```
 MMM
 MMMM       MMM
   MM      M MM
   M      M
  M      M     MMMMMMMMM
 MM     MM    MMMM    MMM
MMM     MM      MM    MMM
MMM    MMM      MM   MMM
MMMMMMMMMM       MMMMMMM          MM
 MMMMMMM MMMM    MMM   MM       MMMMM
          MMM    MM    MMM     M   MM
                 M     MMM    M    M
                M  MM  MMM   MM
            MMMM   MMMMMM   MMM
            MMM             MMM
                            MMM
                             MMM   M
                              MMMMM
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*   A New Notation for Derivations   *
*  in Chomsky's Generative Grammars  *
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A New Notation for Derivations in Chomsky's Generative Grammars

by

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ABSTRACT

Ordered directed graphs (generalizing ordered trees) are defined and used in a new formal definition of grammatical derivation. The latter is shown equivalent to the currently accepted definition. The new scheme is illustrated by detailed proofs of two familiar results: the equivalence of the notions "context-sensitive" and "length-nondecreasing" as applied to grammars, and an important lemma in the theory of deterministic context-free parsing.

0. INTRODUCTION

The mathematical study of formal languages, initiated by Noam Chomsky, has as a primary object of study the process of generation of language elements (words or sentences). An instance of this process, a derivation of a language element, has heretofore usually been presented as a sequence of words (or sentential forms), much like a proof in formal logic. To qualify as a derivation, the sequence must begin and end with words from specified sets, and must proceed from term to term by specified substitutions of subwords. Here I propose a different presentation for the same notion. Although my proposal is fundamentally nothing but a notational variant, it seems to me interesting because of its convenience for the construction of more secure proofs than are now customary in this field.[1]

I propose that grammatical derivations be formally defined as directed graphs with edges labeled by the symbols (terminal elements and grammatical class names) occurring in derivations as usually defined, and with edges assigned a left-to-right ordering distinct from the path ordering they possess as components of a directed graph. The

[1] While begging pardon for having indulged myself in a quest for new proofs, and even merely new presentations of proofs, rather than new results, I take the liberty to remind the reader that mathematical proof is itself a type of research apparatus. Indeed, it seems to me that an increase in the fundamental security of our proofs may serve us as well as an increase in the light-gathering power of a telescope serves an astronomer. In each case, the increase permits more convenient and precise delineation of familiar objects and brings into view some entirely new objects of study. In each case, too, the setting for the apparatus limits meaningful increase. For the astronomer, it is the atmosphere and gravity of our planet which are limiting. For the student of formal languages, it is the extent of his willingness to confine his attention to a rigorously conventional system.

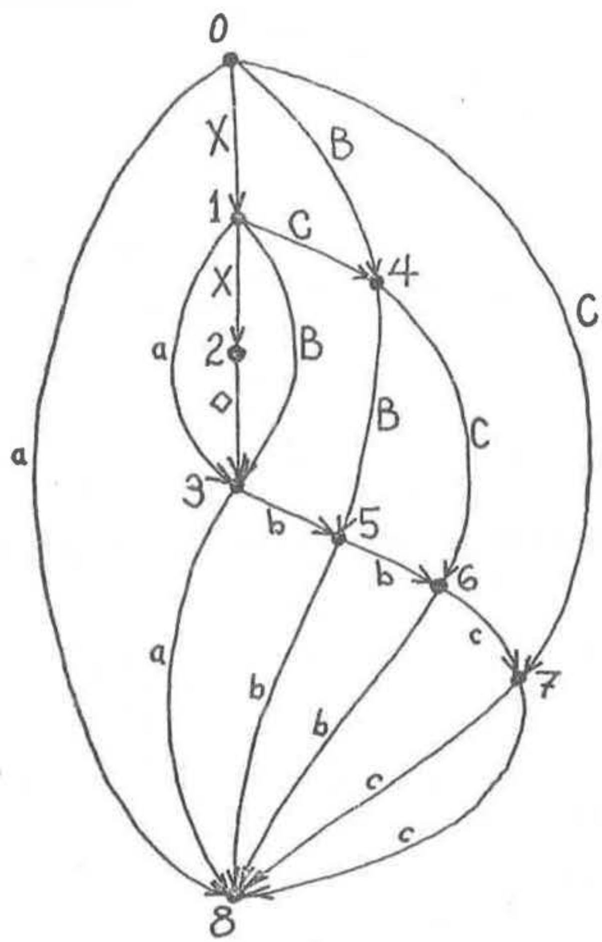
systematic substitutions which characterize grammatical derivations are to be presented as occurring at the nodes of the graph: The sequence of labels on the input edges at a node, taken in the specified left-to-right order, and the similarly obtained sequence of labels on output edges at the same node must make up one of the specified substitutions of the grammar. There are to be distinguished initial and terminal nodes, with no input and output edges, respectively. A derivation will be said to be <u>of</u> the input sequence at the terminal node and <u>from</u> the output sequence at the initial node. For example, figure 1 gives old- and new-style derivations of "aabbcc" from "aXBC" according to the indicated grammar.

X → aXBC
X → <>
CB → BC
aB → ab
bB → bb
bC → bc
cC → cc

(a)

aXBC
aaXBCBC
aaXBBCC
aaBBCC
aabBCC
aabbCC
aabbcC
aabbcc

(b)



(c)

Figure 1. (a) A grammar (substitutions). (b) Old-style derivation. (c) New-style derivation.

An important advantage of the proposed definition is that the notion of single occurence of a grammatical symbol appears explicitly. In the old-style derivation given in fugure 1(b), there are 48 occurrences of symbols, but many of these are "the same occurrence". The occurrences of "X" in the second and third lines are "the same", for example, but "different" from the occurrence of "X" in the first line. In the new-style derivation (figure 1(c)), the 18 "distinct" occurrences of symbols appear explicitly as directed edges of the graph.

Everyone who has looked into the subject, I suppose, feels there is a definite sense in which two (old-style) derivations can be said to have the same underlying structure. In the case of contect-free grammars (those in which the substring replaced in a derivation step is always a single symbol), such structure is well presented by the familiar derivation trees and a corresponding canonical set of derivations, the leftmost derivations. (See, for example, Hopcroft and Ullman (1969).)

Unfortunately, no reasonable general specification of a canonical set of derivations can be made so long as derivations are defined merely as sequences of words. This is clear from Griffiths (1968) and Walters (1970), in each of which a canonical set of derivations is obtained using a definition of "derivation" augmented to include specification of the positions in the terms of a derivation-defining sequence at which subword substitutions are made. Eickel and Loeckx (1972) points out the same shortcoming of the usual definition.

Hart (1976) takes a different view of leftmost derivations, generalizing them not as canonical derivations, but as one-dimensional descriptions of derivation trees[2]. Both the two last-mentioned papers assume

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[2]The relation of Hart's derivation words and "derivation" as defined here is indicated in a remark following definition 2.07 below.

that the notion of structure underlying a derivation is captured by the graphic notation introduced by Loeckx (1970), which is very similar to that used here.

My proposal shares with all the presentations described above the advantage that it specifies the positions at which substitutions occur in a derivation. In abandoning the requirement that formal specification of a derivation be based on a linear ordering of the substitutions occurring in it, my proposal shares with the presentations of Loeckx (1970) and Hart (1976) the further advantage that the structure underlying a derivation is revealed by the formal specification of the derivation itself rather than by the derivation's membership in an equivalence class. Like Hart (1976), the present paper includes a formalization of something like the syntactical graphs presented informally in Loeckx (1970). Here, however, the underlying graphs are characterized intrinsically, apart from their use in derivations, as they are not in Hart (1976).

In the first section of this paper, the ordered directed graphs described above are defined, and some of their properties useful in grammatical derivations are established. In the second section, the new definitions of grammar and derivation are given, along with proof of their equivalence to those currently accepted. In the third section, proofs of two familiar results are presented in the new notation to demonstrate the security it permits.

Ordinary notation for sets and functions is used here. $\mathfrak{N}$ is the set $\{0,1,2...\}$ of <u>natural numbers</u>. If X and Y are sets, then $X \setminus Y$ is the <u>difference</u> $\{x \mid x \in X \text{ and } x \notin Y\}$. X^* is the set of strings over X as alphabet (the free monoid generated by X). $\langle\rangle$ is the empty string (the identity of X^*), X^+ is the set of nonempty strings over X. (Thus, $X^+ = X^* \setminus \{\langle\rangle\}$.) For $x \in X^*$, $|x|$ is the <u>length</u> of x. ($|x| = 0$ if and only if $x = \langle\rangle$.)

1. ORDERED DIRECTED GRAPHS

The reader is assumed to be familiar with the notion of a directed graph. For completeness and to establish notation, the following definition is included. Notice that directed graphs are here permitted to have loops and multiple edges.

1.01. A directed graph D comprises:

D_Q, a nonempty set, the nodes of D;

D_E, a set, the edges of D;

$D_h : D_E \to D_Q$, the head function; and

$D_t : D_E \to D_Q$, the tail function.

When confusion seems unlikely, D_Q, D_E, D_h, D_t will be denoted Q, E, h, t, respectively. If directed graphs D', $\overline{D}$, etc. are being discussed, Q' may be used to denote D'_Q, $\overline{E}$ to denote $\overline{D}_E$, etc.

1.02. If p and q are nodes of a directed graph D, than a path in D from p to q is a sequence $e_1 \ldots e_n$ in E such that $h(e_i) = t(e_{i+1})$ for all $i \in \{1, \ldots, n-1\}$ and $p = t(e_1)$, $q = h(e_n)$ or $n = 0$ (so $e_1 \ldots e_n = \langle\rangle$) and $p = q$.

1.03. If e and f are edges of a directed graph D, than a path in D from e to f is a path in D from h(e) to t(f).

1.04. An ordered directed graph D comprises:

D_Q, D_E, D_h, D_t as in (1.01) together with $D_<$, a transitive relation on D_E satisfying the following trichotomy condition: for any pair of edges e,f of D, exactly one of $\{eD_<f,\ fD_<e,$ some path in D includes both e and f$\}$ is true.

As for Q,E, h and t, $<$, $<'$, etc. may be used to denote $D_<$, $D'_<$ etc. "$e<f$" should be thought of as "e is left of f." In grammatical derivations, we will have $e<f$ only if there are nodes p, q such that e and f lie on disjoint paths from p to q. Intuitively speaking, then, the relation "$<$" may be thought of as "is left of" in a strict and unambiguous sense.

Corollary. In an ordered directed graph D,

(i) $D_<$ is irreflexive and antisymmetric.

(ii) If $e<f$ and some path in D includes e and e', then $f \not< e'$.

(iii) If $e<f$ and some path in D includes f and f', then $f' \not< e$.

1.05. A [n ordered] directed graph D is *finite* if and only if D_Q and D_E are finite.

1.06. A [n ordered] directed graph D is *acyclic* if and only if $\langle\rangle$ is the only path from q to q for all $q \in Q$.

1.07. *Lemma*. If q is a node of an acyclic ordered directed graph D, then each of $h^{-1}(q)$, $t^{-1}(q)$ is totally ordered by $D_<$.

Proof: Otherwise there is a path ee' . . . f'f in D with $h(e) = h(f)$, in which case e' . . . f is a path from h(e) to h(e), or with $t(e) = t(f)$, in which case e . . . f' is a path from t(e) to t(e).

1.08. A [n ordered] directed graph D is *closed* if and only if, for each pair p, q of nodes of D, there are nodes s, t of D such that there are paths to p and q from s and from p and q to t.

1.09. *Lemma*. If D is an acyclic closed [ordered] directed graph in which there is no infinite sequence $e_1 \, e_2 \, . \, . \, .$ in D_E

with $h(e_i) = t(e_{i+1})$ for all i, then there are unique nodes S, T of D such that, for each node q of D, there are paths from S to q and from q to T.

Proof: Define an ordering among paths in D by $x \prec y \equiv x$ is a substring of y, considering x and y as elements of E^*. By the hypothesis disallowing infinite sequences, every $\prec$-chain has an upper bound. Therefore, by Zorn's Lemma, there is a $\prec$-maximal path $e_1 \ldots e_n$ in D.

Let $S=t(e_1)$, $T=h(e_n)$, and consider any $q \in Q$. Since D is closed, there is $s \in Q$ with paths from s to q and to S. By maximality of $e_1 \ldots e_n$, s=S. Similarly, there is a path from q to T.

Corollary. If D is a finite acyclic closed [ordered] directed graph, then there are unique nodes S and T of D such that, for each node q of D, there are paths from S to q and from q to T.

1.10. In an acyclic closed [ordered] directed graph D satisfying the hypothesis of (1.09), D_S, D_T are defined to be the unique nodes S, T (respectively) specified there.

As usual, S, T', etc. may be used to denote D_S, D'_T, etc.

1.11. Lemma. If $e_1<e_2<e_3$ in a closed ordered directed graph D, and e_1, e_3 each occur in paths from [respectively to] some node q of D, then e_2 also occurs in a path from [resp. to] q.

Proof: Since D is closed, there is a node p of D with paths from p to q and to $t(e_2)$. If there is no edge f of D with h(f)=g, then necessarily p=q. Otherwise, let h(f)=q. If $f<e_2$, then $f<e_3$. If $e_2<f$, then $e_1<f$. Thus, assuming there are paths from q to $t(e_1)$ and to $t(e_3)$,

neither $f<e_2$ nor $e_2<f$ is possible, nor is it possible that there is a path from e_2 to f. In any case, then, there is a path from q to $t(e_2)$. [Similarly, assuming there are paths from $t(e_1)$ and from $t(e_3)$ to q, there is a path from $t(e_2)$ to q.]

As the reader will have anticipated, our definition of a grammatical derivation will be based on that of a finite acyclic closed ordered directed graph. The propositions given so far in this section should provide a basis for a reasonably good intuition concerning the consequences of the definitions we have adopted.

In proofs relating to grammars, there frequently appear constructions in which grammatical substitution rules are eliminated or replaced. To determine that such changes to a grammar result in suitable changes in the language it determines, one usually proves that the changes in the grammar are equivalent to a uniform system of modifications in the permissible grammatical derivations. In the rest of this section, there are exhibited three important types of modification under which some classes of ordered directed graphs are closed, and to which appeal is frequently made in grammar-related constructions.

1.12. Lemma. If e is an edge of an ordered directed graph D and there are edges e', e" with $h(e')=h(e)$ and $t(e")=t(e)$, then D', obtained by eliminating e, is also an ordered directed graph. If D is acyclic then so is D', and if D is closed then so is D'. (More precisely, D' is specified by: $Q'=Q$, $E'=E\setminus\{e\}$; h', t', and <' are the restrictions of h, t, and < (respectively) to E'.)

Proof: All paths in D' are paths in D.

1.13. Lemma. If $\hat{e}$ is an edge of a closed ordered directed graph D, then D', obtained by eliminating $\hat{e}$ and identifying $h(\hat{e})$ and $t(\hat{e})$, is a closed, ordered directed graph. If D is acyclic and $\hat{e}$ is <- least [resp. <- greatest] in $h^{-1}(h(e))$ and $\hat{e}$ is <-greatest [resp. <-least] in $t^{-1}(t(\hat{e}))$, then D' is acyclic. (More precisely, D' is specified by:

$$Q' = (Q\setminus\{h(\hat{e}), t(\hat{e})\}) \cup \{q\}, \text{ where } q \notin Q;$$

$$E' = E\setminus\{\hat{e}\};$$

$$h'(e) = \left\{\begin{matrix} q \text{ if } h(e) \in \{h(\hat{e}), t(\hat{e})\} \\ h(e) \text{ otherwise} \end{matrix}\right\} \quad ;$$

$$t'(e) = \left\{\begin{matrix} q \text{ if } t(e) \in \{h(\hat{e}), t(\hat{e})\} \\ t(e) \text{ otherwise} \end{matrix}\right\} \quad ;$$

$e <' f \equiv e < f$ and no path in D' includes both e and f.

Proof: It is clear that Q', E', h', t' make up a closed directed graph, and that <' satisfies the trichotomy condition of (1.04).

To see that <' is transitive, suppose $e_1 < e_2 < e_3$ and that some path in D' includes both e_1 and e_3. Necessarily either there are paths from e_1 to $h(\hat{e})$ and from $t(\hat{e})$ to e_3 or there are paths from e_3 to $h(\hat{e})$ and from $t(e)$ to e_1. In the first case, by (1.11) if $e_2 < \hat{e}$, then there is a path from e_2 to $h(\hat{e})$ in D and if $\hat{e} < e_2$, then there is a path from $t(e)$ to e_2 in D. Similarly in the second case. In either case then, either there is a path in D which includes both e_1 and e_2 or there is one which includes both e_2 and e_3. It follows that <' is transitive.

Finally, suppose the hypothesis "D is acyclic and ... " holds, and that there is a path in D from p to p for some $p \in Q'$. Since D is acyclic, such a path must include some edges e, f with $t(e) = t(\hat{e})$ and $h(f) = h(\hat{e})$. By the hypothesis, then, either $e < \hat{e} < f$ or $f < \hat{e} < e$.

This contradicts the fact that some one path in D includes both e and f. It follows that D' is acyclic.

1.14. Lemma. If q is a node of an ordered directed graph D, card $h^{-1}(q) \geq 2$, card $t^{-1}(q) \geq 2$, $\hat{e}$ is <- least [resp. <- greatest] in $h^{-1}(q)$, and $\hat{f}$ is <- least [resp. <- greatest] in $t^{-1}(q)$; then D', obtained by replacing $\hat{e}$ and $\hat{f}$ with a new edge g from $t(\hat{e})$ to $h(\hat{f})$, is an ordered directed graph. If D is acyclic then so is D', and if D is closed than so is D'. (More precisely, D' is specified by: $Q'=Q$; $E'= (E\setminus\{\hat{e}, \hat{f}\}) \cup \{g\}$, where $g \notin E$; $h'=h$ and $t'=t$ except that $h'(g)=h(\hat{f})$ and $t'(g)=t(\hat{e})$; $e<'f \equiv e<f$ or $e=g$ and ($\hat{e}<f$ or $\hat{f}<f$)

or $f=g$ and ($e<\hat{e}$ or $e<\hat{f}$)

or one path in D includes both e and f but every path in D from e to f includes $\hat{e}$ but not $\hat{f}$ [resp. $\hat{f}$ but not $\hat{e}$] and every path in D from f to e includes f but not e [resp. $\hat{e}$ but not $\hat{f}$].

Proof: Define the following relations in E':

$eR_1f \equiv e<f$ or $e=g$ and ($\hat{e}<f$ or $\hat{f}<f$)

or $f=g$ and ($e<\hat{e}$ or $e<\hat{f}$).

$eR_2f \equiv fR_1e$.

$eR_3f \equiv$ there is a path in D which includes both e and f and includes neither or both of $\hat{e}$, $\hat{f}$;

or $e=g$ and there is a path in D from f to $\hat{e}$ or from $\hat{f}$ to f

or $f=g$ and there is a path in D from e to $\hat{e}$ or from $\hat{f}$ to e.

$eR_4f \equiv$ there is a path in D from e to f, and every such path includes $\hat{e}$ but not $\hat{f}$.

$eR_5f \equiv$ there is a path in D from e to f, and every such path includes $\hat{f}$ but not $\hat{e}$.

$eR_6f \equiv$ there is a path in D from f to e, and every path includes $\hat{e}$ but not $\hat{f}$.

$eR_7f \equiv$ there is a path in D from f to e, and every such path includes $\hat{f}$ but not $\hat{e}$.

We first prove that $\{R_1, R_2 \ldots, R_7\}$ is a partition of $E' \times E'$. If e, f are edges of D and there are two paths from e to f, one including $\hat{e}$ and not $\hat{f}$, the other including $\hat{f}$ and not $\hat{e}$, then clearly eR_3f. Similarly for paths from f to e. It follows that, if some path in D includes both e and f, then $\langle e,f\rangle \varepsilon R_3 \cup R_4 \ldots \cup R_7$. It then follows by (1.04) that $E' \times E' \subset R_1 \cup R_2 \ldots \cup R_7$.

Directly by (1.04) and its corollary, R_1 and R_2 are disjoint and each is disjoint from each of R_3, $R_4 \ldots$, R_7. If edges e and f are such that there are paths in D from e to f and from f to e and any of these includes q, then, for some $\bar{e} \varepsilon h^{-1}(q)$, some path in D includes both $\hat{e}$ and $\bar{e}$, contradicting the hypothesis that $\hat{e}<\bar{e}$ [resp. $\bar{e} < \hat{e}$]. This shows that R_3 is disjoint from each of $R_4 \ldots$, R_7, and that each of R_4, R_5 is disjoint from each of R_6, R_7. On purely logical grounds, R_4 and R_5 are disjoint, as are R_6 and R_7.

By definition of D', $D'_< = R_1 \cup R_4 \cup R_7$ [resp. $R_1 \cup R_5 \cup R_6$], its converse is $R_2 \cup R_5 \cup R_6$[resp. $R_2 \cup R_4 \cup R_7$], and one path in D' includes both e and f if and only if eR_3f. Thus $D'_<$ satisfies the trichotomy condition of (1.04).

We next prove that $D'_<$ is transitive. Suppose $e_1R_ie_2$ and $e_2R_je_3$, where $\{i,j\} \subset \{1, 4, 7\}$ [resp. $\{1, 5, 6\}$]. If i=j=1 and $e_2 \neq g$, then it is clear that $e_1R_1e_3$. If i=j=1 and e_2=g, then either $e_1<e_3$

or there are paths in D from e_1 to $\hat{f}$ and from $\hat{e}$ to e_3 or there are paths in D from e_3 to $\hat{f}$ and from $\hat{e}$ to e_1. Either of the last two alternatives contradicts the hypothesis on $\hat{e},\hat{f}$.

If $i=1$, $j=4$ [resp. $i=6$, $j=1$], then either $e_1<e_3$ or there is a path in D from e_1 to e_3 [resp. from e_3 to e_1]. Let P be such a path. By hypothesis, P does not include $\hat{f}$. Let $\bar{e}$ be the last edge of P which does not occur in any path from e_2 to e_3 [to e_1]. Necessarily $\bar{e}<e_2$ [resp. $e_2<\bar{e}$]. From this it follows that there is a path in D from $\bar{e}$ to $\hat{f}$, and it then follows by definition of $\bar{e}$ that P includes $\hat{e}$, whence $e_1R_4e_3$ [resp. $e_1R_6e_3$].

Similar arguments dispose of $\langle i,j\rangle = \langle 1,7\rangle$, $\langle 4,1\rangle$, $\langle 7,1\rangle$ [resp. $\langle 5,1\rangle$, $\langle 1,6\rangle$, $\langle 1,5\rangle$]. (For $\langle 4,1\rangle$, $e' \in h^{-1}(q)\backslash\{\hat{e}\}$ plays the role of $\hat{f}$ in the above argument. Similarly for $\langle 7,1\rangle$.) $\langle i,j\rangle = \langle 4,7\rangle$, $\langle 7,4\rangle$ [resp. $\langle 5,6\rangle$, $\langle 6,5\rangle$] are each self-contradictory. $i=j \in \{4,7\}$ [resp. $\{5,6\}$] are impossible by the hypothesis on $\hat{e},\hat{f}$.

Finally, we note that if there is a path in D' from p to q with $p, q \in Q$, then there is a path in D from p to q. This shows that if D is acyclic then so is D', and that if D is closed then so is D'.

2. GRAMMARS AND DERIVATIONS

2.01. A grammar G comprises G_P, G_S, G_T, where, for some set V:

G_P is a finite relation in V* (the productions). $<x,y> \in G_P$ is denoted $x \to y$. $<> \to y$ is not permitted.

G_S is a finite subset of V* (the starting words).

$G_T \subset V$ (the terminal vocabulary).

we also define

G_V to be the least inclusive set V as above (the vocabulary), and

$G_N = G_V \backslash G_T$ (the non-terminal vocabulary).

Corollary. In any grammar G, G_N is finite.

2.02. If G is a grammar and x, $y \in G_V^*$, then a G-derivation of y from x comprises a finite closed acyclic ordered directed graph D_Q, D_E, D_h, D_t, $D_<$, together with a function $D_\ell : E \to G_V \cup \{<>\}$ satisfying (i, ii, iii) below, for the specification of which we also define:

D_I [resp. D_O]: $Q \to G_V^*$: $q \to D_\ell(e_1) \ . \ . \ . \ D_\ell(e_n)$, where $e_1 \ . \ . \ . < e_n$ and $h^{-1}(q)$ [resp. $t^{-1}(q)$] $= \{e_1 \ . \ . \ . \ , e_n\}$, (this definition being justified by (1.07).)

(i) $D_O(S) = x$ (necessarily $D_I(S) = <>$).

(ii) $D_I(T) = y$ (necessarily $D_O(T) = <>$).

(iii) For all $q \in Q \backslash \{S,T\}$, $<D_I(q), D_O(q)> \in G_P$.

As before, ℓ, I, O, ℓ', etc. may be used to denote D_ℓ, D_I, D_O, D'_ℓ, etc.

To illustrate these definitions, we may formalize the new material of figure 1 thus:

2.03 <u>Example</u>. Let $G_S = \{aXBC, <>\}$, $G_T = \{a,b,c\}$, and let G_P comprise the productions of figure 1(a). Then G is a grammar and D, specified by tables I, II, and III, is the G-derivation of aabbcc from aXBC shown in figure 1(c). For this D, Q = {0,1,2,3,4,5,6,7,8}, S=0, T=8, and E is the set of headings e in table I. e<f only for those pairs indicated in table II. (This relation is the transitive closure of the left-to-right ordering of the edges at the nodes of figure 1(c) as augmented by 08<13, 13'<45, 35<14, 46<07.)

Table I

e	08	01	13	12	23	13'	38	35	14	04	45	58	56	46	68	67	07	78	78'
t(e)	0	0	1	1	2	1	3	3	1	0	4	5	5	4	6	6	0	7	7
h(e)	8	1	3	2	3	3	8	5	4	4	5	8	6	6	8	7	7	8	8
ℓ(e)	a	X	a	X	<>	B	a	b	C	B	B	b	b	C	b	c	C	c	c

Table II

e \ f	01	13	12	23	13'	38	35	14	04	45	58	56	46	68	67	07	78	78'
08	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<
01									<							<		
13			<	<	<			<	<	<			<			<		
12					<			<	<	<			<			<		
23					<			<	<	<			<			<		
13'								<	<	<			<			<		
38							<	<	<	<	<	<	<	<	<	<	<	<
35								<	<	<			<			<		
14									<							<		
04																<		
45													<			<		
58												<	<	<	<	<	<	<
56													<			<		
46																<		
68															<	<	<	<
67																<		
07																		
78																		<

Table III

q	0	1	2	3	4	5	6	7	8
I(q)	<>	X	X	aB	BC	bB	bC	cC	aabbcc
O(q)	aXBC	aXBC	<>	ab	CB	bb	bc	cc	<>

2.04. If G is a grammar, then L(G), the language determined by G, is the set of strings $y \in G_T^*$ of which there is a G-derivation from some string $x \in G_S$.

For the grammar of example 2.03, L(G) is evidently $\{a^i b^i c^i \mid i \geq 0\}$.

It is clear that an old-style derivation can be read off from a sequence of cut sets in a new-style derivation, beginning with the cut set $t^{-1}(S)$, proceeding at each step by replacement of $h^{-1}(q)$ by $t^{-1}(q)$ for some node q, and ending with $h^{-1}(T)$. Thus, an old-style derivation is essentially a particular traversal from S to T of (the nodes of) a new-style derivation. The following is a formal development of this correspondence.

2.05. Lemma. If D is a finite closed acyclic ordered directed graph with $t^{-1}(S) = \{e_1 \ldots, e_n\}$, $e_1 \ldots < e_n$, then there is a node q of D with $h^{-1}(q) = \{e_j \ldots, e_k\}$ for some j, k with $1 \leq j \leq k \leq n$.

Proof: Let $f_1 \ldots, f_m$ be a longest path in D from S to T, and define $q = h(f_1)$. Clearly, every path in D from S to q has length 1. If q=T, then j=1, k=n will do. Otherwise let j be least with $h(e_j)=q$, and let k be greatest. By (1.11), if $j < \ell < k$, then e_ℓ occurs in a path from S to q, so $h(e_\ell)=q$.

2.06. Lemma. If G is a grammar, D a G-derivation, and $n, e_1 \ldots, e_n$, j, k, and q are as in (2.05) with $q \in \{S,T\}$, then

D' is also a G-derivation, where : $Q'=Q\setminus\{q\}$, $E'=E\setminus\{e_j, \ldots, e_k\}$, $S'=S$, $T'=T$, $t'(e) = \begin{cases} S \text{ if } t(e)=q \\ t(e) \text{ otherwise} \end{cases}$, and $<'$, h', ℓ' are the restrictions of $<$, h, ℓ (respectively) to E'.

Proof: It is sufficient to observe that $f_1 \ldots f_m$ is a path in D if and only if $f_1 \ldots f_m$ is a path in D' or $f_1 \in \{e_j, \ldots, e_k\}$ and ($m=1$ or $f_2 \ldots f_m$ is a path in D' and $t(f_2)=q$).

(It may be helpful to think of D' above as obtained by trying to pull D through a knothole, S first.)

2.07. If G is a grammar and D a G-derivation, then a sequence $\langle u_o, v_o, w_o\rangle \ldots \langle u_m, v_m, w_m\rangle$ in $(G_V^*)^3$ is a trace [resp. preorder trace, reverse preorder trace] of D if and only if

(i) $Q=\{S,T\}$, $m=0$, $u_o=w_o=\langle\rangle$, and $v_o=O(S)=I(T)$ or (inductively)

(ii) n, $e_1, \ldots, e_n$, j, k, q, D' are as in (2.06) [with j,k small as possible (for preorder trace), with j,k large as possible (for reverse preorder trace)], $u_o= \ell(e_1) \ldots \ell(e_{j-1})$, $v_o=\ell(e_j) \ldots \ell(e_k)$, $w_o=\ell(e_{k+1}) \ldots \ell(e_n)$, and $\langle u_1, v_1, w_1\rangle \ldots \langle u_m, v_m, w_m\rangle$ is a trace [resp. preorder trace, reverse preorder trace] of D'.

The traces of derivation D of example 2.03 are exhibited in figure 2. The traversals of the nodes of D shown there generate these traces in the following way: in passing from q to q' in a traversal, (i) construct a new trace element $\langle u, v, w\rangle$ by analysis of the current string uvw (initially O(S)) so that u comprises the sequence of labels on a suitable set of edges left of q' and $v=I(q')$; (ii) update the current string to $uO(q')w$.

(a)

(i)	(ii)	(iii)
< a,X,BC >	< a,X,BC >	< a,X,BC >
< aa,X,BCBC >	< aa,X,BCBC >	< aaXB,CB,C >
< a,aB,CBC >	< aaB,CB,C >	< aa,X,BBCC >
< aab,CB,C >	< a,aB,BCC >	< a,aB,BCC >
< aa,bB,CC >	< aa,bB,CC >	< aa,bB,CC >
< aab,bC,C >	< aab,bC,C >	< aab,bC,C >
< aabb,cC,<> >	< aabb,cC,<> >	< aabb,cC,<> >
< <>,aabbcc,<> >	< <> aabbcc,<> >	< <> aabbcc,<> >

(b)

(i)	0	1	2	3	4	5	6	7	8
(ii)	0	1	2	4	3	5	6	7	8
(iii)	0	1	4	2	3	5	6	7	8

Figure 2. (a) Traces of D (example 2.03). (i) Preorder trace. (iii) Reverse preorder trace. (b) Corresponding traversals of nodes of D.

Supposing $t^{-1}(S) = \{e_1 \ldots, e_n\}$, $e_1 \ldots < e_n$, and $h(e_i)=q_i$ for $i=1 \ldots n$, the traversals generating the preorder and reverse preorder traces may be specified in the manner of Knuth (1975, sec. 2.3.1) thus:

preorder traversal

visit S
for $i=1, 2 \ldots, n$:
if e_i is rightmost in $h^{-1}(q_i)$,

then traverse the subgraph rooted at e_i.

reverse preorder traversal

visit S
for $i=n, n-1 \ldots, 1$:
if e_i is leftmost in $h^{-1}(q_i)$,

then traverse the subgraph rooted at e_i.

Assuming each $q \in D_Q \setminus \{S,T\}$ to be labelled with (a name for) $D_I(q) \rightarrow D_O(q)$, the <u>derivation word</u> of Hart (1975) corresponding to D is simply the sequence of labels of nodes visited and edges passed over in a preorder traversal of D. For D as in (2.03), for example, it is

aX1aX2B3abCB4B5bbC6bcC7cc.

<u>Corollary</u>. If G is a grammar and D a G-derivation, then there is a trace of D, a unique preorder trace of D, and a unique reverse preorder trace of D.

2.08. If G is a grammar, then G-derivations D, D' are <u>isomorphic</u> if and only if there are one-to-one correspondences $f_Q: Q \rightarrow Q'$, $f_E: E \rightarrow E'$

such that $f_Q \circ h = h' \circ f_E$, $f_Q \circ t = t' \circ f_E$, $\ell' \circ f_E = \ell$, and $e_1 < e_2 \Leftrightarrow f_E(e_1) <' f_E(e_2)$. (Necessarily $f_Q(S)=S'$, $f_Q(T)=T'$.)

2.09. Theorem. If G is a grammar and D, D' are isomorphic G-derivations, then z is a trace [resp. preorder trace, reverse preorder trace of D if and only if z is a trace [resp. preorder trace, reverse preorder trace] of D'.

The result is evident from the definitions involved.

The converse of (2.09) does not hold, but only for insignificant reasons, namely that definition 2.02 does not exert much control over edges labelled "<>", and that such edges are not represented in traces. For example, if an edge 15 is added to the derivation of example 2.03 with $t(15)=1$, $h(15)=5$, $13' < 15 < 45$ (etc.), and $\ell(15)=<>$, then the result is still a G-derivation of aabbcc from aXBC, and its traces are again those shown in figure 2. The following addition to definition 2.02 is sufficient to remedy this defect.

2.10. If G is a grammar, than a G-derivation D is normal if and only if, whenever $\ell(e) = <>$, we have

(i) $t^{-1}(t(e)) = \{e\}$, and

(ii) $h(e) = T$ or there are edges e', e'' with $e' < e < e''$ and $h(e')=h(e'')=h(e)$.

2.11. Lemma. If G is a grammar and $z=(<u_o, v_o, w_o>, \ldots, <u_m, v_m, w_m>)$ is a sequence in $(G_V^*)^3$ with $u_m = w_m = <>$ and, for all $i \in \{1, \ldots, m\}$, $u_i v_i w_i = u_{i-1} v' w_{i-1}$ for some v' such that $v_i \to v'$ in G, then there is a normal G-derivation D of v_m from $u_o v_o w_o$ such that z is a trace of D.

Proof: We may proceed by induction on m. If m=0, there is clearly a normal G-derivation as required, with two nodes and min $\{1, |v_0|\}$ edges.

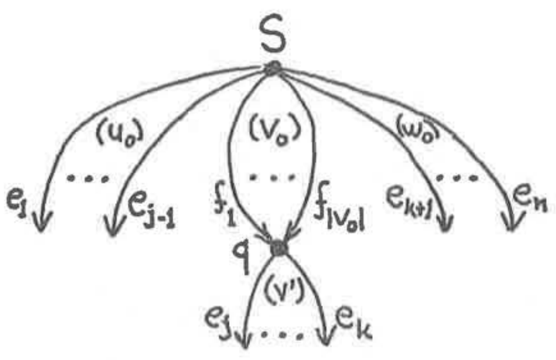
If m > 0, there is, by inductive hypothesis, a normal G-derivation D' of v_m from $u_1v_1w_1$ with $\langle u_1, v_1, w_1\rangle \ldots \langle u_m, v_m, w_m\rangle$ a trace of D'. Also, there is some v' with $v_o \rightarrow v'$ in G and $u_1v_1w_1 = u_ov'w_o$. Letting $t'^{-1}(S') = \{e_1 \ldots, e_n\}$ with $e_1 \ldots <' e_n$, we have unique (since D' is normal) j,k with $u_o = \ell'(e_1) \ldots \ell'(e_{j-1})$, $v' = \ell'(e_j) \ldots \ell'(e_k)$, and $w_o = \ell'(e_{k+1}) \ldots \ell'(e_n)$.

If $v' \neq \langle\rangle$ or m=1 and $v_m = \langle\rangle$, we have $j \leq k$. In this case, we define D thus: $Q = Q' \cup \{q\}$ $(q \notin Q')$; $E=E' \cup \{f_1 \ldots, f_{|v_o|}\}$ $(f_i \notin E')$; $t(f_1) \ldots = t(f_{|v_o|})=S=S'$, $t(e_j) \ldots = t(e_k)=q$, $t(e)=t'(e)$ otherwise; $h(f_1) \ldots = h(f_{|v_o|}) = q$, $h(e)=h'(e)$ otherwise; $\ell(f_1) \ldots \ell(f_{|v_o|})=v_o$, $\ell(e)=\ell'(e)$ otherwise; $e < f \Longleftrightarrow (e<'f)$ or $(e=f_{i_1}$, $f=f_{i_2}$, and $i_1<i_2)$ or $(e<'e_j$ and $f=f_i$ for some i) or $(e=f_i$ for some i and $e_k<'f)$). (See figure 3(a).)

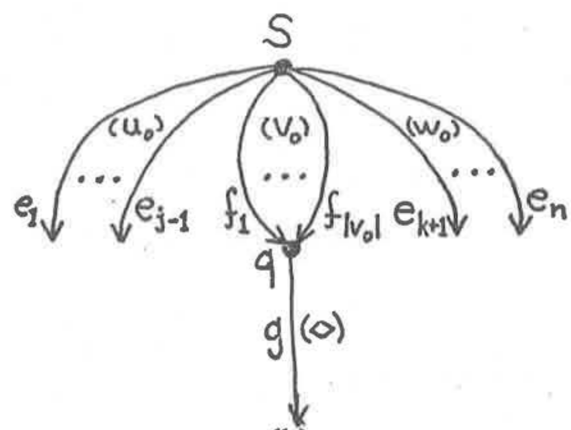
If $v' = \langle\rangle$ and (m > 1 or $v_m \neq \langle\rangle$), we have j=k+1. In this case, we define D thus: $Q=Q' \cup \{q\}$ $(q \notin Q')$ $E=E' \cup \{f_1 \ldots, f_{|v_o|}, g\}$ $(f_i, g \notin E')$; $t(f_1) \ldots = t(f_{|v_o|})=S=S'$, $t(g)=q$, $t(e)=t'(e)$ otherwise; $h(f_1) \ldots = h(f_{|v_o|}) = q$, $h(g)=T=T'$ if j=1 or k=n, otherwise h(g) the node of D' nearest to S' which occurs on both the rightmost path in D' from $h'(e_{j-1})$ to T' and the leftmost path in D' from $h'(e_{k+1})$ to T', $h(e)=h'(e)$ if $e \in E'$; $\ell(f_1) \ldots \ell(f_{|v_o|}) = v_o$, $\ell(g) = \langle\rangle$, $\ell(e) = \ell'(e)$ otherwise; $e < f \Longleftrightarrow ((e <' f)$ or $(e=f_{i_1}$, $f=f_{i_2}$, and $i_1 < i_2)$ or $((e <' e_{k+1}$ or k=n and $e \in E')$

and ($f=g$ or $f=f_i$ for some i)) or (($e=g$ or $e=f_i$ for some i) and ($e_{j-1} < f$ or $j=1$ and $f \in E'$)). (See figure 3(b).)

In either case, D is as required.



(a)



(b)

Figure 3. Construction of D in Lemma 2.11.

We may now make the usual definition of $\overset{*}{\Rightarrow}$ and exhibit the equivalence of the new and old styles.

2.12. If G is a grammar, define a relation $\Rightarrow_G$ in G_V^* thus: $z \Rightarrow_G z'$ if and only if there are u,v,v',w such that $z=uvw$, $z'=uv'w$, and $v \Rightarrow v'$ in G. As usual, we denote the reflexive-transitive closure of $\Rightarrow_G$ by $\overset{*}{\Rightarrow}_G$.

2.13. Theorem. If G is a grammar, then the following are equivalent for any $x,y \in G_V^*$:

(i) there is a G-derivation of y from x

(ii) there is a normal G-derivation of y from x.

(iii) $x \overset{*}{\Rightarrow}_G y$.

Proof: (i)$\Rightarrow$ (iii): Corollary to 2.07. (iii) $\Rightarrow$(ii): 2.11. (ii)$\Rightarrow$ (i): definition.

Finally, let us establish a converse for (2.09) and so prove that, for any grammar G, there is a unique sequence z as in 2.11 which is the [reverse] preorder trace of a G-derivation of y from x if and only if all normal G-derivations of y from x are isomorphic.

2.14. Lemma. If G is a grammar, D a G-derivation, and z a trace of D, then z is a trace of D', obtained by deleting one edge e with $\ell(e) = \langle\rangle$ and $t(e) = S$, provided $\{e\} \neq E$.

Proof: It is clear that D' is a G-derivation. Since $O'(S') = Q(S)$, it follows that z is a trace of D'.

2.15. Theorem. If G is a grammar, D and $\overline{D}$ are normal G-derivations, and z is a trace of D and of $\overline{D}$, then D and $\overline{D}$ are isomorphic.

Proof: We may proceed by induction on m, where $z = \langle u_o, v_o, w_o\rangle \ldots \langle u_m, v_m, w_m\rangle$. For m=0, the result is evident.

For $m > 0$, if $t(e)=S$, then $\ell(e) \neq \langle\rangle$ since D is normal. Similarly for $\overline{D}$. Thus the specification of $n, e_1 \ldots, e_n, j, k, q, D'$, and $\overline{e}_1 \ldots, \overline{e}_n, \overline{q}, \overline{D}'$ as in (2.07(ii)) is unique.

If $u_1 v_1 w_1 = u_o w_o$, then $t'^{-1}(S') = \{e_1 \ldots, e_{j-1}, f, e_{k+1} \ldots, e_n\}$ with $e_{j-1} <' f <' e_{k+1}$ and $\ell(f) = \langle\rangle$, and similarly for $\overline{D}'$ (with $\langle\rangle$ - edge $\overline{f}$). By (2.12), deleting f from D' and $\overline{f}$ from $\overline{D}'$ leads to G-derivations with trace $\langle u_1, w_1, w_1\rangle \ldots \langle u_m, v_m, w_m\rangle$. Clearly, each of these derivations is normal.

If $u_1 v_1 w_1 \neq u_o w_o$, then D' and $\overline{D}'$ are normal G-derivations with trace $\langle u_1, v_1, w_1\rangle \ldots \langle u_m, v_m, w_m\rangle$.

In either case, the inductive hypothesis yields an ismorphism which is clearly extendible to $D \to \overline{D}$.

3. APPLICATION: TWO IMPROVED PROOFS

It is somewhat misleading to describe the proofs given below for (3.07) and (3.10) as improvements, since equivalent propositions are usually presented without proof, as "clear" or with the proof "left as an exercise." However, detailed proof of propositions such as these seems to require reference to occurrences of symbols as, in some sense, "the same occurrence" in successive terms of an old-style derivation, and such reference is very awkward at best. Thus, supposing such reference to be truly necessary, the proofs below are improvements, in this matter of detail, on any proofs possible with the usual notation.[3]

The first proof to be considered is of a property (3.07) of context-free grammars which is useful in connection with deterministic parsing.

3.01. A grammar G is context-free if and only if $|x| = 1$ whenever $x \to y$.

3.02. A grammar G has pure terminal vocabulary if and only if $x \in G_N^*$ whenever $x \to y$.

3.03. If G is a grammar, then $G^\bullet$ is the grammar with $G^\bullet_P = G_P \cup \{S \to x \mid x \in G_S\}$, $G^\bullet_S = \{S\}$, $G^\bullet_T = G_T$, all where $S \notin G$.

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[3]Materially, a proof is a social act--a mathematical communication. Strictly speaking, the claim I am making for my proposal in comparison with accepted usage is not that it offers improved proofs, but that, by its more refined conventions of mathematical language, it enlarges the field of action for those who wish to communicate through proofs.

3.04. If x and y are strings and $k \in \mathfrak{N}$, then $x \overset{k}{=} y$ if and only if $x=y$ or (for some u, x', y') $x=ux'$, $y=uy'$, and $|u|=k$.

3.05. If X is a set and $k \in \mathfrak{N}$, then $X^{\leq k} = \{x \in X^* \mid |x| \leq k\}$.

3.06. If G is a context-free grammar with pure terminal vocabulary, $k \in \mathfrak{N}$, $A \to v$ in $G^\bullet$, and $w \in G_T^{\leq k}$, then

$$R_{G,k}(A \to v, w) = \{uvw \in G_V^* \mid \langle u,A,w' \rangle \text{ occurs in the reverse preorder trace of a } G^\bullet\text{-derivation from S for some } w' \text{ with } w' \overset{k}{=} w\}.$$

(This set represents the set of configurations of a shift-reduce bottom-up parser with k-symbol lookahead in which reduction by $A \to v$ is an appropriate next move. Deterministic parsing is possible if these sets satisfy a condition on disjointness of initial substrings for distinct pairs $\langle A \to v, w \rangle$.)

3.07. Theorem. If $G,k,A \to v$, and w are as at (3.06), then $R_{G,k}(A \to v,w)$ is a regular language over G_V.

Proof: Define a grammar G' with $G'_N \subset G_V^* \times G_T^{\leq k}$,

$$G'_P = \{\langle B,y \rangle \to r\langle C,z \rangle \mid y,z \in G_T^{\leq k},\ C \in G_N, \text{ and } B \to rCs \text{ in G for some } s \in G_V^* \text{ such that } s \overset{*}{\Rightarrow} z' \in G_T^* \text{ for some } z' \text{ with } z'y \overset{k}{=} z\} \cup \{\langle A,w \rangle \to vw\},$$

$$G'_S = \{\langle S, \langle\rangle \rangle\},\ G'_T = G_V.$$

L(G') is regular by the form of the productions of G'. That L(G') $= R_{G,k}(A \to v,w)$ follows immediately from the fact that $\langle S, \langle\rangle \rangle \overset{*}{\Rightarrow}_{G'} u\langle C,z \rangle$ if and only if $\langle u,C,z' \rangle$ occurs in the reverse preorder trace of a $G^\bullet$-derivation from S for some z' with $z' \overset{k}{=} z$, which we now prove.

Suppose $\langle S, \langle\rangle \rangle \Rightarrow_{G'} u \langle C,z \rangle$ and proceed by induction on the size of such a G'-derivation. For size 0, we have $u = z = \langle\rangle$, $C = S$, and $\langle \langle\rangle, S, \langle\rangle \rangle$ occurs as required. For size > 0, suppose $\langle S, \langle\rangle \rangle \overset{*}{\Rightarrow}_{G'} u'\langle B,y \rangle \Rightarrow_{G'} u \langle C,z \rangle$. Then (by definition of G_P) there are r,s,z' such that

$B \to rCs$ in $G^\bullet$, $u=u'r$, $s \overset{*}{\Rightarrow}_G z'$ and $z'y \overset{k}{=} z$. By inductive hypothesis, there is $y' \in G_T^*$ such that $y' \overset{k}{=} y$ and $\langle u',B,y'\rangle$ occurs in the reverse preorder trace of a $G^\bullet$-derivation from S. It follows that $\langle u,C,z'y'\rangle$ occurs as required.

Conversely, let D be a $G^\bullet$-derivation in the reverse preorder trace of which $\langle u,c,z'\rangle$ occurs. This occurrence is by virtue of the fact that $\ell(e)=C$ for some particular edge e of D. Proceed by induction on the distance of e from the initial node D_S. For distance 0, we have $u=z'=\langle\rangle$, $C=S$, so $\langle S, \langle\rangle \rangle \overset{*}{\Rightarrow}_{G'} u \langle C, \langle\rangle \rangle$ as required. For distance > 0, we have (since G is context-free) $h^{-1}(t(e)) = \{e'\}$ with $\ell(e')=B$ and $B \to rCs$ in $G^\bullet$ for some B,r,s. By virtue of this, the inductive hypothesis applies to $\langle u',B,y'\rangle$, where $u'r=u$ and $\bar{y}y'=z'$ for some $\bar{y} \in G_T^*$ such that $s \overset{*}{\Rightarrow}_G \bar{y}$. Let $y,z \in G_T^{\leqslant k}$ with $y \overset{k}{=} y'$ and $z \overset{k}{=} z'$. Then $\langle B,y\rangle \to r \langle C,z\rangle$ in G', and by inductive hypothesis $\langle S, \langle\rangle \rangle \overset{*}{\Rightarrow}_{G'} u' \langle B,y\rangle$, so all is as required.

As mentioned before, proofs are not usually given for (3.07) or its equivalent. Apart from that, nothing is unusual about the above proof except for the last paragraph. Usual arguments for the point of that paragraph[4] proceed by induction on the length of an old-style derivation equivalent to D. This induction requires reference to the first component (sentential form) of that derivation in which a given occurrence of an element of G_V appears, and so appeals to the notion that several apparently distinct occurrences may really be "the same." To believe such a proof, one must have accepted its appeal to the "same occurrence" idea as based in fact--must, that is, have come to understand the structure of context-free derivations quite well. I feel that the proof offered here tends to build up understanding rather than to require it, and so is more communicative.

4 Hopcroft and Ullman (1969, Lemma 12.4), for example.

It might seem that the well-developed notion of trees for context-free derivations could be used to present the proof given above for (3.07) without adopting the wholesale notational changes suggested here, and so it can. Unfortunately, however, the careful definition of "derivation" as "tree" is no easier than the definition of "derivation" as done here, and the work of establishing the correspondence between trees and old-style derivations is essentially the same as defining "trace." Thus, in a complete presentation of generative grammars, the new notation is no more awkward and time-consuming than the old. Furthermore, trees are insufficient to deal with the second proof we consider, in which the notions "context-sensitive" and "length-nondecreasing" are shown to be equivalent.

3.08. A grammar G is context-sensitive if and only if each production of G is of the form $uAv \to uxv$, where $A \in G_V$, $u,v \in G_V^*$, and $x \in G_V^+$.

3.09. A grammar G is length-nondecreasing if and only if $|x| \leq |y|$ whenever $x \to y$.

3.10. Theorem. If G is a length-nondecreasing grammar, then there is a context-sensitive grammar G' with $L(G')=L(G)$.

Proof: For a grammatical production $p=(x' \to y')$, define the torsion of p to be the smallest $t \in \mathfrak{N}$ such that $x'=uxv$, $y'=uyv$, and $|x|=t+1$ for some strings u,v,x,y. To prove the theorem, we proceed by induction, primarily on the maximum torsion of any $p \in G_P$, secondarily on the number of elements of G_P with maximum torsion. If the maximum torsion of any $p \in G_P$ is 0, then $G'=G$ will do. Otherwise, let $p=(uAxv \to uyBzv)$ have maximum torsion in G_P, with $A,B \in G_V$, $|x| = |y| =$ (the torsion of p) > 0.

Define $\hat{P} = \{uAxv \to uCxv,\ Cx \to CwD,\ CwD \to yD,\ uyDv \to uuBzv\}$, where $|w| = |x|-1$ and C,D, and the components of w are all distinct and disjoint from G_V.

Define grammars H, H' by $H_P = G_P \cup \hat{P}$, $H'_P = H_P \backslash \{p\}$; $H_S = H'_S = G_S$; $H_T = H'_T = G_T$. Obviously $L(G) \subset L(H') \subset L(H)$.

We prove $L(H) \subset L(G)$: Let D be a normal H-derivation of $t \in G_T^*$ from $s \in G_S$. We show by induction on the number of edges e of D with $\ell(e)$ among (C,D, and the components of w) that there is a G-derivation of t from s. If the number of such edges is 0, then D is as required. Otherwise, there are nodes q_1, q_2, q_2', q_3 of D with $I(q_1)=uAxv$, $O(q_1)=uCxv$, $I(q_2)=Cx$, $O(q_2)=I(q_2')=CwD$, $O(q_2')=yD$, $I(q_3)=uyDv$, and $O(q_3)=uyBzv$, all as in figure 4(a). (In figure 4, "$\Rightarrow$ " denotes a sequence of edges with common head or tail.) More specifically, since H is length-nondecreasing and D is normal, there are no edges e of D with $\ell(e) = \langle\rangle$, and we may, as a matter of notation, specify $h^{-1}(q_1)= \{e_{11} \ldots, e_{1,|uAxv|}\}$, $h^{-1}(q_2)= \{e_{2,|uA|} \ldots, e_{2,|uAx|}\}$, $h^{-1}(q_3) = \{e_{31} \ldots, e_{3,|uAxv|}\}$, $t^{-1}(q_1)= \{f_{11} \ldots ,f_{1,|uAxv|}\}$, $t^{-1}(q_2') = \{f_{2,|uA|} \ldots, f_{2,|uAx|}\}$, and $t^{-1}(q_3) = \{f_{31} \ldots, f_{3,|uyBzv|}\}$, with $e_{ij} < e_{ik}$ and $f_{ij} < f_{ik}$ wherever $j < k$ in all cases. As indicated in figure 4(a), the q_i have the following properties: $f_{1,|uC|}= e_{2,|uC|}$ (labelled "C"), $f_{2,|uyD|}= e_{3,|uyd|}$ (labelled "D"), and $t^{-1}(q_2) = h^{-1}(q_2')$ (labelled "CwD").

We now construct $D^{(1)}$, $D^{(2)}$, $D^{(3)}$ satisfying (2.02) except that $D^{(1)}$ and $D^{(2)}$ may not satisfy (2.02 (iii)) at node q_2.

$D^{(1)}$ = D except that $Q^{(1)} = Q\backslash\{q_2'\}$, $E^{(1)}= E\backslash h^{-1}(q_2')$, $t^{(1)}(f_{2j}) = q_2$ for all $j = |uA|, |uA| + 1 \ldots, |uAx|$, and $<^{(1)}$ is the restriction of $<$ to $E^{(1)}$. (All as in figure 4(b).) Since $D^{(1)}$ may be obtained from D by $|Cw|$ applications of (1.12) and one of (1.13), it is as claimed.

$D^{(2)}$, as shown in figure 4(c), is obtained from $D^{(1)}$ by $|uy|$ applications of (1.14), with $\langle\hat{e},\hat{f}\rangle = \langle e_{31}, f_{31}\rangle \ldots \langle e_{3,|uy|}, f_{3,|uy|}\rangle$ in turn followed by an application of (1.13) with $\hat{e} = e_{3,|uyD|}$ (labelled "D" in figure 4(b)).

$D^{(3)}$, as shown in figure 4(d), is obtained from $D^{(2)}$ by $|xv|$ applications of (1.14), with $\langle\hat{e},\hat{f}\rangle = \langle e_{1,|uAxv|}, f_{1,|uAxv|}\rangle \ldots \langle e_{1,|uA|+1}, f_{1,|uA|+1}\rangle$ in turn, followed by an application of (1.13) with $\hat{e} = f_{1,|uC|}$ (labelled "C" in figure 4(b,c)).

Since $D^{(3)}$ is a normal H-derivation of t from s with fewer edges labelled by (C,D, and the components of w), the inductive proof that there is a G-derivation of t from s is complete. This in turn completes the proof that $L(H) \subset L(G)$, so $L(H') = L(G)$.

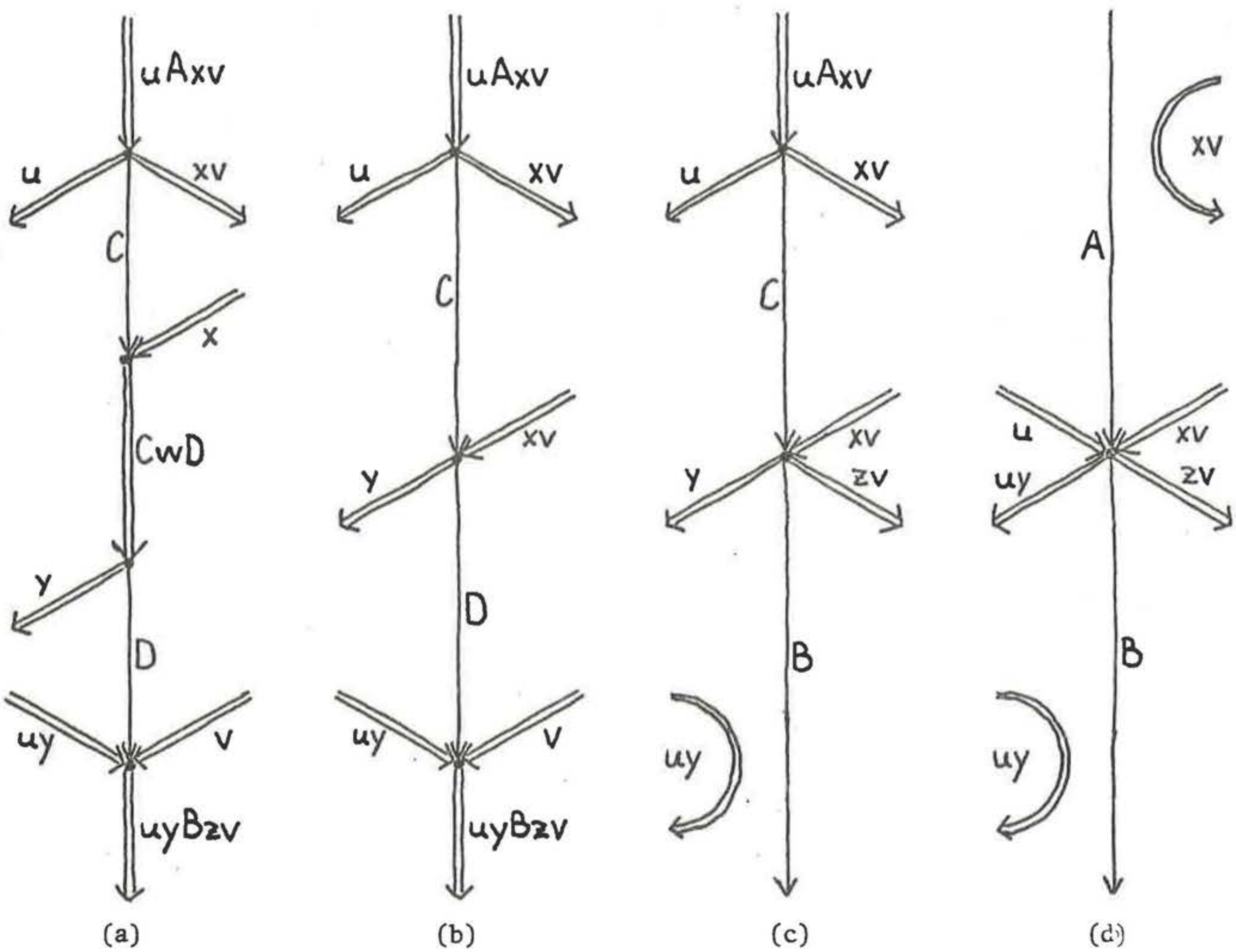

Figure 4. Construction of $D^{(1)}$, $D^{(2)}$, $D^{(3)}$ in proof of theorem 3.10.

Since the elements of $\hat{P}$ have smaller torsion than p, either H'_p has fewer elements with the torsion of p than G_p, or the maximum torsion of any element of H'_p is less than that of p. The proof of the theorem is thus complete.

The construction given in the above proof is pretty much the usual one, but I think no proof with the detail given above has previously been published. Indeed, I believe a proof with that detail presented in the usual notation would be either so dense or so lengthy as to communicate the basis of the construction's validity quite badly.

I will close by defending my proposal against two objections I anticipate. It may be objected: first, that the criticism of density or (more probably) lengthiness applies to the proof given here for (3.10), particularly if the necessary lemmas (1.12, 1.13, 1.14) are included; and second, that not even the amount of detail given in the proof of (3.10) itself is needed to communicate the underlying idea. I agree with the second point, provided it is understood that the "underlying idea" includes not merely the substitution of $\hat{P}$ for {p}, but also some key to the equivalence of the resulting grammar with its original. In particular, it is clear to me that for most purposes the proof of (3.10) should include no more detail than is necessary to describe figure 4 informally. However, even that detail, since it depends on the notion of single occurrence of a grammatical symbol in a new-style derivation, would be unsupportably tedious in the usual notation, so I find no objection to my proposal in the second point. To the first point, I respond further that the details included here to establish the foundations for my proposal are strictly optional for any ordinary presentation of grammatical matters, and that the ability to vary in reasonably fine steps the amount of detail expressed is a convenience this notation affords.

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