TYPED RECURSION THEOREMS

by

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ABSTRACT

In recursion theory, recursion theorems are usually considered for effective functions over an effective universal set, like the set N of all natural numbers or the set RE of all recursively enumerable sets.

We observe that certain effective subsets of these effective universes have rich structure, and we study recursion theorems for these effective subsets.

§1. Introduction

We consider types as effective subsets of some effective universal set.

Finitary types, i.e. types composed of finite objects can naturally be considered as recursive languages over a nonempty finite set Σ, or equivalently as recursive subsets of N. In §2 we study the space of "typed partial recursive functions" $f:R^k \to R$ where R is an arbitrary finite type. We show that both Rogers and Kleene second recursion theorems hold for this typed function space.

Infinitary types can be considered as suitable recursively enumerable subsets of some complete numeration which is rich enough to be a universal set. For example, the Post numbering $W:N \to RE$ of the set of all recursively enumerable sets would be an interesting universal set. In §3, we study Ersov recursion theorem for these infinitary types in a general setting.

In §4, we present an alternative to the approach taken in §3. Instead of a complete universe, we assume a universe which satisfies an analogue to Kleene 2nd recursion theorem, we call it K-recursion theorem, and we show that retracts of such universe satisfy the K-recursion theorem.

Throughout we assume basic concepts and facts of recursive function theory. Readers are refered to standard textbooks. We also use basic results of the numeration theory. We briefly overview a small part of this theory, which is needed in this paper. For details see Ersov [2].

Definition 1.1

A *numeration (of a set X)* is a surjection $\chi:N\to X$. Given two numerations $\alpha:N\to A$ and $\beta:N\to B$, a *morphism from* α *to* β is a function $h:A\to B$ such that there is a recursive function $r_h:N\to N$ satisfying:

$$h\cdot\alpha=\beta\cdot r_h.$$

We say such r_h *realizes* h. Hom(α,β) denotes the set of all morphisms from α to β.

Lemma 1.2

The collection of numerations and the collection of morphisms form a category. We denote this category by Num.

Definition 1.3

Given numerations $\alpha_1:N\to A_1$, ..., $\alpha_k:N\to A_k$, we define a numeration $\alpha_1\times\cdots\times\alpha_k:N\to A_1\times\cdots\times A_k$ by

$$\alpha_1\times\cdots\times\alpha_k(<x_1,\ldots,x_k>)=(\alpha_1(x_1),\ldots,\alpha_k(x_k))$$

where $<x_1,\ldots,x_k>:N^k\to N$ is the standard bijection.

Definition 1.4

A numeration $\alpha:N\to A$ is *precomplete* if for every partial recursive function $f:N\to N$ there exists a total recursive $g:N\to N$ such that

$$f(i)\downarrow \quad \text{implies} \quad \alpha(g(i))=\alpha(f(i)).$$

Such α is *complete* if there exists an element $e\in A$ such that

$$\alpha(g(i))=\begin{cases}\alpha(f(i)) & \text{if } f(i)\downarrow\\ e & \text{otherwise}\end{cases}$$

Such e is called a *special element* of α.

Proposition 1.5 (Ersov [2])

$\alpha: N \to A$ is precomplete iff there exists a recursive function fix:$N \to N$ such that

$$\psi_i^{(1)}(\mathrm{fix}(i))\downarrow \quad \text{implies} \quad \alpha(\psi_i^{(1)}(\mathrm{fix}(i)))=\alpha(\mathrm{fix}(i)).$$

where $\psi^{(k)}$ is the Kleene numbering of partial recursive k-ary functions.

In definition 1.4, g *totalizes* f *modulo* α.

§2. Recursive Sets and Typed Recursion Theorems

Definition 2.1

A function $h:X\to X$ is an *idempotent* if $h\cdot h=h$. A subset $R\subset X$ is a retract of X if $R=\{h(x)\mid x\in X\}=h(X)$ for some idempotent $h:X\to X$.

Lemma 2.2 (Meseguer [3])

A nonempty subset of N is recursive iff it is a recursive retract of N.

Proof Assume $R=h(N)$ for some recursive idempotent $h:N\to N$. Then

$$x\in X \quad \text{iff} \quad h(x)=x.$$

Thus R is a recursive set. Conversely assume R is recursive. Define $h:N\to N$ by,

$$\begin{array}{ll} h(x)=x & \text{if } x\in X \\ \qquad x_0 & \text{otherwise} \end{array}$$

where x_0 is the smallest element in R. Then h is a recursive idempotent. Also $R=h(N)$.

Definition 2.3

Let R be a recursive set and $r:N\to N$ be a recursive idempotent such that $R=r(N)$. We define a numeration $\gamma_r:N\to R$ by

$$\gamma_r(n)=r(n).$$

A partial function $f:R\to R$ is γ_r-*partial recursive* if there exists a partial recursive function $\tilde{f}:N^k\to N$ such that the following diagram commutes:

$$\begin{tikzcd} N^k \arrow[r, "\tilde{f}"] \arrow[d, "{\gamma_r}^k"'] & N \arrow[d, "\gamma_r"] \\ R^k \arrow[r, "f"] & R \end{tikzcd}$$

where for any function $g:X\to X$,

$$g^k = \underbrace{g\times \cdots\cdots \times g}_{k\text{-times}} : X^k \to X^k$$

is the following function:

$$g^k(x_1,\ldots,x_k)=(g(x_1),\ldots,g(x_k)).$$

Theorem 2.4

Let R be a recursive nonempty set and $r_1,r_2:N\to N$ be recursive idempotents such that

$$r_1(N)=r_2(N)=R.$$

Then a partial function $f:R^k\to R$ is γ_{r_1}-partial recursive iff it is γ_{r_2}-partial recursive.

Proof Assume $f:R^k\to R$ is γ_{r_1}-partial recursive. Then there is a partial recursive function $f_1:N^k\to N$ such that for all $\bar{n}\in N^k$,

$$r_1\cdot f_1(\bar{n})=f\cdot r_1{}^k(\bar{n}) \qquad \text{and}$$

$$f_1(\bar{n})\downarrow \quad \text{iff} \quad f\cdot r_1{}^k(\bar{n})\downarrow.$$

From this f_1 we can construct a partial recursive function $f_2:N^k\to N$ such that

$$r_2\cdot f_2(\bar{n})=f\cdot r_2{}^k(\bar{n}) \qquad \text{and}$$

$$f_2(\bar{n})\downarrow \quad \text{iff} \quad f\cdot r_2{}^k(\bar{n})\downarrow$$

as follows:

Define f_2 by

$$f_2(\bar{n})=r_1\cdot f(\mu\bar{m}.[r_1{}^k(\bar{m})=r_2{}^k(\bar{n})]).$$

Since both r_1 and r_2 are recursive functions f_2 is partial recursive, and $f_2(\bar{n})\downarrow$ iff $f\cdot r_2(\bar{n})\downarrow$. Furtheremore $f_2(\bar{n})\in R$ because r_1 is an idempotent for R. Thus we have:

$$r_2\cdot f_2(\bar{n})=f\cdot r_2{}^k(\bar{n}).$$

This theorem states that the concept of "partial recrsiveness" of partial functions $f:R^k\to R$ where R is a recursive nonempty set is independent of the choice of recursive idempotents for R.

Definition 2.5

Let R be a nonempty recursive set. A partial function $f:R^k\to R$ is *partial recursive* if f is γ_r-partial recursive for some recursive idempotent such that $R=r(N)$. Furtheremore for each nonempty recursive set R, the recursive idempotent $h:N\to N$ defined by

$$h(x)=\begin{cases} x & \text{if } x\in R \\ x_0 & \text{otherwise} \end{cases}$$

where x_0 is the smallest element of R, is called the *standard idempotent of* R and the numeration $\gamma_h:N\to R$ is called the *standard numeration of* R.

Due to the theorem 2.4, without loss of generality, we can restrict our discussion to standard idempotents and standard numerations of nonempty recursive sets.

Definition 2.6

Let R be a nonempty recursive set and $PRR^{(k)}$ be the set of all partial recursive functions $R^k\to R$. We define a numeration $\sigma^{(k)}:N\to PRR^{(k)}$ as follows:

$$\sigma_i^{(k)} = h \cdot \psi_i^{(k)} \upharpoonright R^k$$

where h is the standard idempotent for R.

The definition of $\sigma^{(k)}$ indicates that this numeration inherits many properties of the numeration $\psi^{(k)}$.

Theorem 2.7 (Typed Rogers 2nd Recursion Theorem)

Let R be a nonempty recursive set with the standard numeration γ. For every partial recursive function $f:N \to N$, there exists a number $n_f \in N$ such that

$$f(n_f)\downarrow \text{ implies}$$

$$\sigma^{(k)}_{f(n_f)}(\gamma(x_1),\ldots,\gamma(x_k)) = \sigma^{(k)}_{n_f}(\gamma(x_1),\ldots,\gamma(x_k))$$

Proof By the Rogers 2nd recursion theorem there exists a number n_f such that

$$f(n_f)\downarrow \text{ implies}$$

$$\psi^{(k)}_{f(n_f)}(x_1,\ldots,x_k) = \psi^{(k)}_{n_f}(x_1,\ldots,x_k).$$

Obviously this n_f satisfies the theorem.

When a property of $\psi^{(k)}$ involves arguments of indexed functions, such property may not be passed directly to $\sigma^{(k)}$.

Theorem 2.8 (Typed S-m-n Theorem)

Let R be a nonempty recursive set and γ be the standard enumeration of R. There exists a recursive function $st^m_n : N^{m+1} \to N$ such that

$$\sigma_i^{(m+n)}(\gamma(y_1),\ldots,\gamma(y_m),\gamma(z_1),\ldots,\gamma(z_n))$$
$$= \sigma^{(n)}_{st^m_n(i,y_1,\ldots,y_m)}(\gamma(z_1),\ldots,\gamma(z_n)).$$

Proof Define st^m_n by,

$$st^m_n(i,y_1,\dots,y_m)=s^m_n(i,\gamma(y_1),\dots,\gamma(y_m))$$

where s^m_n is the s-m-n function for the Kleene numbering. It can readily be seen that this st^m_n establishes the theorem.

In the above proof, st^m_n is not primitive recursive in general because γ need not be so. Thus the primitive recursiveness of the s-m-n function is lost in typed case.

The following theorem is concerned with a property of the Kleene numbering which requires some elaboration to be passed to $\sigma^{(k)}$.

Theorem 2.9 (Typed Kleene 2nd Recursion Theorem)

Let R be a nonempty recursive set and let γ be the standard numeration of R. For any partial recursive function $f:N^{k+1}\to N$ there exists a number m_f such that

$$\sigma^{(k)}_{m_f}(\gamma(x_1),\dots,\gamma(x_k))$$
$$=\gamma(f(m_f,\gamma(x_1),\dots,\gamma(x_k)).$$

Proof Let $\tilde{f}:N^{k+2}\to N$ be the following partial recursive function

$$\tilde{f}(i,y,x_1,\dots,x_k)=f(st^1_k(i,\gamma(y)),x_1,\dots,x_k).$$

Due to the Kleene 2nd recursion theorem for $\psi^{(k+1)}$, we have a number $e\in N$ such that

$$\psi^{(k+1)}_e(y,x_1,\dots,x_k)=\tilde{f}(e,y,x_1,\dots,x_k)$$
$$=f(st^1_k(e,\gamma(y)),x_1,\dots,x_k)$$

Let $m_f=st^1_k(e,\gamma(e))$. Then we have

$$\begin{aligned}&\sigma^{(k)}_{m_f}(\gamma(x_1),\ldots,\gamma(x_k))\\ &\quad=\sigma^{(k)}_{st^1_k(e,\gamma(e))}(\gamma(x_1),\ldots,\gamma(x_k))\\ &\quad=\gamma\cdot\psi^{(k+1)}_e(\gamma(e),\gamma(x_1),\ldots,\gamma(x_k))\\ &\quad=\gamma\cdot f(st^1_k(e,\gamma(e)),\gamma(x_1),\ldots,\gamma(x_k))\\ &\quad=\gamma\cdot f(m_f,\gamma(x_1),\ldots,\gamma(x_k)).\end{aligned}$$

It is evident that the same results as in this section hold for a more general case of partial recursive word functions of Asser [1].

For technical simplicity, we considered only partial recursive functions $f:R^k\to R$. It is evident that we can generalize our results to partial recursive functions $f:R_1\times\cdots\times R_k\to R_0$ where R_i $(i\leq k)$ are recursive sets.

§3. Complete Universes and Typed Ersov Recursion Theorem

Definition 3.1

Let $\alpha:N\to A$ be a numeration. A subset $B\subset A$ is *weakly enumerable* if $B=\alpha(X)$ for some recursively enumerable set $X\subset N$. Let $\alpha\Gamma(A)$ be the collection of all weakly enumerable subsets of A and let $\rho:N\to\alpha\Gamma(A)$ be the following numeration:

$$\rho(n)=\alpha(W_n)$$

where W is the Post numbering of recursively enumerable sets.

Lemma 3.2

ρ is precomplete. Thus the Ersov recursion theorem holds for ρ.

Proof Immediate.

This obvious lemma states that we can recursively define weakly enumerable sets.

Definition 3.3

Let $\alpha:N\to A$ be a complete numeration with a special element $e\in A$. A weakly enumerable subset $B\subset A$ such that $e\in B$ is called an *α-type*

Due to Rice Theorem recursive subsets of A are trivial.

Lemma 3.4

Let $\alpha:N\to A$ be a complete numeration with a special element $e\in A$. Every α-type B has a complete numeration $\tau:N\to B$ with a special element e.

Proof Since α is complete there is a total recursive function $h:N\to N$ which totalizes

$$g(i)=f(\psi_i^{(1)}(0))$$

modulo α where f is a total recursive function $N \to N$ such that

$$B=\{f(i) \mid i \in N\}.$$

Define $\tau : N \to B$ as

$$\tau(i)=\alpha \cdot h(i).$$

It can readily be seen that this τ is complete with a special element $e \in B$.

Theorem 3.5 (Typed Ersov 2nd Recursion Theorem)

Let $\alpha : N \to A$ be complete with a special element $e \in A$. Let $\tau : N \to B$ the complete numeration of an α-type B with a special element e as above. Then for each partial recursive function $f : N \to N$, there is a number $n_f \in N$ such that:

$$f(n_f)\downarrow \quad \text{implies} \quad \tau(f(n_f))=\tau(n_f).$$

Proof By 3.4 and the Ersov recursion theorem .

This theorem states that every α-type admits recursive definition of its elements, thus provides a typed recursion theorem.

Finding a good characterization of those partial recursive functions (recursive definitions) whose ρ-fixpoints are α-types is left open.

In sammary, lemma 3.2 and theorem 3.5 are numeration theoretic analogue to the retract calculus of Scott [4], which uses Tarski fixpoint theorem instead.

§4. Abstraction Universes and Typed K-recursion Theorem

Definition 4.1

Let $\alpha:N\to A$ be a numeration. A morphism $h\in\mathrm{Hom}(\alpha,\alpha)$ is an *idempotent of* α if $h=h\cdot h$. The numeration $\gamma:N\to h(A)$ such that

$$\gamma(i)=h(\alpha(i))$$

is called a *retract of* α (*via* h).

Lemma 4.2

If h is an idempotent of a numeration $\alpha:N\to A$ then $h(A)$ is weakly enumerable.

Proof trivial.

Definition 4.3

A pair (α,β) of numerations has the *abstraction property* if there exists a numeration $(\alpha\to\beta):N\to\mathrm{Hom}(\alpha,\beta)$ such that for every $f\in\mathrm{Hom}((\alpha\to\beta)\times\alpha,\beta)$ there exists a morphism c_f from $(\alpha\to\beta)$ to itself such that

$$f((\alpha\to\beta)(i),\alpha(j))=c_f((\alpha\to\beta)(i))(\alpha(j)).$$

Theorem 4.4 (The K-recursion Theorem)

Assume that (α,β) has the abstraction property and $(\alpha\to\beta)$ is precomplete, then for all $f\in\mathrm{Hom}((\alpha\to\beta)\times\alpha,\beta)$ there exists a number m_f such that

$$f((\alpha\to\beta)(m_f),\alpha(j))=((\alpha\to\beta)(m_f))(\alpha(j))$$

Proof Since (α,β) has the abstraction property, for some $c_f\in\mathrm{Hom}((\alpha\to\beta),(\alpha\to\beta))$ we have:

$$f((\alpha\to\beta)(i),\alpha(j))=c_f((\alpha\to\beta)(i))(\alpha(j)).$$

Let $g=r_{c_f}$. Since $(\alpha\to\beta)$ is precomplete there is a number n_g

such that

$$(\alpha\to\beta)(g(n_g))=(\alpha\to\beta)(n_g).$$

Thus we have:

$$\begin{aligned} f((\alpha\to\beta)(n_g),\alpha(j)) &= c_f((\alpha\to\beta)(n_g))(\alpha(j)) \\ &= (\alpha\to\beta)(g(n_g))(\alpha(j)) \\ &= (\alpha\to\beta)(n_g)(\alpha(j)). \end{aligned}$$

Set $m_f=n_g$.

This theorem is a numeration theoretic analogue to the Kleene 2nd recursion theorem.

Lemma 4.5

Let $\alpha:N\to A$ be a numeration and γ,γ' be retracts of α via h,h' respectively. Then $f:h(A)\to h'(A)$ is a morphism from γ to γ' iff there is a morphism $F\in\mathrm{Hom}(\alpha,\alpha)$ such that

$$f\cdot h=h'\cdot F. \qquad \dots\dots\dots\dots \quad \text{(E)}$$

Proof Assume that f is a morphism from γ to γ'. Since h and h' are idempotents of α, $F:A\to A$ defined by

$$F=h'\cdot f\cdot h$$

satisfies (E). Since f,h,h' are morphisms, due to 1.2, F is a morphism from α to α.

Conversely assume f is a function $h(A)\to h'(A)$ such that (E) holds for some $F\in\mathrm{Hom}(\alpha,\alpha)$. Let $g=h\restriction h(A)$. Then $r:N\to N$ defined by

$$r=r_h\cdot r_h$$

is a recursive function which realizes g, for we have:

$$g(\gamma(i))=h(\alpha(r_h(i))=\alpha(r_h\cdot r_h(i)).$$

Since h is an idempotent of α, we have

$$f=h'\cdot F\cdot g.$$

Thus f is a morphism from γ to γ'.

Notice that the above proof also states that for any morphism $F\in\mathrm{Hom}(\alpha,\alpha)$, $f:h(A)\to h'(A)$ defined by

$$f=h'\cdot F\cdot g$$

is a morphism from γ to γ'. Thus we have the following definition:

Definition 4.6

Let $\alpha:N\to A$ and $(\alpha\to\alpha):N\to\mathrm{Hom}(\alpha,\alpha)$ be numerations. Let γ,γ' be retracts of α via h,h' respectively. We define a numeration $(\gamma\to\gamma'):N\to\mathrm{Hom}(\gamma,\gamma')$ by

$$(\gamma\to\gamma')(i)=h'\cdot((\gamma\to\gamma)(i))\cdot g$$

where $g=h\restriction h(A)$.

Theorem 4.7

Let $\alpha,(\alpha\to\alpha),\gamma,\gamma',(\gamma\to\gamma')$ be as in the definition 4.6. If $(\alpha\to\alpha)$ is precomplete then $(\gamma\to\gamma')$ also is precomplete.

Proof For any partial recursive function $f:N\to N$, the recursive function $g:N\to N$ which totalizes f modulo $(\alpha\to\alpha)$ totalizes f modulo $(\gamma\to\gamma')$.

Theorem 4.8

Let $\alpha,(\alpha\to\alpha),\gamma,\gamma',(\gamma\to\gamma')$ be as in the definition 4.6. If (α,α) has the abstraction property, then so does (γ,γ').

Proof Let $t\in\mathrm{Hom}((\gamma\to\gamma')\times\gamma,\gamma')$. Define $T:\mathrm{Hom}(\alpha,\alpha)\times A\to A$ by:

$$T(F,a)=t(h'\cdot F\cdot g,h(a)).$$

Then T is a morphism from $(\alpha\to\alpha)\times\alpha$ to α, for $r:N\to N$ such that

$$r(\langle i,j\rangle)=r_t(r_{h'}\cdot r_F\cdot r_g(i),r_h(j))$$

realizes T. Since (α,α) has the abstraction property, for some $c_{\mathcal{T}} \in \mathrm{Hom}((\alpha\to\alpha),(\alpha\to\alpha))$,

$$\mathcal{T}((\alpha\to\alpha)(i),\alpha(j))=c_{\mathcal{T}}((\alpha\to\alpha)(i))(\alpha(j)).$$

Define $c_t:\mathrm{Hom}(\gamma,\gamma')\to\mathrm{Hom}(\gamma,\gamma')$ by:

$$c_t(f)=h'\cdot c_{\mathcal{T}}(h'\cdot f\cdot h)\cdot g.$$

Then c_t is a morphism from $(\gamma\to\gamma')$ to itself because the identity function $N\to N$ realizes c_t. But we have:

$$\begin{aligned} & c_t((\gamma\to\gamma')(i))(\gamma(j)) \\ &\qquad =(h'\cdot c_{\mathcal{T}}(h'\cdot((\gamma\to\gamma')(i))\cdot h)\cdot g)(\gamma(j)) \\ &\qquad =h'\cdot\mathcal{T}(h'\cdot((\alpha\to\alpha)(i))\cdot h)\cdot g,\gamma(j)) \\ &\qquad =h'\cdot t(h'\cdot((\alpha\to\alpha)(i))\cdot g,h(\gamma(j))) \\ &\qquad =h'\cdot t((\gamma\to\gamma')(i),\gamma(j)) \\ &\qquad =t((\gamma\to\gamma')(i),\gamma(j)). \end{aligned}$$

Lemma 4.9

Let $\alpha:N\to A$ be a numeration and $\gamma_1,\ldots,\gamma_k$ be retracts of α via $h_1,\ldots,h_k$ respectively. Then $\gamma_1\times\cdots\times\gamma_k$ is a retract of $\alpha^k=\underbrace{\alpha\times\cdots\times\alpha}_{k\text{-times}}$.

Proof Define $h=h_1\times\cdots\times h_k$. Then $h=h\cdot h$. Also h is a morphism from α^k to itself because $r:N\to N$ defined by

$$r(<x_1,\ldots,x_k>)=<r_{h_1}(x_1),\ldots,r_{h_k}(x_k)>$$

realizes h. But obviously

$$\gamma_1\times\cdots\times\gamma_k(<x_1,\ldots,x_k>)=h(\alpha^k(<x_1,\ldots,x_k>)).$$

Theorem 4.10 (Typed K-recursion Theorem)

Assume (α^k,α) has the abstraction property. Also assume that $(\alpha^k\to\alpha)$ is precomplete. Then for each retracts $\gamma_1,\dots,\gamma_{k+1}$ of α, if $f\in\mathrm{Hom}((\gamma_1\times\cdots\times\gamma_k\to\gamma_{k+1})\times\gamma_1\times\cdots\times\gamma_k,\gamma_{k+1})$ then there exists $m_f\in N$ such that

$$f((\gamma_1\times\cdots\times\gamma_k\to\gamma_{k+1})(m_f),\gamma_1(x_1),\dots,\gamma_k(x_k))$$

$$=(\gamma_1\times\cdots\times\gamma_k\to\gamma_{k+1})(m_f)(\gamma_1(x_1),\dots\gamma_k(x_k)).$$

Proof Immediate from 4.4,4.7,4.8 and 4.9.

A numeration $\alpha:N\to A$ which satisfies the condition of the theorem 4.10 for every $k\geq 1$ is called an *abstraction universe.* Retracts of such α is called α-K-types.

In words, if α is an abstraction universe then α-K-types satisfy the typed K-recursion theorem.

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