RETRACTS OF NUMERATIONS

Akira Kanda
Department of Computer Science
University of British Columbia
Vancouver, B.C.
Canada

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ABSTRACTS

In this paper we study some important properties of numerations which can be passe to their retracts. Furthermore we show a sufficient condition for a category $Ret(\alpha)$ of retracts of a numeration α and morphisms to be Cartesian closed, in terms of α.

§1. Introduction

The retracts have been uesd for the study of substructure. In this paper, we study retracts of numerations.

In §2, we study some interesting properties of numerations which can be passed to their retracts. Especially we study a few properties, yielding recursion theorems, which can be passed to their retracts.

In §3, we study a category $Ret(\alpha)$ of retracts of a numeration α and morphisms, which is a full subcategory of the category *Num* of numerations and morphisms. One of the main results is a sufficient condition for $Ret(\alpha)$ to be Cartesian closed, in terms of the numeration α.

Before we finish this introductory section, we briefly overview a small part of the theory of numerations developed by Eršov and Mal'cev. For details and further exposure readers and referred to Eršov [1] and Mal'cev [4].

Definition 1.1.

A *numeration (of a set A)* is a surjection $\alpha : N \to A$. Let $\alpha : N \to A$, $\beta : N \to B$ be numerations. A *morphism f from α to β* is a function $f : A \to B$ such that there exists a recursive function $r_f : N \to N$ which makes the following diagram commute:

$$\begin{array}{ccccc} & A & \xrightarrow{\;f\;} & B & \\ \alpha & \uparrow & & \uparrow & \beta \\ & N & \xrightarrow{\;r_f\;} & N & \end{array}$$

We say r_f *realizes f.*

$\square$

Lemma 1.2.

Numerations and morphisms among them form a category. We denote it by *Num.*

□

Definition 1.3.

Let $\alpha: N \rightarrow A$ be a numeration. It is *precomplete* if for every partial recursive function $f: N \rightarrow N$ there is a recursive function $g: N \rightarrow N$ such that

$$f(i) \quad \textit{implies} \quad \alpha(g(i)) = \alpha(f(i)).$$

We say g *totalizes* f *modulo* α. It is *complete* if there exists an element $e \in A$ such that for every partial recursive $f: N \rightarrow N$ there exists a recursive $g: N \rightarrow N$ satisfying:

$$\alpha(g(i)) = \begin{cases} \alpha(f(i)) & \text{if } f(i)\downarrow \\ e & \textit{otherwise.} \end{cases}$$

□

Theorem 1.4. (Eršov Recursion Theorem [1])

A numeration $\alpha: N \rightarrow A$ is precomplete iff there exists a recursive function $fix: N \rightarrow N$ such that

$$\varphi_i^{(1)}(fix(i))\downarrow \quad \textit{implies} \quad \alpha(\varphi_i^{(1)}(fix(i))) = \alpha(fix(i)).$$

where $\varphi^{(k)}$ is the Kleene numbering of partial recursive k-ary functions. We call $fix(i)$ a *fixpoint of* $\phi_i^{(1)}$ *modulo* α.

□

Corollary 1.5.

A numeration $\alpha: N \rightarrow A$ is precomplete if there exists a recursive function $total: N \rightarrow N$ such that:

$$\varphi^{(1)}_{total(i)} \quad \textit{totalizes} \quad \varphi^{(1)}_i \ \textit{modulo}\ \alpha.$$

□

Definition 1.6.

Let $\alpha:N \to A$, $\beta:N \to B$ be numerations. A numeration $\tau:N \to Hom(\alpha,\beta)$ is ***realizable*** if there exists a recursive function $real:N \to N$ such that:

$$\varphi^{(1)}_{real(i)} \qquad \textit{realizes} \qquad \tau(i).$$

It is *enumerable* if there exists a recursive function $enum:N \to N$ such that:

$$\textit{if}\ \varphi^{(1)}_i\ \textit{realizes}\ f \in Hom(\alpha,\beta) \qquad \textit{then}\ f = \tau(enum(i))$$

It is *acceptable* if it is both realizable and enumerable.

□

Theorem 1.7. (see [2])

Let $\tau,\tau':N \to Hom(\alpha,\beta)$ be acceptable. Then there exists a recursive isomorphism $h:N \to N$ such that

$$\tau = \tau' \cdot h.$$

□

Definition 1.8.

Given numerations $\alpha_1:N \to A_1,\ldots,\alpha_k:N \to A_k$, we define a numeration $\alpha_1 \times \ldots \times \alpha_k:N \to A_1 \times \ldots \times A_k$ by

$$\alpha_1 \times \ldots \times \alpha_k(<x_1,\ldots,x_k>)$$
$$= (\alpha_1(x_1),\ldots,\alpha_k(x_k))$$

where $<x_1,\ldots,x_k>:N^k \to N$ is the standard bijection.

□

Definition 1.9.

Let $\alpha:N\rightarrow A$, $\beta:N\rightarrow B$ be numerations. A numeration $(\alpha\rightarrow\beta):N\rightarrow Hom(\alpha,\beta)$ is *abstract* if for every $f\in Hom((\alpha\rightarrow\beta)\times\alpha,\beta)$ there exists a morphism $c_f\in Hom((\alpha\rightarrow\beta),(\alpha\rightarrow\beta))$ satisfying:

$$f((\alpha\rightarrow\beta)(i),\alpha(j))$$
$$=c_f((\alpha\rightarrow\beta)(i))(\alpha(j)).$$

□

Theorem 1.10. (K-recursion Theorem [3])

Assume $(\alpha\rightarrow\beta)$ is abstract and precomplete. For all $f\in Hom((\alpha\rightarrow\beta)\times\alpha,\beta)$ there exists a number $m_f\in N$ such that

$$f((\alpha\rightarrow\beta)(m_f),\alpha(j))$$
$$=((\alpha\rightarrow\beta)(m_f))(\alpha(j)).$$

Proof. (Outline) A fixpoint of c_f modulo $(\alpha\rightarrow\beta)$ is the desired number.

□

Theorem 1.11. (see [2])

Let α,β,γ be numerations such that $(\alpha\times\beta\rightarrow\gamma):N\rightarrow Hom(\alpha\times\beta,\gamma)$, $(\beta\rightarrow\gamma):N\rightarrow Hom(\beta,\gamma)$ and $(\alpha\rightarrow(\beta\rightarrow\gamma)):N\rightarrow Hom(\alpha,(\beta\rightarrow\gamma))$ are acceptable. Then $(\alpha\times\beta\rightarrow\gamma)\cong(\alpha\rightarrow(\beta\rightarrow\gamma))$.

□

§2. Retracts of Numerations

Definition 2.1.

Let $\alpha:N \to A$ be a numeration. A morphism $h \in Hom(\alpha,\alpha)$ is an *idempotent of* α if $h = h \cdot h$. The numeration $\gamma:N \to h(A)$ such that

$$\gamma(i) = h(\alpha(i))$$

is called a *retract of* α *(via* h*).*

□

Lemma 2.2.

Let α be a numeration and γ be a retract of α via h. Then $x = \gamma(i)$ for some $i \in N$ iff x is a fixpoint of h.

Proof. trivial.

□

Theorem 2.3.

Let $\alpha:N \to A$, $\beta:N \to B$ be numerations and γ, γ' be retracts of α, β via h, h' respectively. $f:h(A) \to h'(B)$ is a morphism from γ to γ' iff there exists a morphism $\tilde{f} \in Hom(\alpha,\beta)$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\tilde{f}} & B \\ h \downarrow & & \downarrow h' \\ h(A) & \xrightarrow[f]{} & h'(B) \end{array}$$

Proof. Assume $f \in Hom(\gamma,\gamma')$. Define $\tilde{f}:A \to B$ by

$$\tilde{f} = h' \cdot f \cdot h.$$

This $\tilde{f}$ makes the above diagram commute and it is a morphism from α to β due to 1.2. Conversely assume $f : h(A) \rightarrow h'(B)$ and for some $\tilde{f} \in Hom(\alpha,\beta)$ the above diagram commutes. Let $g = h \restriction h(A)$. This g is a morphism from γ to α because

$$g(\gamma(i)) = h(\alpha(r_h(i))) = \alpha(r_h \cdot r_h(i)).$$

By the diagram we have:

$$f = h' \cdot \tilde{f} \cdot g.$$

Thus $f \in Hom(\gamma,\gamma')$.

□

Theorem 2.4.

Let γ be a retract of α. Then we have:

(1) If α is precomplete then so is γ.

(2) If α is complete then so is γ.

Proof. trivial.

□

Definition 2.5.

Let γ, γ' be retracts of α, β via h, h' respectively. Also let $(\alpha\rightarrow\beta)$ be a numeration of $Hom(\alpha,\beta)$. Define a numeration $(\gamma\rightarrow\gamma')$ of $Hom(\gamma,\gamma')$ by

$$(\gamma\rightarrow\gamma')(i) = h' \cdot ((\alpha\rightarrow\beta)(i)) \cdot g$$

where $g = h \restriction h(A)$.

□

Due to the theorem 2.3, this $(\gamma \rightarrow \gamma')$ is well-defined.

Theorem 2.6.

Let γ, γ' be retracts of α, β via h, h' respectively. Also assume that $(\alpha \rightarrow \beta)$ is a numeration of $Hom(\alpha,\beta)$.

(1) if $(\alpha \rightarrow \beta)$ is precomplete then so is $(\gamma \rightarrow \gamma')$.

(2) if $(\alpha \rightarrow \beta)$ is complete then so is $(\gamma \rightarrow \gamma')$.

Proof. trivial.

$\square$

Theorem 2.7.

Let γ, γ' be retracts of α, β via h, h' respectively. Also assume that $(\alpha \rightarrow \beta)$ is a numeration of $Hom(\alpha,\beta)$.

(1) if $(\alpha \rightarrow \beta)$ is realizable then so is $(\gamma \rightarrow \gamma')$.

(2) if $(\alpha \rightarrow \beta)$ is enumerable then so is $(\gamma \rightarrow \gamma')$.

(3) if $(\alpha \rightarrow \beta)$ is acceptable then so is $(\gamma \rightarrow \gamma')$.

Proof.

(1) $(\gamma \rightarrow \gamma')(i) = h' \cdot ((\alpha \rightarrow \beta)(i)) \cdot g$. But $(\alpha \rightarrow \beta)(i)$ is realized by $\varphi^{(1)}_{real(i)}$. Thus $(\gamma \rightarrow \gamma')(i)$ is realized by

$$r = r_{h'} \cdot \varphi^{(1)}_{real(i)} \cdot r_g .$$

By $s-m-n$ theorem, $r = \varphi^{(1)}_{z(i)}$ for some recursive $z: N \rightarrow N$.

(2) Assume $\varphi^{(1)}_i$ realizes $f \in Hom(\gamma,\gamma')$. Then $\tilde{f} = h' \cdot f \cdot h$ is realized by

$$r = r_{h'} \cdot \varphi^{(1)}_i \cdot r_h = \varphi^{(1)}_{t(i)}$$

where t is a recursive function due to the $s-m-n$ theorem. Since $\tilde{f} \in Hom(\alpha,\beta)$, and $(\alpha\rightarrow\beta)$ is enumerable,

$$\tilde{f} = (\alpha\rightarrow\beta)(enum(t(i))).$$

Thus we have

$$f = h' \cdot ((\alpha\rightarrow\beta)(enum(t(i)))) \cdot g$$
$$= (\gamma\rightarrow\gamma')(enum \cdot t(i))$$

(3) immediate from (1) to (2).

□

Theorem 2.8. (see [3])

Let γ, γ' be retracts of α, α' via h, h' respectively. Assume $(\alpha\rightarrow\beta)$ is abstract, then so is $(\gamma\rightarrow\gamma')$.

Proof. For every $t \in Hom((\gamma\rightarrow\gamma')\times\gamma,\gamma')$, we define $T: Hom(\alpha,\beta)\times A \rightarrow B$ as

$$T(f,a) = t(h' \cdot f \cdot g, h(a))$$

where $g = h \restriction h(A)$. It can readily be seen that T is a morphism from $(\alpha\rightarrow\beta)\times\alpha$ to β. Since $(\alpha\rightarrow\beta)$ is abstract, for some $C_T \in Hom((\alpha\rightarrow\beta),(\alpha\rightarrow\beta))$

$$T((\alpha\rightarrow\beta)(i),\alpha(j)) = C_T((\alpha\rightarrow\beta)(i))(\alpha(j)).$$

Define $C_t: Hom(\gamma,\gamma')\rightarrow Hom(\gamma,\gamma')$ by

$$C_t(f) = h' \cdot C_T(h' \cdot f \cdot h) \cdot g.$$

Using that h', h are idempotents, we can show

$$C_T((\gamma\rightarrow\gamma')(i))(\gamma(j))$$

$$= t((\gamma \to \gamma')(i), \gamma(j)).$$

□

Theorem 2.9.

Let $\alpha_i : N \to A_i$ be a numeration and let γ_i be retracts of α_i via h_i respectively $(1 \leq i \leq k)$. Then $\gamma_1 \times ... \times \gamma_k$ is a retract of $\alpha_1 \times ... \times \alpha_k$. Also if α_i, $1 \leq i \leq k$ are precomplete then so is $\gamma_1 \times ... \times \gamma_k$. Furthermore if α_i, $1 \leq i \leq k$ are complete then so is $\gamma_1 \times ... \times \gamma_k$.

Proof. Define $h = h_1 \times ... \times h_k$. Then $h = h \cdot h$. Also h is a morphism from $\alpha_1 \times ... \times \alpha_k$ to itself, for

$$r(<x_1,...,x_k>) = <r_{h_1}(x_1),...,r_{h_k}(x_k)>$$

realizes h. But obviously

$$\gamma_1 \times ... \times \gamma_k (<x_1,...,x_k>)$$
$$= h(\alpha_1 \times ... \times \alpha_k(<x_1,...,x_k>))$$

Thus $\gamma_1 \times ... \times \gamma_k$ is a retract of $\alpha_1 \times ... \times \alpha_k$ via h. Assume α_i $(1 \leq i \leq k)$ are precomplete then by 2.4, γ_i are precomplete. Let $f : N \to N$ be a partial recursive function. Define $f_i : N \to N$ by:

$$f_i(x) = y_i \qquad \textit{where} \quad f(x) = <y_1,...,y_k>.$$

Then f_i are partial recursive, $1 \leq i \leq k$. Let g_i totalizes f_i moduldo γ_i. Define $g : N \to N$ by

$$g(x) = <g_1(x),...,g_k(x)>.$$

Then g is recursive. Also we have:

$$\gamma_1\times\ldots\times\gamma_k(g(x))$$
$$=\gamma_1\times\ldots\times\gamma_k(<g_1(x),\ldots,g_k(x)>)$$
$$=\gamma_1\times\ldots\times\gamma_k(f(x)) \qquad \text{if } f(x)\downarrow.$$

Thus g totalizes f modulo $\gamma_1\times\ldots\times\gamma_k$. Similarly if α_i $(1\leq i\leq k)$ are complete then so is $\gamma_1\times\ldots\times\gamma_k$.

$\square$

Theorem 2.10.

Assume $(\alpha_1\times\ldots\times\alpha_k\rightarrow\alpha_0):N\rightarrow Hom(\alpha_1\times\ldots\times\alpha_k,\alpha_0)$ is abstract and precomplete. Then for each retracts γ_i of α_i $(0\leq i\leq k)$, if $f\in Hom((\gamma_1\times\ldots\times\gamma_k\rightarrow\gamma_0)\times\gamma_1\times\ldots\times\gamma_k,\gamma_0)$ then there exists $m_f\in N$ such that

$$f((\gamma_1\times\ldots\times\gamma_k\rightarrow\gamma_0)(m_f),\gamma_1(x_1),\ldots,\gamma_k(x_k))$$
$$=(\gamma_1\times\ldots\times\gamma_k\rightarrow\gamma_0)(m_f)(\gamma_1(x_1),\ldots,\gamma_k(x_k)).$$

Proof. Immediate from 1.10, 2.6, 2.8 and 2.9.

$\square$

Lemma 2.11.

(1) If γ is a retract of $\alpha:N\rightarrow A$ then there is a pair (f,g) of morphisms $f\in Hom(\alpha,\gamma)$, $g\in Hom(\gamma,\alpha)$ such that $f\cdot g=id_\gamma$ where id_γ is the identity morphism over γ.

(2) If there exists a pair (f,g) of morphisms $f\in Hom(\alpha,\beta)$, $g\in Hom(\beta,\alpha)$, such that $f\cdot g=id_\beta$ then there is a retract γ of α such that $\gamma\cong\beta$ in Num.

Proof.

(1) Assume γ is a retract of α via h. Define $g=h\restriction h(A)$. Then $g\in Hom(\gamma,\alpha)$ and $h\cdot g=id_\gamma$.

(2) Define $h=g\cdot f$. Then $h\in Hom(\alpha,\alpha)$. Also $h\cdot h=g\cdot f\cdot g\cdot f=g\cdot f=h$. Let γ be a retract of α via h. Define $\Phi=f\cdot(h\restriction h(A))$ and $\Psi=h\cdot g$. Then $\Phi\in Hom(\gamma,\beta)$ and $\Psi\in Hom(\beta,\gamma)$, because $h\in Hom(\alpha,\gamma)$ and $h\restriction h(A)\in Hom(\gamma,\alpha)$. Furthermore we have:

$$\begin{aligned}
\Phi\cdot\Psi &= f\cdot g\cdot(f\restriction h(A))\cdot h\cdot g\\
&= f\cdot g\cdot f\cdot h\cdot g\\
&= f\cdot g\cdot f\cdot g\cdot f\cdot g\\
&= f\cdot g\cdot f\cdot g\\
&= id_{\beta}.
\end{aligned}$$

$$\begin{aligned}
\Psi\cdot\Phi &= g\cdot f\cdot g\cdot f\cdot(h\restriction h(A))\\
&= g\cdot f\cdot g\cdot f\cdot g\cdot f\restriction h(A)\\
&= g\cdot f\restriction h(A)\\
&= h\restriction h(A)\\
&= id_{\gamma}.
\end{aligned}$$

□

Note. If there exists a pair (f,g) of morphisms $f\in Hom(\alpha,\beta)$ and $g\in Hom(\beta,\alpha)$ such that $f\cdot g=id_{\beta}$, we write $\alpha\rhd\beta$. Also (f,g) is called a *retraction pair*.

§3. Categories of Retracts

Lemma 3.1.

Every numeration is a retract of itself via the identity morphism. Therefore the *Num* is the same as the category of retracts (of numerations) and morphisms.

Proof. trivial

□

Lemma 3.2.

Retracts of a numeration $\alpha:N\rightarrow A$ and morphisms among them form a category, which is denoted by $Ret(\alpha)$. This is a full subcategory of *Num* .

Proof. Immediate from 1.2 and 2.3.

□

Lemma 3.3.

$Ret(\alpha)$ has a final object.

Proof. Let $h_0:A\rightarrow A$ be the following function:

$$h_0(x)=a \qquad \textit{for all } x$$

where a is an element of A . Then $h_0\cdot h_0=h_0$. Also $r:N\rightarrow N$ such that

$$r(i)=j \qquad \textit{for all } i$$

where $\alpha(j)=a$, realizes h_0. Thus h_0 is an idempotent of α. Let $\rho:N\rightarrow\{a\}$ be the following numeration:

$$\rho(i)=h_0(\alpha(i)).$$

Then $\rho \in Ret(\alpha)$. Let $\delta \in Ret(\alpha)$. Then for some idempotent h of α,

$$\delta(i)=h(\alpha(i)).$$

Let f be the only function from $h(A)$ to $\{a\}$. Obviously $g:N\to N$ such that

$$g(i)=0$$

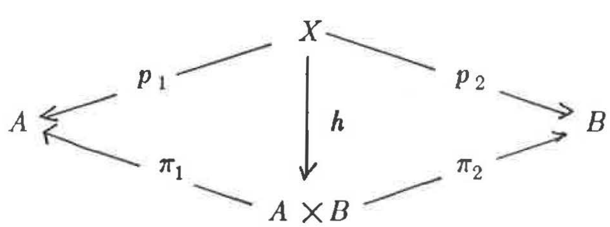
is a recursive function which realizes f. Thus $\{f\}=Hom(\delta,\rho)$. Thus ρ is a final object in $Ret(\alpha)$.

$\square$

Lemma 3.4. (Eršov [1])

Num is a Cartesian category.

Proof. Let $\alpha:N\to A$ and $\beta:N\to B$ be numerations. Furthermore let $\chi:N\to X$ be a numeration and $p_1 \in Hom(\chi,\alpha)$, $p_2 \in Hom(\chi,\beta)$. Since the category *Set* of sets and functions is Cartesian there is a unique function $h:X\to A\times B$ such that the following diagram in *Set* commutes:



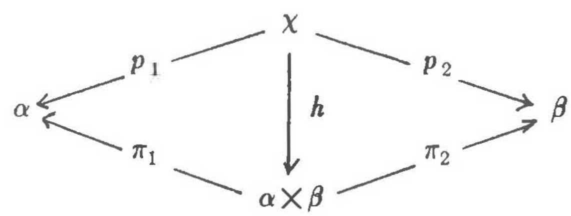
In fact $h(x)=(p_1(x),p_2(x))$. But this $h:X\to A\times B$ can be realized by the following recursive function $r:N\to N$;

$$r(i)=<r_{p_1}(i),r_{p_2}(i)>$$

because

$$h(\chi(i)) = (p_1(\chi(i)), p_2(\chi(i)))$$
$$= (\alpha(r_{p_1}(i)), \beta(r_{p_2}(i)))$$
$$= \alpha\times\beta(<r_{p_1}(i), r_{p_2}(i)>).$$

Thus $h \in Hom(\chi, \alpha\times\beta)$. Also $\pi_1{:}A\times B\rightarrow A$, $\pi_2{:}A\times B\rightarrow B$ can be realized by $<>_1$, $<>_2$ where $<>_1$, $<>_2$ are the recursive inverse of $<x_1,x_2>$. Thus $\pi_1 \in Hom(\alpha\times\beta,\alpha)$, $\pi_2\in Hom(\alpha\times\beta,\beta)$. Therefore h is the unique morphism from χ to $\alpha\times\beta$ which makes the following diagram in *Num* commute:



Obviously *Num* has the final object.

□

Theorem 3.5.

Let $\alpha{:}N\rightarrow A$ be a numeration. If $\alpha\cong\alpha\times\alpha$ in *Num* then $Ret(\alpha)$ is a Cartesian category.

Proof. Let $[-,-]$ be a pairing morphism and $[]_1$, $[]_2$ be the inverses of $[-,-]$ such that

$$[\alpha(i),\alpha(j)]_1 = \alpha(i)$$
$$[\alpha(i),\alpha(j)]_2 = \alpha(j)$$

Assume γ_i is a retract of α via h_i $(i=1,2)$. By 2.9, $\gamma_1\times\gamma_2$ is a retrct of α via $h_1\times h_2$. Define $h:A\rightarrow A$ by,

$$h(a) = [h_1\times h_2([a]_1,[a]_2)] = [h_1([a]_1), h_2([a])_2]$$

Then h is realized by a recursive function:

$$r(j) = r_{[-,-]}(r_{h_1 \times h_2}(<r_{[]_1}(j), r_{[]_2}(j)>)).$$

Furthermore we have:

$$\begin{aligned} h \cdot h([a,b]) &= [h_1 \times h_2(h_1(a), h_2(b))] \\ &= [h_1 \cdot h_1(a), h_2 \cdot h_2(b)] \\ &= [h_1(a), h_2(b)] \\ &= h([a,b]). \end{aligned}$$

Thus h is an idempotent of α. Let γ be a retract of α via h. Define $\rho_i : h(A) \rightarrow h_i(A)$ $(i=1,2)$ by:

$$\rho_i(h(a)) = h_i([a]_i).$$

Then $\rho_i \in Hom(\gamma, \gamma_i)$ because the recursive functions $r_{[]_i}$ realizes ρ_i. We calim that (γ, ρ_1, ρ_2) is the product of γ_1 and γ_2 in $Ret(\alpha)$. To prove this claim, assume δ is a retract of α via g and $p_i \in Hom(\delta, \delta_i)$ $i=1,2$. Define a function $u : g(A) \rightarrow h(A)$ by:

$$u(x) = [p_1(x), p_2(x)]$$

This u is a morphism from δ to γ because

$$r_u(j) = r_{[-,-]}(<r_{h_1} \cdot r_{p_1}(j), r_{h_2} \cdot r_{p_2}(j)>)$$

realizes it as

$$\begin{aligned} u(\delta(n)) &= [\delta_1(r_{p_1}(n)), \delta_2(r_{p_2}(n))] \\ &= [\alpha(r_{h_1} \cdot r_{p_1}(n)), \alpha(r_{h_2} \cdot r_{p_2}(n))] \\ &= \alpha(r_{[-,-]}(<r_{h_1} \cdot r_{p_1}(n), r_{h_2} \cdot r_{p_2}(n)>)) \\ &= h(\alpha(r_u(n))) \end{aligned}$$

$$= \gamma(r_u(j))$$

Furthermore we have:

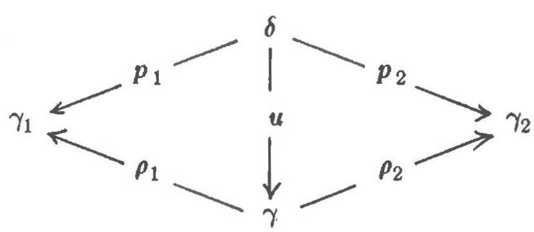
$$\rho_i \cdot u(x) = \rho_i([p_1(x), p_2(x)])$$

$$= \rho_i(h([p_1(x), p_2(x)]))$$

$$= h_i(p_i(x))$$

$$= p_i(x) \qquad i=1,2.$$

Thus u makes the following diagram in $Ret(\alpha)$ commutes:



Assume $v \in Hom(\delta,\gamma)$ also makes the above diagram commutes. Then

$$\rho_i \cdot v = p_i \qquad i=1,2.$$

Thus

$$P_i(x) = h_i([v(x)]_i)$$

$$= [v(x)]_i$$

Therefore $v(x)=[p_1(x),p_2(x)]=u(x)$.

$\Box$

Note. We can prove this theorem in an alternative way as follows: We can show $\gamma \cong \gamma_1 \times \gamma_2$ in *Num*. Since ρ_i are isomorphic image of $\pi_i \in Hom(\gamma_1 \times \gamma_2, \gamma_i)$ an $Ret(\alpha)$ is a full subcategory of *Num*, using lemma 3.4, we can show that (γ, ρ_1, ρ_2) is a product of γ_1 and γ_2 in $Ret(\alpha)$.

Lemma 3.6.

Let $\alpha:N\to A$ be a numeration such that $\alpha \rhd (\alpha\to\alpha)$ for some acceptable numeration $(\alpha\to\alpha):N\to Hom(\alpha,\alpha)$. Then for every retracts γ_1, γ_2 of α, there is a retract γ of α such that $\gamma\cong(\gamma_1\to\gamma_2)$ in *Num*.

Proof. Let γ_1, γ_2 be retracts of α via h_1, h_2 respectively. Let $\Psi\in Hom((\alpha\to\alpha),\alpha)$, $\Phi\in Hom(\alpha,(\alpha\to\alpha))$ be such that $\Phi\cdot\Psi=id_{(\alpha\to\beta)}$. Define $h:A\to A$ as

$$h(x)=\Psi(h_2\cdot\Phi(x)\cdot h_1).$$

Then we have:

$$\begin{aligned} h\cdot h(x) &= \Psi(h_2\cdot\Phi(\Psi(h_2\cdot\Phi(x)\cdot h_1))\cdot h_1) \\ &= \Psi(h_2\cdot\Phi(x))\cdot h_1) \\ &= h(x) \end{aligned}$$

Since *Num* is a category, $h\in Hom(\alpha,\alpha)$. Thus h is an idempotent of α. Note $(\gamma_1\to\gamma_2)(i)=h_2\cdot(\alpha\to\alpha)(i)\cdot g$ where $g=h_1\restriction h_1(A)$. Let $F:Hom(\gamma_1,\gamma_2)\to h(A)$ and $G:h(A)\to Hom(\gamma_1,\gamma_2)$ be as follows:

$$\begin{aligned} F(f) &= \Psi(h_2\cdot h_2\cdot f\cdot h_1\cdot h_1) \\ &= \Psi(h_2\cdot f\cdot h_1) \end{aligned}$$

$$\begin{aligned} G(h(x)) &= h_2\cdot\Phi(h(x))\cdot g \\ &= h_2\cdot h_2\cdot\Phi(x)\cdot h_1\cdot g \\ &= h_2\cdot\Phi(x)\cdot g. \end{aligned}$$

Then $F\in Hom((\gamma_1\to\gamma_2),\gamma)$ and $G\in Hom(\gamma,(\gamma_1\to\gamma_2))$. Also we have:

$$F\cdot G(h(x))$$

$$= \Psi(h_2 \cdot h_2 \cdot \Phi(x) \cdot g \cdot h_1)$$

$$= \Psi(h_2 \cdot \Phi(x) \cdot h_1)$$

$$= h(x)$$

$G \cdot F(f)$

$$= h_2 \cdot \Phi(\Psi(h_2 \cdot f \cdot h_1)) \cdot g$$

$$= h_2 \cdot f \cdot g$$

$$= f \qquad (\because \; f \in Hom(\gamma_1, \gamma_2))$$

Therefore $\gamma \cong (\gamma_1 \rightarrow \gamma_2)$ in Num.

□

Lemma 3.7.

Let $(\alpha \rightarrow \gamma): N \rightarrow Hom(\alpha, \gamma)$ be a numeration and $\alpha \cong \beta$ in Num. Define $(\beta \rightarrow \gamma): N \rightarrow Hom(\beta, \gamma)$ by:

$$(\beta \rightarrow \gamma)(i) = ((\alpha \rightarrow \gamma)(i)) \cdot g$$

where $(f \in Hom(\alpha, \beta), g \in Hom(\beta, \alpha))$ is an isomorphism pair. Then we have:

$$(\alpha \rightarrow \gamma) \cong (\beta \rightarrow \gamma)$$

in Num.

Proof. Define $F: Hom(\alpha, \gamma) \rightarrow Hom(\beta, \gamma)$ and $G: Hom(\beta, \gamma) \rightarrow Hom(\alpha, \gamma)$ by:

$$F((\alpha \rightarrow \gamma)(i)) = (\beta \rightarrow \gamma)(i)$$

$$G((\beta \rightarrow \gamma)(i)) = (\alpha \rightarrow \gamma)(i).$$

Obviously $F \in Hom((\alpha \rightarrow \gamma), (\beta \rightarrow \gamma))$ and $G \in Hom((\beta \rightarrow \gamma), (\alpha \rightarrow \gamma))$. Furthermore we have:

$$F \cdot G = id_{(\beta\to\gamma)} \qquad \textit{and}$$

$$G \cdot F = id_{(\alpha\to\gamma)}.$$

Thus $(\alpha\to\gamma)\cong(\beta\to\gamma)$ in Num.

□

Lemma 3.8.

Let $(\alpha\to\beta):N\to Hom(\alpha,\beta)$ be a numeration and $\beta\cong\gamma$ in Num. Define a numeration $(\alpha\to\gamma):N\to Hom(\alpha,\gamma)$ by:

$$(\alpha\to\gamma)(i) = f \cdot((\alpha\to\beta)(i))$$

where $(f \in Hom(\beta,\gamma), g \in Hom(\gamma,\beta))$ is an isomorphism pair. Then

$$(\alpha\to\gamma)\cong(\alpha\to\beta).$$

Proof. Define $G:Hom(\alpha,\gamma)\to Hom(\alpha,\beta)$ and $F:Hom(\alpha,\beta)\to Hom(\alpha,\gamma)$ by

$$G((\alpha\to\gamma)(i)) = g \cdot((\alpha\to\gamma)(i))$$

$$F((\alpha\to\beta)(i)) = f \cdot((\alpha\to\beta)(i))$$

Then

$$G((\alpha\to\gamma)(i)) = g \cdot f \cdot((\alpha\to\beta)(i))$$

$$= (\alpha\to\beta)(i) \qquad \textit{and}$$

$$F((\alpha\to\beta)(i)) = (\alpha\to\gamma)(i).$$

Thus $G \in Hom((\alpha\to\gamma),(\alpha\to\beta))$ and $F \in Hom((\alpha\to\beta),(\alpha\to\gamma))$. Also obviously

$$G \cdot F = id_{(\alpha\to\beta)} \qquad F \cdot G = id_{(\alpha\to\gamma)}$$

□

Lemma 3.9.

Let $(\alpha\to\beta):N\to Hom(\alpha,\beta)$ be acceptable and let $(\alpha'\to\beta'):N\to Hom(\alpha',\beta')$ be a numeration such that $(\alpha\to\beta)\cong(\alpha'\to\beta')$. Then $(\alpha'\to\beta')$ is acceptable.

Proof. Let $(f\in Hom((\alpha\to\beta),(\alpha'\to\beta')),\ g\in Hom((\alpha'\to\beta'),(\alpha\to\beta)))$ be an isomorphism pair. Assume $\varphi_k^{(1)}$ realizes $(\alpha'\to\beta')(i)$. Then

$$g((\alpha'\to\beta')(i))$$

$$=(\alpha\to\beta)(enum\cdot t(k))$$

where t is a recursive function satisfying:

$$\varphi_{t(k)}^{(1)}=r_g\cdot\varphi_k^{(1)}.$$

Thus

$$(\alpha'\to\beta')(i)=f((\alpha\to\beta)(enum\cdot t(k)))$$

$$=(\alpha'\to\beta')(r_f\cdot enum\cdot t(k))$$

Hence $(\alpha'\to\beta')$ is enumerable. Since

$$g((\alpha'\to\beta')(i))=(\alpha\to\beta)(r_g(i))$$

is realized by $\varphi_{real(r_g(i))}^{(1)}$.

$$(\alpha'\to\beta')(i)=f((\alpha\to\beta)(r_g(i)))$$

is realized by $\varphi_{t(i)}^{(1)}$ where t is a recursive function s.t.

$$\varphi_{t(i)}^{(1)}=r_f\cdot\varphi_{real(r_g(i))}^{(1)}.$$

Theorem 3.10.

Let $\alpha:N\rightarrow A$ be a numeration such that in *Num* $\alpha\times\alpha\cong\alpha$ and $\alpha\triangleright(\alpha\rightarrow\alpha)$, for some acceptable numeration $(\alpha\rightarrow\alpha):N\rightarrow Hom(\alpha,\alpha)$. Then the category $Ret(\alpha)$ is Cartesian closed.

Proof. Let γ_1, γ_2, γ_3 be retracts of α via h_1, h_2, h_3 respectively. Due to the proof of 3.5 there exists a retract δ of α such that

$$\delta\cong\gamma_1\times\gamma_2$$

in *Num*. Also by 2.7, $(\gamma_2\rightarrow\gamma_3)$ as defined in 2.5 is acceptable. Furthermore by 3.6, there is a retract γ of α such that $\gamma\cong(\gamma_2\rightarrow\gamma_3)$ in *Num*. By 2.7, $(\gamma_1\rightarrow\gamma)$ is acceptable and by 3.8,

$$(\gamma_1\rightarrow\gamma)\cong(\gamma_1\rightarrow(\gamma_2\rightarrow\gamma_3)).$$

Thus by 3.9, $(\gamma_1\rightarrow(\gamma_2\rightarrow\gamma_3))$ is acceptable. By 2.7, $(\delta\rightarrow\gamma_3)$ is acceptable. By 3.6, there is a retract σ of α such that $\sigma\cong(\delta\rightarrow\gamma_3)$. Also by 3.7, in Num we have:

$$(\delta\rightarrow\gamma_3)\cong(\gamma_1\times\gamma_2\rightarrow\gamma_3).$$

Thus by 3.9, $(\gamma_1\times\gamma_2\rightarrow\gamma_3)$ is acceptable. Therefore by 1.11 we have:

$$(\gamma_1\rightarrow(\gamma_2\rightarrow\gamma_3))\cong(\gamma_1\times\gamma_2\rightarrow\gamma_3).$$

Since $Ret(\alpha)$ is a full subcategory of *Num*, by 3.2 and 3.5 we can conclude that $Ret(\alpha)$ is Cartesian closed.

$\square$

Corollary 3.11.

If $\alpha:N\rightarrow A$ is a numeration such that in *Num*

$$\alpha\times\alpha\cong\alpha$$

$$\alpha \triangleright (\alpha \rightarrow \alpha)$$

for some acceptable numeration $(\alpha \rightarrow \alpha): N \rightarrow Hom(\alpha,\alpha)$, then we have: for every retracts γ_1, γ_2 of α_1:

(1) $(\gamma_1 \rightarrow \gamma_2)$ is abstract

(2) $(\gamma_1 \rightarrow \gamma_2)$ is acceptable.

Thus both K-recursion theorem and Eršov recursion theorem hold for $(\gamma_1 \rightarrow \gamma_2)$.

Proof. Immediate

□

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