PRODUCTIVE SETS AND CONSTRUCTIVELY NONPARTIAL-RECURSIVE FUNCTIONS

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Technical Report 86-15

August 1986

Revised November 1986

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Horowitz [1] called a partial function f:**N**→**N** *constructively nonpartial-recursive* if for some recursive function h:**N**→**N**, $f(h(n)) \not\cong \phi_n(h(n))$ where $\cong$ is equality for partial functions. He related this concept to the productiveness of the domain of such functions. His main result is as follows:

Proposition 1. (Horowitz)

(1) If the domain **D**f of a partial recursive function f:**N**→**N** is productive, then f is constructively nonpartial-recursive.

(2) If f:**N**→**N** is constructively nonpartial-recursive via h:**N**→**N** such that $\phi_x(h(x))$ is defined implies $f(h(x))$ undefined, then **D**f is productive.

□

In this short note, we characterize productive sets in terms of constructively nonpartial-recursive functions. Hence we further the study of the intimate connection between the theory of constructively nonpartial-recursive functions and that of productive sets.

A constructively nonpartial-recursive function f is *strongly constructively nonpartial-recursive* if there is a recursive function $e:N \to N$ such that $f(h(n))$ defined implies $e(n)=f(h(n))$.

Let $<\_,\_>$ be a standard pairing function with the inverse witnessed by π_1 and π_2. For any partial function f:N→N, Gf denotes the graph of f, more precisely,

$$Gf=\{<m,n>|f(m)=n\}$$

There are recursive functions g:N→N and a:N→N satisfying

- $G\phi_n = W_{g(n)}$
- If W_n is single valued then $W_n = G\phi_{a(n)}$.

Theorem 2.

(1) Gf is productive only if f is constructively nonpartial-recursive.

(2) Gf is productive if f is strongly constructively nonpartial-recursive.

Proof. (only if) Assume that Gf is completely productive via a recursive function h:N→N. If $h(g(n))\in Gf$ where g is as above, then $h(g(n)) \notin W_{g(n)}$. Thus, $\phi_n(\pi_1 \cdot h \cdot g(n))$ is undefined. But $f(\pi_1 \cdot h \cdot g(n))$ is defined. If $h(g(n)) \notin Gf$, then $h(g(n)) \in W_{g(n)}$. Thus $\phi_n(\pi_1 \cdot h \cdot g(n)) = \pi_2 \cdot h \cdot g(n)$. But $f(\pi_1 \cdot h \cdot g(n))$ is either undefined or not equal to $\pi_2 \cdot h \cdot g(n)$. Therefore, f is constructively nonpartial-recursive via $\pi_1 \cdot h \cdot g$.

(if) Assume that f is strongly constructively nonpartial-recursive. Let $W_n \subseteq Gf$. Then $W_n = G\phi_{a(n)}$. If $<h(a(n)),m> \in Gf$ for some m then $<h(a(n)),m> \notin W_n$. If $<h(a(n)),k> \notin Gf$ for all k, then $<h(a(n)),k> \in W_n$ for some k. Then $<h(a(n)),k> \in Gf$. This is a contradiction. Therefore Gf

is productive via $h\prime:\mathbf{N}\rightarrow\mathbf{N}$ such that

$$h\prime(n)=<h(a(n)),m>.$$

□

Let $i_A:\mathbf{N}\rightarrow\mathbf{N}$ be the following partial function:

$$i_A(x)= \begin{cases} x & \textit{if } x\in A \\ \textit{undefined} & \textit{otherwise.} \end{cases}$$

Theorem 3. (The Characterization Theorem)

A is productive iff i_A is constructively nonpartial recursive.

Proof. (only if) If A is productive then $\mathbf{D}i_A$ is productive. Thus i_A is constructively nonpartial-recursive.

(if) If i_A is constructively nonpartial-recurisve then it is strongly constructively nonpartial recursive thus $\mathbf{G}i_A=\{<x,x>|x\in A\}$ is productive. But $\mathbf{G}i_A\leq_m A$ via

$$f(z) = \begin{cases} \pi_1(z) & \textit{if } \pi_1(z)=\pi_2(z) \\ e \notin A & \textit{otherwise} \end{cases}$$

Thus A is productive.

□

Acknowledgement

The author thanks D. Spreene for finding an error in the earlier version of this paper.

References

[1] Horowitz, B.M., 'Constructively Nonpartial Recursive Functions', Notre Dame Journal of Formal Logic, Vol. XXI, Number 2, 1980.