiv a1 =_{aC2} a2)$

(i) $C2 =_e C1,\ [\forall f,g{:}Ar_{C1}](\langle f,g\rangle{:}F2 \equiv f =_{aC1} g)$

$\rightarrow *[\forall f,g{:}Ar_{C2}](\langle f,g\rangle{:}F2 \equiv f =_{aC2} g)$

(ii) $\rightarrow C2 =_e C2$ (lemma 5.1.1)

(iii) $D2 =_e C1,\ C2 =_e C1 \rightarrow D2 =_e C2$ (lemma 5.1.1)

$C2 =_e C1,\ D2 =_e C1,\ [\forall f,g{:}Ar_{C1}](\langle f,g\rangle{:}F2 \equiv f =_{aC1} g)$

$\rightarrow *(C2 =_e C2 \wedge D2 =_e C2 \wedge [\forall f,g{:}Ar_{C2}](\langle f,g\rangle{:}F2 \equiv f =_{aC2} g))$

$*\langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle\rangle{:}\mathbb{S}r \rightarrow *\langle\langle F2,C2,D2\rangle, \langle F2,C2,D2\rangle\rangle{:}\mathbb{S}r$

5.4.1.6.2 The derivation of (2) is similar to that of (1) and is omitted.

5.4.1.6.3 To derive (3) it is necessary to derive the following two sequents:

(a) $\langle F1,C1,D1\rangle{:}\mathbb{A}r,\ D2 =_e C1,\ [\forall f,g{:}Ar_{C1}](\langle f,g\rangle{:}F2 \equiv f =_{aC1} g),$

$a1{:}Ar_{C1},\ a2{:}\ Ar_{D1} \rightarrow *(\langle a1,a2\rangle{:}F1 \supset [\exists h{:}Ar_{D2}](\langle a1,h\rangle{:}F2 \wedge \langle h,a2\rangle{:}F1))$

(b) $\langle F1,C1,D1\rangle{:}\mathbb{A}r,\ C2 =_e C1,\ D2 =_e C1,\ *[\forall f,g{:}Ar_{C1}](\langle f,g\rangle{:}F2 \equiv f =_{aC1} g),$

$a1{:}Ar_{C1},\ a2{:}\ Ar_{D1} \rightarrow *([\exists h{:}Ar_{D2}](\langle a1,h\rangle{:}F2 \wedge \langle h,a2\rangle{:}F1) \supset \langle a1,a2\rangle{:}F1)$

5.4.1.6.3.1 First we derive (a) using a1 and a2 as first order parameters.

(i) $a1{:}Ar_{C1},\ D2 =_e C1 \rightarrow a1{:}Ar_{D2}$ (lemma 5.1.1)

(ii) $\langle a1,a1\rangle{:}F2 \rightarrow \langle a1,a1\rangle{:}F2$ (axiom)

(iii) $\langle a1,a2\rangle{:}F1 \rightarrow \langle a1,a2\rangle{:}F1$ (axiom)

(i) $D2 =_e C1,\ \langle a1,a1\rangle{:}F2,\ a1{:}Ar_{C1}, \langle a1,a2\rangle{:}F1$

$\rightarrow *\,[\exists h{:}Ar_{D2}]*(\langle a1,h\rangle{:}F2 \wedge \langle h,a2\rangle{:}F1)$

(ii) $C1{:}Cat,\ a1{:}Ar_{C1} \rightarrow a1 =_{aC1} a1$ (direct consequence of c1)

---

thinning

(i) $\langle F1,C1,D1\rangle{:}Ar,\ D2 =_e C1,\ *(a1 =_{aC1} a1 \supset \langle a1,a1\rangle{:}F2),$

$a1{:}Ar_{C1},\ a2{:}\ Ar_{D1}, \langle a1,a2\rangle{:}F1 \rightarrow [\exists h{:}Ar_{D2}](\langle a1,h\rangle{:}F2 \wedge \langle h,a2\rangle{:}F1)$

(ii) $a1{:}Ar_{C1} \rightarrow a1{:}Ar_{C1}$ (axiom, used twice)

---

thinning

$\langle F1,C1,D1\rangle{:}Ar,\ D2 =_e C1,\ *[\forall f,g{:}Ar_{C1}](\langle f,g\rangle{:}F2 \equiv f =_{aC1} g),$

$a1{:}Ar_{C1},\ a2{:}\ Ar_{D1}, \langle a1,a2\rangle{:}F1 \rightarrow [\exists h{:}Ar_{D2}](\langle a1,h\rangle{:}F2 \wedge \langle h,a2\rangle{:}F1)$

$\langle F1,C1,D1\rangle{:}Ar,\ D2 =_e C1,\ [\forall f,g{:}Ar_{C1}](\langle f,g\rangle{:}F2 \equiv f =_{aC1} g),$

$a1{:}Ar_{C1},\ a2{:}\ Ar_{D1} \rightarrow *(\langle a1,a2\rangle{:}F1 \supset [\exists h{:}Ar_{D2}](\langle a1,h\rangle{:}F2 \wedge \langle h,a2\rangle{:}F1))$

5.4.1.6.3.2 The above sequent (b) is derivable.
In the following derivation a1, a2 and a3 are first order parameters:

(i) $a1{:}Ar_{C1} \rightarrow a1{:}Ar_{C1}$ (axiom)

(ii) $a3{:}Ar_{C1} \rightarrow a3{:}Ar_{C1}$ (axiom)

(iii) $a1 =_{aC1} a3 \rightarrow a1 =_{aC1} a3$ (axiom)

(iv) $a3 =_{aC1} a1 \rightarrow a3 =_{aC1} a1$ (axiom)

$*[\forall f,g{:}Ar_{C1}]*(f =_{aC1} g \supset g =_{aC1} f),\ a1{:}Ar_{C1},\ a3{:}Ar_{C1},$

$a1 =_{aC1} a3 \rightarrow a3 =_{aC1} a1$

(i) $*C1{:}Cat,\ a1{:}Ar_{C1},\ *\,a3{:}Ar_{C1},\ a1 =_{aC1} a3 \rightarrow a3 =_{aC1} a1$ (c2 & thin)

(ii) $a3{:}Ar_{D2},\ D2 =_e C1 \rightarrow *\,a3{:}Ar_{C1}$ (lemma 5.1.1)

---

cut

(i) $C1{:}Cat,\ D2 =_e C1,\ a1{:}Ar_{C1},\ a3{:}Ar_{D2},\ a1 =_{aC1} a3 \rightarrow a3 =_{aC1} a1$

(ii) $\langle a3,a2\rangle F1 \rightarrow \langle a3,a2\rangle F1$ (axiom)

(iii) $\langle a1,a2\rangle F1 \rightarrow \langle a1,a2\rangle F1$ (axiom)

(i) $C1{:}Cat,\ {*}(a3 =_{aC1} a1 \wedge \langle a3,a2\rangle F1 \supset \langle a1,a2\rangle F1),\ D2 =_e C1,$

$a1 =_{aC1} a3,\ a1{:}Ar_{C1},\ a3{:}Ar_{D2},\ \langle a3,a2\rangle{:}F1 \rightarrow \langle a1,a2\rangle{:}F1$

(ii) $a3{:}Ar_{D2},\ D2 =_e C1 \rightarrow a3{:}Ar_{C1}$ (lemma 5.1.1)

(iii) $a1{:}Ar_{C1} \rightarrow a1{:}Ar_{C1}$ (axiom)

(iv) $a2{:}\ Ar_{D1} \rightarrow a2{:}\ Ar_{D1}$ (axiom)

$C1{:}Cat,\ {*}[\forall fc,gc{:}Ar_{C1}]{*}[\forall fd{:}Ar_{D1}](fc =_{aC1} gc \wedge \langle fc,fd\rangle F1 \supset \langle gc,fd\rangle F1),$

$D2 =_e C1,\ a1 =_{aC1} a3,$

$a1{:}Ar_{C1},\ a2{:}\ Ar_{D1},\ a3{:}Ar_{D2},\ \langle a3,a2\rangle{:}F1 \rightarrow \langle a1,a2\rangle{:}F1$

(i) ${*}\langle F1,C1,D1\rangle{:}Ar,\ D2 =_e C1,\ a1 =_{aC1} a3,$

$a1{:}Ar_{C1},\ a2{:}\ Ar_{D1},\ a3{:}Ar_{D2},\ \langle a3,a2\rangle{:}F1 \rightarrow \langle a1,a2\rangle{:}F1$ (f5 & thin)

(ii) $a1{:}Ar_{C1} \rightarrow a1{:}Ar_{C1}$ (axiom)

(iii) $a3{:}Ar_{D2} \wedge D2 =_e C1 \rightarrow a3{:}Ar_{C1}$ (lemma 5.1.1)

(iv) $\langle a1,a3\rangle{:}F2 \rightarrow \langle a1,a3\rangle{:}F2$ (axiom)

$\langle F1,C1,D1\rangle{:}Ar,\ D2 =_e C1,\ {*}[\forall f,g{:}Ar_{C1}]{*}(\langle f,g\rangle{:}F2 \supset f =_{aC1} g),$

$a1{:}Ar_{C1},\ a2{:}\ Ar_{D1},\ a3{:}Ar_{D2},\ \langle a1,a3\rangle{:}F2,\ \langle a3,a2\rangle{:}F1 \rightarrow \langle a1,a2\rangle{:}F1$

$\langle F1,C1,D1\rangle{:}Ar,\ D2 =_e C1,\ [\forall f,g{:}Ar_{C1}](\langle f,g\rangle{:}F2 \equiv f =_{aC1} g),$

$a1{:}Ar_{C1},\ a2{:}\ Ar_{D1},\ {*}[\exists h{:}Ar_{D2}](\langle a1,h\rangle{:}F2 \wedge \langle h,a2\rangle{:}F1) \rightarrow \langle a1,a2\rangle{:}F1$ (thin)

$\langle F1,C1,D1\rangle{:}Ar,\ D2 =_e C1,\ [\forall f,g{:}Ar_{C1}](\langle f,g\rangle{:}F2 \equiv f =_{aC1} g),$

$a1{:}Ar_{C1},\ a2{:}\ Ar_{D1} \rightarrow {*}([\exists h{:}Ar_{D2}](\langle a1,h\rangle{:}F2 \wedge \langle h,a2\rangle{:}F1) \supset \langle a1,a2\rangle{:}F1)$

5.4.1.6.3.3 A derivation of (3) from (a) and (b) follows:

(a) (b)

$\langle F1,C1,D1\rangle{:}Ar,\ D2 =_e C1,\ [\forall f,g{:}Ar_{C1}](\langle f,g\rangle{:}F2 \equiv f =_{aC1} g),$

$a1{:}Ar_{C1},\ a2{:}\ Ar_{D1} \rightarrow {*}(\langle a1,a2\rangle{:}F1 \equiv [\exists h{:}Ar_{D2}](\langle a1,h\rangle{:}F2 \wedge \langle h,a2\rangle{:}F1))$

(i) $\langle F1,C1,D1\rangle{:}Ar,\ D2 =_e C1,\ [\forall f,g{:}Ar_{C1}](\langle f,g\rangle{:}F2 \equiv f =_{aC1} g)$

$\rightarrow {*}[\forall f{:}Ar_{C1}]{*}[\forall g{:}\ Ar_{D1}](\langle f,g\rangle{:}F1 \equiv [\exists h{:}Ar_{D2}](\langle f,h\rangle{:}F2 \wedge \langle h,g\rangle{:}F1))$

(ii) $C2 =_e C1 \rightarrow C2 =_e C1$ (lemma 5.1.1)

(iii) $D2 =_e C1 \rightarrow D2 =_e C1$ (lemma 5.1.1)

$\langle F1,C1,D1\rangle$:Ar, $C2 =_e C1$, $D2 =_e C1$, $[\forall f,g:Ar_{C1}](\langle f,g\rangle:F2 \equiv f =_{aC1} g)$

$\rightarrow$ *($C2 =_e C1 \wedge D2 =_e C1 \wedge D1 =_e D1 \wedge$

$[\forall f:Ar_{C1}][\forall g: Ar_{D1}](\langle f,g\rangle:F1 \equiv [\exists h:Ar_{D2}](\langle f,h\rangle:F2 \wedge \langle h,g\rangle:F1))$)

$\langle F1,C1,D1\rangle$:Ar, *$\langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle\rangle$:Sr $\rightarrow$

*$\langle\langle F2,C2,D2\rangle, \langle F1,C1,D1\rangle, \langle F1,C1,D1\rangle\rangle$:Cp

5.4.1.6.4 Finally, T1 can be obtained from (1), (2) and (3) by the following simple derivation.

(1) (2) (3)

--- thinning

$\langle F1,C1,D1\rangle$:Ar, $\langle F2,C2,D2\rangle$:Ar, $\langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle\rangle$:Sr $\rightarrow$

($\langle\langle F2,C2,D2\rangle, \langle F2,C2,D2\rangle\rangle$:Sr $\wedge$ $\langle\langle F2,C2,D2\rangle, \langle F2,C2,D2\rangle\rangle$:Tg $\wedge$

$\langle\langle F2,C2,D2\rangle, \langle F1,C1,D1\rangle, \langle F1,C1,D1\rangle\rangle$:Cp)

$\rightarrow$ *$[\forall f,a:Ar]$*($\langle f,a\rangle$:Sr $\supset$ $\langle a,a\rangle$:Sr $\wedge$ $\langle a,a\rangle$:Tg $\wedge$ $\langle a,f,f\rangle$:Cp)

5.4.1.7 The sequent T1 is derivable when Ax is the axiom c20.

In the following derivation c, d, d1 and d2 are first order parameters, and the D and C notations previously introduced to represent five tuples is used again. Further the notation

$\langle F(12)3, C(12)3, D(12)3\rangle$

represents a functor resulting from first composing functors

$\langle F1, C1, D1\rangle$ and $\langle F2, C2, D2\rangle$

and then composing the result with the functor

$\langle F3, C3, D3\rangle$.

The triple

$\langle F1(23), C1(23), D1(23)\rangle$

has a similar meaning.

(i) $d1:Ar_{D12}, D2 =_e D12 \rightarrow d1:Ar_{D2}$ (lemma 5.1.1)

(ii) $\langle d2,d1\rangle{:}F2 \rightarrow \langle d2,d1\rangle{:}F2$ (axiom)

(iii) $\langle d1,d\rangle{:}F3 \rightarrow \langle d1,d\rangle{:}F3$ (axiom)

(i) $\langle d2,d1\rangle{:}F2,\ d1{:}Ar_{D12},\ \langle d1,d\rangle{:}F3,\ D2 =_{e} D12$

$\rightarrow\ {*}[\exists h{:}Ar_{D2}]{*}(\langle d2,h\rangle{:}F2 \wedge \langle h,d\rangle{:}F3)$

(ii) $\langle d2,d\rangle{:}F23 \rightarrow \langle d2,d\rangle{:}F23$ (axiom)

(i) $\langle d2,d1\rangle{:}F2,\ {*}([\exists h{:}Ar_{D2}](\langle d2,h\rangle{:}F2 \wedge \langle h,d\rangle{:}F3) \supset \langle d2,d\rangle{:}F23),$

$d1{:}Ar_{D12},\ \langle d1,d\rangle{:}F3,\ D2 =_{e} D12 \rightarrow \langle d2,d\rangle{:}F23$

(ii) $d2{:}Ar_{D1},\ D1 =_{e} C23 \rightarrow d2{:}Ar_{C23}$ (lemma 5.1.1)

(iii) $d{:}Ar_{D(12)3},\ D3 =_{e} D23,\ D3 =_{e} D(12)3 \rightarrow d{:}Ar_{D23}$ (lemma 5.1.1)

--- thinning

(i) $d2{:}Ar_{D1},\ \langle d2,d1\rangle{:}F2,$

${*}[\forall f{:}Ar_{C23}]{*}[\forall g{:}Ar_{D23}]([\exists h{:}Ar_{D2}](\langle f,h\rangle{:}F2 \wedge \langle h,g\rangle{:}F3) \equiv \langle f,g\rangle{:}F23),$

$d1{:}Ar_{D12},\ \langle d1,d\rangle{:}F3,\ d{:}Ar_{D(12)3},$

$D2 =_{e} D12,\ D1 =_{e} C23,\ D3 =_{e} D23,\ D3 =_{e} D(12)3$

$\rightarrow \langle d2,d\rangle{:}F23$

(ii) $d2{:}Ar_{D1} \rightarrow d2{:}Ar_{D1}$ (axiom)

(iii) $\langle c,d2\rangle{:}F1 \rightarrow \langle c,d2\rangle{:}F1$ (axiom)

$d2{:}Ar_{D1},\ \langle c,d2\rangle{:}F1,\ \langle d2,d1\rangle{:}F2,$

$[\forall f{:}Ar_{C23}][\forall g{:}Ar_{D23}]([\exists h{:}Ar_{D2}](\langle f,h\rangle{:}F2 \wedge \langle h,g\rangle{:}F3) \equiv \langle f,g\rangle{:}F23),$

$d1{:}Ar_{D12},\ \langle d1,d\rangle{:}F3,\ d{:}Ar_{D(12)3},$

$D2 =_{e} D12,\ D1 =_{e} C23,\ D3 =_{e} D23,\ D3 =_{e} D(12)3$

$\rightarrow\ {*}[\exists h{:}Ar_{D1}]{*}(\langle c,h\rangle{:}F1 \wedge \langle h,d\rangle{:}F23)$

(i) ${*}[\exists h{:}Ar_{D1}]{*}(\langle c,h\rangle{:}F1 \wedge \langle h,d1\rangle{:}F2),$

$[\forall f{:}Ar_{C23}][\forall g{:}Ar_{D23}]([\exists h{:}Ar_{D2}](\langle f,h\rangle{:}F2 \wedge \langle h,g\rangle{:}F3) \equiv \langle f,g\rangle{:}F23),$

$d1{:}Ar_{D12},\ \langle d1,d\rangle{:}F3,\ d{:}Ar_{D(12)3},$

$D2 =_{e} D12,\ D1 =_{e} C23,\ D3 =_{e} D23,\ D3 =_{e} D(12)3$

$\rightarrow [\exists h{:}Ar_{D1}](\langle c,h\rangle{:}F1 \wedge \langle h,d\rangle{:}F23)$

(ii) $\langle c,d1\rangle:F12 \rightarrow \langle c,d1\rangle:F12$ (axiom)

(i) $*(\langle c,d1\rangle:F12 \supset [\exists h:Ar_{D1}](\langle c,h\rangle:F1 \wedge \langle h,d1\rangle:F2)),$

$[\forall f:Ar_{C23}][\forall g:Ar_{D23}]([\exists h:Ar_{D2}](\langle f,h\rangle:F2 \wedge \langle h,g\rangle:F3) \equiv \langle f,g\rangle:F23),$

$d1:Ar_{D12},\ \langle c,d1\rangle:F12,\ \langle d1,d\rangle:F3,\ d:Ar_{D(12)3},$

$D2 =_e D12,\ D1 =_e C23, D3 =_e D23,\ D3 =_e D(12)3$

$\rightarrow [\exists h:Ar_{D1}](\langle c,h\rangle:F1 \wedge \langle h,d\rangle:F23)$

(ii) $c:Ar_{C(12)3},\ C12 =_e C(12)3 \rightarrow c:Ar_{C12}$ (lemma 5.1.1)

(iii) $d1:Ar_{D12} \rightarrow d1:Ar_{D12}$ (axiom)

---

thinning

$*[\forall f:Ar_{C12}]*[\forall g:Ar_{D12}](\langle f,g\rangle:F12 \equiv [\exists h:Ar_{D1}](\langle f,h\rangle:F1 \wedge \langle h,g\rangle:F2)),$

$[\forall f:Ar_{C23}][\forall g:Ar_{D23}]([\exists h:Ar_{D2}](\langle f,h\rangle:F2 \wedge \langle h,g\rangle:F3) \equiv \langle f,g\rangle:F23),$

$d1:Ar_{D12},\ \langle c,d1\rangle:F12,\ \langle d1,d\rangle:F3,$

$D2 =_e D12,\ D1 =_e C23, D3 =_e D23,\ D3 =_e D(12)3,\ \ C12 =_e C(12)3,$

$c:Ar_{C(12)3},\ d:Ar_{D(12)3} \rightarrow [\exists h:Ar_{D1}](\langle c,h\rangle:F1 \wedge \langle h,d\rangle:F23)$

(i) $[\forall f:Ar_{C12}][\forall g:Ar_{D12}](\langle f,g\rangle:F12 \equiv [\exists h:Ar_{D1}](\langle f,h\rangle:F1 \wedge \langle h,g\rangle:F2)),$

$[\forall f:Ar_{C23}][\forall g:Ar_{D23}]([\exists h:Ar_{D2}](\langle f,h\rangle:F2 \wedge \langle h,g\rangle:F3) \equiv \langle f,g\rangle:F23),$

$*[\exists h:Ar_{D12}]*(\langle c,h\rangle:F12 \wedge \langle h,d\rangle:F3),$

$D2 =_e D12,\ D1 =_e C23, D3 =_e D23,\ D3 =_e D(12)3,\ \ C12 =_e C(12)3\ ,$

$c:Ar_{C(12)3},\ d:Ar_{D(12)3} \rightarrow [\exists h:Ar_{D1}](\langle c,h\rangle:F1 \wedge \langle h,d\rangle:F23)$

(ii) $\langle c,d\rangle:F(12)3 \rightarrow \langle c,d\rangle:F(12)3$ (axiom)

(iii) $\langle c,d\rangle:F1(23) \rightarrow \langle c,d\rangle:F1(23)$ (axiom)

(i) $[\forall f:Ar_{C12}][\forall g:Ar_{D12}](\langle f,g\rangle:F12 \equiv [\exists h:Ar_{D1}](\langle f,h\rangle:F1 \wedge \langle h,g\rangle:F2)),$

$[\forall f:Ar_{C23}][\forall g:Ar_{D23}]([\exists h:Ar_{D2}](\langle f,h\rangle:F2 \wedge \langle h,g\rangle:F3) \equiv \langle f,g\rangle:F23),$

$*(\langle c,d\rangle:F(12)3 \supset [\exists h:Ar_{D12}](\langle c,h\rangle:F12 \wedge \langle h,d\rangle:F3)),$

$*([\exists h:Ar_{D1}](\langle c,h\rangle:F1 \wedge \langle h,d\rangle:F23) \supset \langle c,d\rangle:F1(23)\ ),$

$D2 =_e D12,\ D1 =_e C23, D3 =_e D23,\ D3 =_e D(12)3,\ \ C12 =_e C(12)3\ ,$

$c:Ar_{C(12)3},\ d:Ar_{D(12)3}, \langle c,d\rangle:F(12)3 \rightarrow \langle c,d\rangle:F1(23)$

(ii) $c{:}Ar_{C(12)3} \to c{:}Ar_{C(12)3}$ (axiom)

(iii) $d{:}Ar_{D(12)3} \to d{:}Ar_{D(12)3}$ (axiom)

(iv) $c{:}Ar_{C(12)3},\ C1 =_e C12,\ C12 =_e C(12)3,\ C1 =_e C1(23) \to c{:}Ar_{C1(23)}$ (lemma 5.1.1)

(v) $d{:}Ar_{D(12)3},\ D3 =_e D(12)3,\ D3 =_e D23,\ D23 =_e D1(23) \to d{:}Ar_{D1(23)}$ (lemma 5.1.1)

---

thinning

$[\forall f{:}Ar_{C12}][\forall g{:}Ar_{D12}](\langle f,g\rangle{:}F12 \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2)),$

$[\forall f{:}Ar_{C23}][\forall g{:}Ar_{D23}]([\exists h{:}Ar_{D2}](\langle f,h\rangle{:}F2 \wedge \langle h,g\rangle{:}F3) \equiv \langle f,g\rangle{:}F23),$

$*[\forall f{:}Ar_{C(12)3}]*[\forall g{:}Ar_{D(12)3}](\langle f,g\rangle{:}F(12)3 \equiv [\exists h{:}Ar_{D12}](\langle f,h\rangle{:}F12 \wedge \langle h,g\rangle{:}F3)),$

$*[\forall f{:}Ar_{C1(23)}]*[\forall g{:}Ar_{D1(23)}]([\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F23) \equiv \langle f,g\rangle{:}F1(23)),$

$C1 =_e C12,\ D1 =_e C2,\ D2 =_e D12,\ C2 =_e C23,\ D2 =_e C3,\ D3 =_e D23,$

$C12 =_e C(12)3,\ D12 =_e C3,\ D3 =_e D(12)3,\ C1 =_e C1(23),\ D1 =_e C23,\ D23 =_e D1(23),$

$c{:}Ar_{C(12)3},\ d{:}Ar_{D(12)3},\ \langle c,d\rangle{:}F(12)3 \to \langle c,d\rangle{:}F1(23)$

(i) $[\forall f{:}Ar_{C12}][\forall g{:}Ar_{D12}](\langle f,g\rangle{:}F12 \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2)),$

$[\forall f{:}Ar_{C23}][\forall g{:}Ar_{D23}]([\exists h{:}Ar_{D2}](\langle f,h\rangle{:}F2 \wedge \langle h,g\rangle{:}F3) \equiv \langle f,g\rangle{:}F23),$

$[\forall f{:}Ar_{C(12)3}][\forall g{:}Ar_{D(12)3}](\langle f,g\rangle{:}F(12)3 \equiv [\exists h{:}Ar_{D12}](\langle f,h\rangle{:}F12 \wedge \langle h,g\rangle{:}F3)),$

$[\forall f{:}Ar_{C1(23)}][\forall g{:}Ar_{D1(23)}]([\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F23) \equiv \langle f,g\rangle{:}F1(23)),$

$C1 =_e C12,\ D1 =_e C2,\ D2 =_e D12,\ C2 =_e C23,\ D2 =_e C3,\ D3 =_e D23,$

$C12 =_e C(12)3,\ D12 =_e C3,\ D3 =_e D(12)3,\ C1 =_e C1(23),\ D1 =_e C23,\ D23 =_e D1(23),$

$\to *[\forall f{:}Ar_{C(12)3}]*[\forall g{:}Ar_{D(12)3}]*(\langle f,g\rangle{:}F(12)3 \supset \langle f,g\rangle{:}F1(23))$

(ii) $[\forall f{:}Ar_{C12}][\forall g{:}Ar_{D12}]([\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2) \equiv \langle f,g\rangle{:}F12),$

$[\forall f{:}Ar_{C23}][\forall g{:}Ar_{D23}](\langle f,g\rangle{:}F23 \equiv [\exists h{:}Ar_{D2}](\langle f,h\rangle{:}F2 \wedge \langle h,g\rangle{:}F3)),$

$[\forall f{:}Ar_{C(12)3}][\forall g{:}Ar_{D(12)3}]([\exists h{:}Ar_{D12}](\langle f,h\rangle{:}F12 \wedge \langle h,g\rangle{:}F3) \equiv \langle f,g\rangle{:}F(12)3),$

$[\forall f{:}Ar_{C1(23)}][\forall g{:}Ar_{D1(23)}](\langle f,g\rangle{:}F1(23) \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F23)),$

$C1 =_e C12,\ D1 =_e C2,\ D2 =_e D12,\ C2 =_e C23,\ D2 =_e C3,\ D3 =_e D23,$

$C12 =_e C(12)3,\ D12 =_e C3,\ D3 =_e D(12)3,\ C1 =_e C1(23),\ D1 =_e C23,\ D23 =_e D1(23),$

$\to [\forall f{:}Ar_{C1(23)}][\forall g{:}Ar_{D1(23)}](\langle f,g\rangle{:}F1(23) \supset \langle f,g\rangle{:}F(12)3)$ (similar to i)

(i) $[\forall f{:}Ar_{C12}][\forall g{:}Ar_{D12}](\langle f,g\rangle{:}F12 \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2))$,

$[\forall f{:}Ar_{C23}][\forall g{:}Ar_{D23}](\langle f,g\rangle{:}F23 \equiv [\exists h{:}Ar_{D2}](\langle f,h\rangle{:}F2 \wedge \langle h,g\rangle{:}F3))$,

$[\forall f{:}Ar_{C(12)3}][\forall g{:}Ar_{D(12)3}](\langle f,g\rangle{:}F(12)3 \equiv [\exists h{:}Ar_{D12}](\langle f,h\rangle{:}F12 \wedge \langle h,g\rangle{:}F3))$,

$[\forall f{:}Ar_{C1(23)}][\forall g{:}Ar_{D1(23)}](\langle f,g\rangle{:}F1(23) \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F23))$,

$C1 =_e C12,\ D1 =_e C2,\ D2 =_e D12,\ C2 =_e C23,\ D2 =_e C3,\ D3 =_e D23,$

$C12 =_e C(12)3,\ D12 =_e C3,\ D3 =_e D(12)3,\ C1 =_e C1(23),\ D1 =_e C23,\ D23 =_e D1(23),$

$\rightarrow\ {*}([\forall f{:}Ar_{C(12)3}][\forall g{:}Ar_{D(12)3}](\langle f,g\rangle{:}F(12)3 \supset \langle f,g\rangle{:}F1(23)) \wedge$

$[\forall f{:}Ar_{C1(23)}][\forall g{:}Ar_{D1(23)}](\langle f,g\rangle{:}F1(23) \supset \langle f,g\rangle{:}F(12)3)\ )$

(ii) $C1 =_e C12,\ C12 =_e C(12)3,\ C1 =_e C1(23) \rightarrow C(12)3 =_e C1(23)$ (uses lemma 5 1 1)

(iii) $D3 =_e D(12)3,\ D3 =_e D23,\ D23 =_e D1(23) \rightarrow D(12)3 =_e D1(23)$ (uses lemma 5.1.1)

$C1 =_e C12,\ D1 =_e C2,\ D2 =_e D12,$

$[\forall f{:}Ar_{C12}][\forall g{:}Ar_{D12}](\langle f,g\rangle{:}F12 \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F2))$,

$C2 =_e C23,\ D2 =_e C3,\ D3 =_e D23,$

$[\forall f{:}Ar_{C23}][\forall g{:}Ar_{D23}](\langle f,g\rangle{:}F23 \equiv [\exists h{:}Ar_{D2}](\langle f,h\rangle{:}F2 \wedge \langle h,g\rangle{:}F3))$,

$C12 =_e C(12)3,\ D12 =_e C3,\ D3 =_e D(12)3,$

$[\forall f{:}Ar_{C(12)3}][\forall g{:}Ar_{D(12)3}](\langle f,g\rangle{:}F(12)3 \equiv [\exists h{:}Ar_{D12}](\langle f,h\rangle{:}F12 \wedge \langle h,g\rangle{:}F3))$,

$C1 =_e C1(23),\ D1 =_e C23,\ D23 =_e D1(23),$

$[\forall f{:}Ar_{C1(23)}][\forall g{:}Ar_{D1(23)}](\langle f,g\rangle{:}F1(23) \equiv [\exists h{:}Ar_{D1}](\langle f,h\rangle{:}F1 \wedge \langle h,g\rangle{:}F23))$

$\rightarrow\ {*}(\ C(12)3 =_e C1(23) \wedge D(12)3 =_e D1(23) \wedge {*}F(12)3 =_e F1(23)\ )$

$\langle F1,C1,D1\rangle{:}Ar,\ \langle F2,C2,D2\rangle{:}Ar,\ \langle F3,C3,D3\rangle{:}Ar,\ \langle F12,C12,D12\rangle{:}Ar,$

$\langle F23,C23,D23\rangle{:}Ar,\ \langle F(12)3,C(12)3,D(12)3\rangle{:}Ar,\ \langle F1(23),C1(23),D1(23)\rangle{:}Ar,$

${*}\langle\langle F1,C1,D1\rangle, \langle F2,C2,D2\rangle, \langle F12,C12,D12\rangle\rangle{:}Cp,$

${*}\langle\langle F2,C2,D2\rangle, \langle F3,C3,D3\rangle, \langle F23,C23,D23\rangle\rangle{:}Cp,$

${*}\langle\langle F12,C12,D12\rangle, \langle F3,C3,D3\rangle, \langle F(12)3,C(12)3,D(12)3\rangle\rangle{:}Cp,$

${*}\langle\langle F1,C1,D1\rangle, \langle F23,C23,D23\rangle, \langle F1(23),C1(23),D1(23)\rangle\rangle{:}Cp$

$\rightarrow\ {*}\langle F(12)3,C(12)3,D(12)3\rangle =_a \langle F1(23),C1(23),D1(23)\rangle$ (thin)

$\rightarrow$ *[∀f,g,h,fg,gh,fg1h,f1gh:**Ar**]*(<f,g,fg>:$\mathbb{C}$**p** ∧ <g,h,gh>:$\mathbb{C}$**p** ∧

<fg,h,fg1h>:$\mathbb{C}$**p** ∧ <f,gh,f1gh>:$\mathbb{C}$**p** ⊃ fg1h $=_{\mathbf{a}}$ f1gh)

End of proof of theorem 5.4.1

6. NATURAL TRANSFORMATIONS and FUNCTOR CATEGORIES

As Eilenberg and Mac Lane observed [MacLane71], "category" has been defined in order to define "functor", and "functor" has been defined in order to define "natural transformation". This notion induces an equivalence relation between categories that allows the comparison of categories that are "alike" but of different "sizes". Moreover, natural transformation is the basic ingredient in the ubiquitous construct of functor categories.

6.1 Natural Transformations

We now proceed with a NaDSet definition of a naturral transformation from one functor to another. In this, $\mathbf{T}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{Ar_B}$, $\mathbf{=_{aB}}$, $\mathbf{Sr_B}$, $\mathbf{Tg_B}$, $\mathbf{Cp_B}$, $\mathbf{Ar_C}$, $\mathbf{=_{aC}}$, $\mathbf{Sr_C}$, $\mathbf{Tg_C}$, $\mathbf{Cp_C}$, $\mathbf{Ar_D}$, $\mathbf{=_{aD}}$, $\mathbf{Sr_D}$, $\mathbf{Tg_D}$, $\mathbf{Cp_D}$ are used as metavariables ranging over second order terms, while $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are used as abbreviations of the tuples $< \mathbf{Ar_B}, \mathbf{=_{aB}}, \mathbf{Sr_B}, \mathbf{Tg_B}, \mathbf{Cp_B} >$, $<\mathbf{Ar_C}, \mathbf{=_{aC}}, \mathbf{Sr_C}, \mathbf{Tg_C}, \mathbf{Cp_C}>$ and $<\mathbf{Ar_D}, \mathbf{=_{aD}}, \mathbf{Sr_D}, \mathbf{Tg_D}, \mathbf{Cp_D}>$ respectively.

If **F** and **G** are functors from **C** to **D**, the following sentences define **T** to be a *natural transformation* fom F to G.

NatTransform[**T**, **F**, **G**, **C**, **D**]   **for** axioms

where axioms consist of the conjunction of the following sentences:

T is a map for functors

$< \mathbf{F}, \mathbf{C}, \mathbf{D} >:\mathrm{Func}$ (t1)

$< \mathbf{G}, \mathbf{C}, \mathbf{D} >:\mathrm{Func}$ (t2)

T is a function from objects in C to arrows in D

$[\forall c:\mathrm{Ob}[\mathbf{C}]]\ [\exists tc:\mathbf{Ar_D}]\ <c,tc>:\mathbf{T}$ (t3)

$[\forall c:\mathrm{Ob}[\mathbf{C}]]\ [\forall tc:\mathbf{Ar_D}]\ (<c,tc>:\mathbf{T} \supset [\exists fc,gc:\mathbf{Ar_D}]($

$<c,fc>:\mathbf{F} \wedge <c,gc>:\mathbf{G} \wedge <tc,fc>:\mathbf{Sr_D} \wedge <tc,gc>:\mathbf{Tg_D}\ ))$ (t4)

$[\forall c1,c2\ \mathrm{Ob}[\mathbf{C}]]\ [\forall tc1,tc2:\mathbf{Ar_D}]($

$c1 =_{\mathbf{aC}} c2 \wedge <c1,tc1>:\mathbf{T} \wedge <c2,tc2>:\mathbf{T} \supset tc1 =_{\mathbf{aD}} tc2\ )$ (t5)

$$[\forall c1,c2{:}Ob[C]]\ [\forall tc{:}\mathbf{Ar_D}](\ c1 =_{\mathbf{aC}} c2 \wedge \langle c1,tc\rangle{:}\mathbf{T} \supset \langle c2,tc\rangle{:}\mathbf{T}\ ) \tag{t6}$$

$$[\forall c{:}Ob[C]]\ [\forall tc1,tc2{:}\mathbf{Ar_D}](\ tc1 =_{\mathbf{aD}} tc2 \wedge \langle c,tc1\rangle{:}\mathbf{T} \supset \langle c,tc2\rangle{:}\mathbf{T}\ ) \tag{t7}$$

<u>For every arrow</u> $\begin{array}{c} c1 \\ {\scriptstyle h}\downarrow \\ c2 \end{array}$ <u>of C, the diagram</u>
$$\begin{tikzcd} Fc1 \arrow[r, "Tc1"] \arrow[d, "Fh"'] & Gc1 \arrow[d, "Gh"] \\ Fc2 \arrow[r, "Tc2"'] & Gc2 \end{tikzcd}$$
<u>commutes</u>

$$\begin{aligned}&[\forall c1,c2\ Ob[C]]\ [\forall h{:}\mathbf{Ar_C}]\ [\forall tc1,tc2,fh,gh{:}\mathbf{Ar_D}](\ \langle h,c1\rangle{:}\mathbf{Sr_C} \wedge \langle h,c2\rangle{:}\mathbf{Tg_C} \wedge \\ &\qquad \langle c1,tc1\rangle{:}\mathbf{T} \wedge \langle c2,tc2\rangle{:}\mathbf{T} \wedge \langle h,fh\rangle{:}\mathbf{F} \wedge \langle h,gh\rangle{:}\mathbf{G} \supset \\ &\qquad [\exists k{:}\mathbf{Ar_D}](\ \langle tc1,gh,k\rangle{:}\mathbf{Cp_D} \wedge \langle fh,tc2,k\rangle{:}\mathbf{Cp_D}\ )\ )\end{aligned} \tag{t8}$$

The set of natural transformations is defined:

NatTrans **for** { <t,f,g,c,d> | NatTransform[t,f,g,c,d] }

Given the functors **F**, **G**: **C**→ **D**, the sets of natural transformations from **F** to **G** can now be defined:

NatTrans[**F**, **G**, **C**, **D**] **for** {t | <t, **F**, **G**, **C**, **D** >:NatTrans }

6.2 Natural Equivalence

A natural transformation is a *natural isomorphism* (or a *natural equivalence*) if each component of it is an isomorphism in the target category:

NatIsomorphism [**F**, **G**, **C**, **D**] **for**

$$\begin{aligned}&\{\ t \mid t{:}\mathrm{NatTrans}[\mathbf{F},\mathbf{G},\mathbf{C},\mathbf{D}]\ \wedge \\ &[\forall c{:}Ob[C]]\ [\forall tc,d1,d2{:}\mathbf{Ar_D}]\ (\langle c,tc\rangle{:}t \wedge \langle tc,d1\rangle{:}\mathbf{Sr_D} \wedge \langle tc,d2\rangle{:}\mathbf{Tg_D} \supset \\ &[\exists h{:}\mathbf{Ar_D}](\ \langle tc,h,d1\rangle{:}\mathbf{Cp_D} \wedge \langle h,tc,d2\rangle{:}\mathbf{Cp_D}\ )\ )\ \}\end{aligned}$$

Given two categories **C** and **D**, the equivalence relation among functors from **C** to **D** is given by:

NatEq[**C**, **D**] **for** {<f,g> | [∃τ:NatTrans[f, g, **C**, **D**] τ:NatIsomorphism[f, g, **C**, **D**] }

An equivalence relation ≅ between categories that meets the requirements mentioned at the beginning of the section, can be given by the following definition in which C, D are used as tuples of abstraction variables.

$\cong$ **for** { <C,D> | [∃F:Func[C,D]] [∃G:Func[D,C]] (

< FC[F,C,D,G,D,C], Id[C] >: NatEq[C, C] ∧

< FC[G,D,C,F,C,D], Id[D] >: NatEq[D, D]) }

where Id[_] and FC[_,_,_,_,_] are the terms defined in lemmas 5.1 and 5.2 respectively.

6.3 Functor Categories

If **C** and **D** are caregories, the category of functors --*functor category* -- from **C** to **D**, denoted by $\mathbf{D}^{\mathbf{C}}$ or FunCat[C,D], is defined as the tuple

$\mathbf{D}^{\mathbf{C}}$ **for** <*Ar*[C,D], $=_a$[C,D], *Sr*[C,D], *Tg*[C,D], *Cp*[C,D]>

of the parameterized terms *Ar*[C,D], $=_a$[C,D], *Sr*[C,D], *Tg*[C,D], *Cp*[C,D] whose definitions follow.

Obviously, the arrows of this category are the natural transformations among functors from **C** to **D**. The reader should note that the objects of this category are the functors themselves. Thus we define

Ar [C,D] **for** {<T,F,G> | NatTransform[T, F, G, C, **D**] }

The identity among the members of *Ar* [C,D] is defined in terms of the extensional identity.

$=_a$ [C,D] **for** {<<T1,F1,G1>, <T2,F2,G2>> |

F1 $=_e$ F2 ∧ G1 $=_e$ G2 ∧ T1 $=_e$ T2 }.

The identity $=_e$ for the terms that represent functors (F's and G's) was defined in section 5; it only remains to give its definition for the terms representing natural transformations:

T1 $=_e$ **T2** **for** [∀c:Ob[C]] [∀d:$\mathbf{Ar}_{\mathbf{D}}$](<c,d>:T1 ≡ <c,d>:T2).

The source and the target of an arrow coinsides with the source and the target functors of the transformation which are viewed as identity natural transformations. Consequently we define

Sr [C,D] **for** {<<T1,F1,G1>, <T2,F2,G2>> |

T2 $=_e$ F1 ∧ F2 $=_e$ F1 ∧ G2 $=_e$ F1 }

and

$\mathcal{Tg}$ [C,D] **for** {<<T1,F1,G1>, <T2,F2,G2>> |

$$T2 =_e G1 \ \wedge\ F2 =_e G1 \ \wedge\ G2 =_e G1 \ \} .$$

Finally, the composition of

$\mathcal{Cp}$ [C,D] **for** {<<T1,F1,G1>, <T2,F2,G2>, <T3,F3,G3>> |

$F1 =_e F3 \ \wedge\ G1 =_e F2 \ \wedge\ G2 =_e G3 \ \wedge$

$[\forall c{:}Ob[C]]\ [\forall d{:}\mathbf{Ar_D}](\ <c,d>{:}T3 \equiv$

$[\exists d1,d2{:}\mathbf{Ar_D}]\ (<c,d1>{:}T1 \wedge <c,d2>{:}T2 \wedge <d1,d2,d>\mathbf{Cp_D})\ )\ \}$.

The following theorem stipulates that for any categories C, D, the set of functors from C to D is itself a category and moreover, this sentence is derivable within NaDSet.

6.3.1 Theorem

The sequence

$\rightarrow\ [\forall X,Y{:}Cat]\ <\mathcal{Ar}\ [X,Y],\ =_a [X,Y],\ \mathcal{Sr}\ [X,Y],\ \mathcal{Tg}\ [X,Y],\ \mathcal{Cp}\ [X,Y]>{:}Cat$

is derivable within NaDSet.

A derivation of the theorem can be obtained if a derivation is provided for each sequence of the form

$Ax[Ar_C, =_{aC}, Sr_C, Tg_C, Cp_C],\ \ Ax[\ Ar_D, =_{aD}, Sr_D, Tg_D, Cp_D\ ]$

$\rightarrow\ Ax[\ \mathcal{Ar}\ [C,D],\ =_a [C,D],\ \mathcal{Sr}\ [C,D],\ \mathcal{Tg}\ [C,D],\ \mathcal{Cp}\ [C,D]\ ]$

where $Ar_C, =_{aC}, Sr_C, Tg_C, Cp_C, Ar_D, =_{aD}, Sr_D, Tg_D, Cp_D$ are second order parameters, Cand D are the tuples $<Ar_C, =_{aC}, Sr_C, Tg_C, Cp_C>, <Ar_D, =_{aD}, Sr_D, Tg_D, Cp_D>$ and $Ax[\mathbf{Ar}, =_\mathbf{a}, \mathbf{Sr}, \mathbf{Tg}, \mathbf{Cp}]$ is one of the axioms c1 to c20. The latter derivations are similar (in structure as well as in length) to those in the proof of theorem 5.3 and are omitted for space reasons.

7. OTHER CONSTRUCTIONS

7.1 Opposites

To each category **C**, we assosiate the *opposite* category, $\mathbf{C^{op}}$, defined to be the term $\langle \mathbf{Ar_C}, \mathbf{=_C}, Sr^{op}[C], Tg^{op}[C], Cp^{op}[C] \rangle$ with components:

$Sr^{op}[C]$ **for** $\{\langle u,v\rangle \mid \langle u,v\rangle{:}\mathbf{Tg_C}\}$

$Tg^{op}[C]$ **for** $\{\langle u,v\rangle \mid \langle u,v\rangle{:}\mathbf{Sr_C}\}$

$Cp^{op}[C]$ **for** $\{\langle u,v,g\rangle \mid \langle u,v,g\rangle{:}\mathbf{Cp_C}\}$

7.1.1 Lemma The sequents

$\rightarrow [\forall X{:}Cat]\ \mathbf{X^{op}}{:}Cat$

$\rightarrow [\forall X{:}Cat]\ (\mathbf{X^{op}})^{op} =_{\mathbf{e}} X$

are derivable.

7.2. Product Categories

Given two categories **B** and **C**, the product of them, **BxC**, is defined to be the term $\langle Ar^x[\mathbf{B,C}], =^x[\mathbf{B,C}], Sr^x[\mathbf{B,C}], Tg^x[\mathbf{B,C}], Cp^x[\mathbf{B,C}]\rangle$ with components:

$Ar^x[\mathbf{B,C}]$ **for** $\{\langle u,v\rangle \mid u{:}\mathbf{Ar_B} \wedge v{:}\mathbf{Ar_C}\}$

$=^x[\mathbf{B,C}]$ **for** $\{\langle\langle u,v\rangle,\langle f,g\rangle\rangle \mid \langle u,f\rangle{:} =_{\mathbf{aB}} \wedge \langle v,g\rangle{:} =_{\mathbf{aC}}\}$

$Sr^x[\mathbf{B,C}]$ **for** $\{\langle\langle u,v\rangle,\langle f,g\rangle\rangle \mid \langle u,f\rangle{:}\mathbf{Sr_B} \wedge \langle v,g\rangle{:}\mathbf{Sr_C}\}$

$Tg^x[\mathbf{B,C}]$ **for** $\{\langle\langle u,v\rangle,\langle f,g\rangle\rangle \mid \langle u,f\rangle{:}\mathbf{Tg_B} \wedge \langle v,g\rangle{:}\mathbf{Tg_C}\}$

$Cp^x[\mathbf{B,C}]$ **for** $\{\langle\langle u1,v1\rangle,\langle u2,v2\rangle,\langle f,g\rangle\rangle \mid \langle u1,u2,f\rangle{:}\mathbf{Cp_B} \wedge \langle v1,v2,g\rangle{:}\mathbf{Cp_C}\}$.

Given two functors **F** and **G** their product, **FxG** is given by:

FxG for $\{\langle\langle u,v\rangle,\langle f,g\rangle\rangle \mid \langle u,f\rangle{:}\mathbf{F} \wedge \langle v,g\rangle{:}\mathbf{G}\}$.

7.2.1. Lemma The sequents

$\rightarrow [\forall W,Z{:}Cat]\ WxZ{:}Cat$

$\rightarrow [\forall W1,W2,Z1,Z2{:}Cat][\forall F{:}Func[W1,Z1]][\forall G{:}Func[W2,Z2]]$
$FxG{:}Func[W1xW2, Z1xZ2]$

are derivable.

7.3. Comma Categories

If **B**,**C** and **D** are categories and **F**:**C**$\rightarrow$B, **G**:**D**$\rightarrow$B functors, the *comma* category **(F,G)** is defined to be the term

$$\langle Ar'[F,G,B,C,D], =' [F,G,B,C,D], Sr'[F,G,B,C,D], Tg'[F,G,B,C,D], Cp'[F,G,B,C,D] \rangle$$

with components:

$$\mathbf{Ar'[F,G,B,C,D]\ \ for}\ \ \{\langle u,v,w,x\rangle \mid u{:}\mathbf{Ar_C} \wedge v{:}\mathbf{Ar_D} \wedge w{:}\mathbf{Ar_B} \wedge x{:}\mathbf{Ar_B} \wedge$$
$$[\exists f,g,h{:}Ar_B](\langle u,f\rangle{:}\mathbf{F} \wedge \langle v,g\rangle{:}\mathbf{G} \wedge \langle f,w,h\rangle{:}\mathbf{Cp_B} \wedge \langle x,g,h\rangle{:}\mathbf{Cp_B})\}$$

$$\mathbf{='[F,G,B,C,D]\ \ for}\ \ \{\langle\langle u1,v1,w1,x1\rangle,\langle u2,v2,w2,x2\rangle\rangle \mid$$
$$\langle u1,u2\rangle{:} =_{\mathbf{aC}} \wedge \langle v1,v2\rangle{:} =_{\mathbf{aD}} \wedge \langle w1,w2\rangle{:} =_{\mathbf{aB}} \wedge \langle x1,x2\rangle{:} =_{\mathbf{aB}}\}$$

$$\mathbf{Sr'[F,G,B,C,D]\ \ for}\ \ \{\langle\langle u1,v1,w1,x1\rangle,\langle u2,v2,w2,x2\rangle\rangle \mid$$
$$\langle u1,u2\rangle{:}\mathbf{Sr_C} \wedge \langle v1,v2\rangle{:}\mathbf{Sr_D} \wedge \langle w1,w2\rangle{:} =_{\mathbf{aB}} \wedge \langle w1,x2\rangle{:} =_{\mathbf{aB}}\}$$

$$\mathbf{Tg'[F,G,B,C,D]\ \ for}\ \ \{\langle\langle u1,v1,w1,x1\rangle,\langle u2,v2,w2,x2\rangle\rangle \mid$$
$$\langle u1,u2\rangle{:}\mathbf{Tg_C} \wedge \langle v1,v2\rangle{:}\mathbf{Tg_D} \wedge \langle x1,w2\rangle{:} =_{\mathbf{aB}} \wedge \langle x1,x2\rangle{:} =_{\mathbf{aB}}\}$$

$$\mathbf{Cp'[F,G,B,C,D]\ \ for}\ \ \{\langle\langle u1,v1,w1,x1\rangle,\langle u2,v2,w2,x2\rangle,\langle u3,v3,w3,x3\rangle\rangle \mid$$
$$\langle u1,u3\rangle{:} =_{\mathbf{aB}} \wedge \langle v1,u2\rangle{:} =_{\mathbf{aB}} \wedge \langle v2,v3\rangle{:} =_{\mathbf{aB}} \wedge$$
$$\langle u1,u2,u3\rangle{:}\mathbf{Cp_C} \wedge \langle v1,v2,v3\rangle{:}\mathbf{Cp_D}\}.$$

The meticulous reader will have already noticed in the last definition a slight deviation from the traditional one. The arrows of a comma category, according to the above definition, are quadruples instead of pairs. Although such a deviation is immaterial (it only affects the representation of the construct not its properties), it has been found necessary to avoid the explicit use of objects and Hom-sets. Nevertheless, it can be shown that a triple $\langle e,d,f\rangle$ is an object of (F,G) as defined in [MacLane 71] iff $\langle e,d,f,f\rangle$ is an object of (F,G) according to our definition. Moreover, an arrow $\langle k,h\rangle : \langle e,d,f\rangle \to \langle e',d',f'\rangle$ in [MacLane 71] is exactly the arrow $\langle k,h,f,f'\rangle$ in our definition. The difference is that in the first case an arrow cannot be determined by the pair $\langle k,h\rangle$ alone without explicitly giving its source and target, while in our presentation the tuple $\langle k,h,f,f'\rangle$ uniqely deternines an arrow in (F,G).

7.3.1. Lemma The sequent

$$\to [\forall X,Y,Z{:}Cat][\forall F{:}Func[X,Y]][\forall G{:}Func[Z,Y]]\ \ (F,G){:}Cat$$

is derivable.

7.4. Universals and Limits

To improve readability, in the next two sections additional abbreviations will be used that resemble the functional notation used in mathematics. Specifically, if **F** is a functor (or transformation) from **B** to **C** then

$\mathbf{F}[\mathbf{x}]_{\mathbf{C}}$ **for** $\{y \mid y{:}\mathbf{Ar}_{\mathbf{C}} \wedge \langle x,y\rangle{:}\mathbf{F}\}$,

$[\mathbf{y}{\to}\mathbf{z}]_{\mathbf{C}}$ **for** $\{w \mid w{:}\mathbf{Ar}_{\mathbf{C}} \wedge \langle w,y\rangle{:}\mathbf{Sr}_{\mathbf{C}} \wedge \langle w,z\rangle{:}\mathbf{Tg}_{\mathbf{C}}\}$,

and combining them

$[\mathbf{y}{\to}\mathbf{F}[\mathbf{x}]]_{\mathbf{C}}$ **for** $\{w \mid w{:}\mathbf{Ar}_{\mathbf{C}} \wedge \langle w,y\rangle{:}\mathbf{Sr}_{\mathbf{C}} \wedge [\exists z{:}\mathbf{F}[\mathbf{x}]_{\mathbf{C}}]\langle w,z\rangle{:}\mathbf{Tg}_{\mathbf{C}}\}$.

Similar definition can be given for $[F[y]{\to}x]_{C}$ and $[F[y]{\to}F[x]]_{C}$. We can proceed now with the definition of universal arrows.

Given a functor **F**:**D**→**C** and an object **c** of **C**, the following term defines the set of *universal arrows* from **c** to **F**.

$\mathrm{UniArrFrom}[\mathbf{F},\mathbf{D},\mathbf{C},\mathbf{c}]$ **for**

$\{ \langle r,u\rangle \mid r{:}\mathrm{Ob}[\mathbf{D}] \wedge u{:}[\mathbf{c}{\to}\mathbf{F}[r]]_{\mathbf{C}} \wedge$

$[\forall d{:}\mathrm{Ob}[\mathbf{D}]]\, [\forall g{:}[\mathbf{c}{\to}\mathbf{F}[d]]_{\mathbf{C}}]\, [\exists g1{:}[r{\to}d]_{\mathbf{D}}]\, [\exists fg1{:}\mathbf{F}[g1]_{\mathbf{C}}](\langle u,fg1,g\rangle{:}\mathbf{Cp}_{\mathbf{C}} \wedge$

$[\forall g2{:}[r{\to}d]_{\mathbf{D}}]\, [\forall fg2{:}\mathbf{F}[g2]_{\mathbf{C}}](\langle u,fg2,g\rangle{:}\mathbf{Cp}_{\mathbf{C}} \supset g1 =_{\mathbf{aD}} g2))\ \}$

By duality, the set of universal arrows from the functor **F** to an object **c** is given by:

$\mathrm{UniArrTo}[\mathbf{F},\mathbf{D},\mathbf{C},\mathbf{c}]$ **for**

$\{ \langle r,u\rangle \mid r{:}\mathrm{Ob}[\mathbf{D}] \wedge u{:}[\mathbf{F}[r]{\to}\mathbf{c}]_{\mathbf{C}} \wedge$

$[\forall d{:}\mathrm{Ob}[\mathbf{D}]]\, [\forall g{:}[\mathbf{F}[d]{\to}\mathbf{c}]_{\mathbf{C}}]\, [\exists g1{:}[d{\to}r]_{\mathbf{D}}]\, [\exists fg1{:}\mathbf{F}[g1]_{\mathbf{C}}](\langle fg1,u,g\rangle{:}\mathbf{Cp}_{\mathbf{C}} \wedge$

$[\forall g2{:}[d{\to}r]_{\mathbf{D}}]\, [\forall fg2{:}\mathbf{F}[g2]_{\mathbf{C}}](\langle fg2,u,g\rangle{:}\mathbf{Cp}_{\mathbf{C}} \supset g1 =_{\mathbf{aD}} g2))\ \}$

A definition of the diagonal functor must preceed a discusion of limits and colimits. In the following definitions **B** and **C** are categories, **c** an object of **C** and **f** an arrow of **C**:

$\mathrm{DF}[\mathbf{B},\mathbf{C},\mathbf{c}]$ **for** $\{\langle u,v\rangle \mid u{:}\mathbf{Ar}_{\mathbf{B}} \wedge v =_{\mathbf{aC}} \mathbf{c}\}$

$\mathrm{DT}[\mathbf{B},\mathbf{C},\mathbf{f}]$ **for** $\{\langle u,v\rangle \mid u{:}\mathrm{Ob}[\mathbf{B}] \wedge v =_{\mathbf{aC}} \mathbf{f}\}$.

The diagonal functor from **C** to $\mathbf{C}^{\mathbf{B}}$ is defined as

$\Delta[\mathbf{B},\mathbf{C}]$ **for** $\{\langle u,y\rangle \mid u{:}\mathbf{Ar}_{\mathbf{C}} \wedge y =_{\mathbf{e}} \mathrm{DT}[\mathbf{B},\mathbf{C},u]\}$

The following lemma justifies these definitions:

7.4.1. Lemma The sequences

$\rightarrow$ [∀J,X:Cat][∀c:Ob[X]] DF[J,X,c]:Func[J,X]

$\rightarrow$ [∀J,X:Cat] [∀c,c':Ar_X] [∀f:$[c\rightarrow c']_X$]

DT[J,X,f]:NatTrans[DF[J,X,c], DF[J,X,c'], J,X]

$\rightarrow$ [∀J,X:Cat] Δ[J,X]:Func[X,X^J]

are derivable.

Definitions of limits and colimits can now be given. Given a functor **F**:**B**→**C**, the *limits* for **F** are given by

Limit[**F**,**B**,**C**] **for** { <u,v> | <u,v>:UniArrowTo[Δ[**B**,**C**], **C**, $\mathbf{C^B}$, **F**] }

and the *colimits* of F by

Colimit[**F**,**B**,**C**] **for** { <u,v> | <u,v>:UniArrowFrom[Δ[**B**,**C**], **C**, $\mathbf{C^B}$, **F**] } .

Products, powers, equalizers, pullbacks and their duals can easily be defined as special cases of limits and colimits respectively.

7.5. Adjoints

Given two categories **C**, **D**, an *adjunction* from **C** to **D** consists of a pair of functors **F**:**C**→**D**, **G**:**D**→**C** and a natural transformation η from the identity functor of **C** to the composition of **F** and **G** with some additional properties given by the folowing definition.

Adjunction[**C**,**D**,**F**,**G**,η] **for**

F:Func[**C**,**D**] ∧ **G**:Func[**D**,**C**] ∧

η:NatTrans[𝕀d[**C**], 𝔽ℂ[**F**,**C**,**D**,**G**,**D**,**C**], **C**, **C**] ∧

[∀x:Ob[**C**]] [∀y:Ob[**D**]] [∀f:$[x\rightarrow \mathbf{G}[y]]_{\mathbf{C}}$] [∃ηx:$\eta[x]_{\mathbf{C}}$]

[∃f1:$[\mathbf{F}[x]\rightarrow y]_{\mathbf{D}}$] [∃gf1:$\mathbf{G}[f1]_{\mathbf{C}}$] (<ηx,gf1,f>:$\mathbf{Cp_C}$ ∧

[∀f2:$[\mathbf{F}[x]\rightarrow y]_{\mathbf{D}}$] [∀gf2:$\mathbf{G}[f2]_{\mathbf{C}}$] (<ηx,gf2,f>:$\mathbf{Cp_C}$ ⊃ f1 $=_{\mathbf{aD}}$ f2))

or equivalently,

Adjunction[**C**,**D**,**F**,**G**,η] **for**

F:Func[**C**,**D**] ∧ **G**:Func[**D**,**C**] ∧

η:NatTrans[𝕀d[**C**], 𝔽ℂ[**F**,**C**,**D**,**G**,**D**,**C**], **C**, **C**] ∧

[∀x:Ob[**C**]] [∃fx:$\mathbf{F}[x]_{\mathbf{D}}$] [∃ηx:$\eta[x]_{\mathbf{C}}$] <fx,ηx>:UniArrFrom[**G**,**D**,**C**,x]

Finally,the set of *adjoint pairs* of functors from **C** to **D** is defined as

Adjoint[C,D] **for**

{<f,g> | [∃η:NatTrans[Id[C], FC[f,C,D, g,D,C], C, C]] Adjunction[C,D, f, g,η] }.

These definitions within NaDSet suggest that the other constructions in category theory as well as in topos theory can be defined and used in a similar way. NaDSet, with its reductionist semantics, can provide a logic for the theory of abstract categories and provide semantics that does not rely on a fixed universe of "constant" sets.

8. CONCLUSION

Category theory has been widely applied in mathematics as well as computer science. The theory has been used to provide an abstract and uniform treatment of many mathematical structures. This suggests that any logical system which aspires to provide a foundation for mathematics has to provide one for abstract categories and functors. In this paper NaDSet has been shown to provide such a foundation.

An important aspect of the formalization of category and other theories within NaDSet is that they do not impose any existential implications on NaDSet. More specifically, the models of NaDSet are not affected by any of the definitions in sections 3 through 7. This is in contrast to the standard formalizations of theories within first order logic; the nonlogical axioms of a consistent theory necessarily restrict the models of the first order theory. In this sense, therefore, a negative answer can be provided to the question posed in [Blass84], whether category theory necessarily involves existential principles that go beyond those of other mathematical disciplines. However, this does not determine the power of the methods that may be needed to construct a particular category. But it does make that the distinction between small and large categories [Mac Lane 71] unnecessary.

Formalizations for most of the main concepts and constructs in category theory have been presented. But in addition, NaDSet definitions for natural transformations, functor categories, an equivalence relation on categories, products, comma categories, universals limits and adjoints have also been provided. This suggests that the variety of constructs defined for categories, sheaves, triples and related theories, [Barr Wells 85] can be defined within NaDSet.

BIBLIOGRAPHY

The numbers in parentheses refer to the date of publication. (xx) is the year 19xx.

Barr, M. & Wells, C.

(85) Toposes, Triples and Theories, Springer-Verlag

Blass, Andreas

(84) The Interaction between Category Theory and Set Theory, in Mathematical Applications of Category Theory, J.W. Gray editor, *Contemporary Mathematics*, 30, American Mathematical Society, 5-29.

Feferman, Solomon

(77) Categorical Foundations and Foundations of Category Theory, Logic, Foundations of Mathematics and Computability Theory, Editors Butts and Hintikka, D. Reidel, 149-169.

(84) Towards Useful Type-Free Theories, I, *Journal of Symbolic Logic*, March, 75-111.

Gentzen, Gerhard

(34,35) Untersuchungen über das logische Scliessen, *Mathematische Zeitschrift*, 39, 176-210, 405-431.

Gilmore, P.C. (Paul C.),

(71) A Consistent Naive Set Theory: Foundations for a Formal Theory of Computation, IBM Research Report RC 3413, June 22.

(80) Combining Unrestricted Abstraction with Universal Quantification, To H.B. Curry: Essays on Combinatorial Logic, Lambda Calculus and Formalism, Editors J.P. Seldin, J.R. Hindley, Academic Press, 99-123. This is a revised version of [Gilmore71].

(86) Natural Deduction Based Set Theories: A New Resolution of the Old Paradoxes, *Journal of Symbolic Logic*, 51, 393-411.

(89) How Many Real Numbers are There?, Department of Computer Science Technical Report TR 89-7, University of British Columbia.

Gray, J.W.

(84) Editor, Mathematical Applications of Category Theory, Contemporary Mathematics, Vol. 30, American Mathematical Society.

Kleene, Stephen Cole

(52) Introduction to Metamathematics, North-Holland.

Kripke, Saul

(75) Outline of a Theory of Truth, *Journal of Philosophy*, November 6, 690-716.

Mac Lane, Saunders

(71) Categories for the Working Mathematician, Springer-Verlag

Lawvere, F.W.

(66) The Category of Categories as a Foundation for Mathematics. Proc. Conf. on Categorical Algebras, pp1-21, Springer

Quine, Willard Van Orman

(51) Mathematical Logic, Harvard.

Schütte, K.

(60) Beweistheorie, Springer

Tarski, Alfred

(56) The Concept of Truth in Formalized Languages appearing in Logic, Semantics, Metamathematics, Papers from 1923 to 1938, Oxford University Press, 152-278.