

# Performance Measures for Robot Manipulators: A Unified Approach\*

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## Abstract

We introduce a formalism for the systematic construction of performance measures of robot manipulators in a unified framework based on differential geometry. We show how known measures arise naturally in our formalism and we construct several new ones, including a non-linearity measure and a class of redundancy measures. The measures are applied to the analysis of two and three link planar arms for illustration.

## 1 Introduction

We introduce a unified approach to the construction of performance measures for robot manipulators. A performance measure is a field defined on the configuration manifold, i.e. the space of all postures of the manipulator, that measures some general property of the manipulator. The availability of such a measure is important for kinematically redundant manipulators, since the inverse kinematics problem generally has an infinite number of solutions. A performance measure allows us to choose the “best” solution.

For example, consider a task that requires arbitrary movements of the end-effector of the manipulator within a region in the workspace. In this case we would like to maximize the “mobility” of the end-effector in this region. On the other hand, if the task would be to push a heavy object in a certain direction one would like to minimize the mobility in this particular direction, allowing an easier application of a large force in that direction. In this case a nearly “singular” posture would be better.

The measures we are interested in are local, i.e. they depend only on the momentary posture of the manipulator. Global measures can be obtained by integrating local measures over some region in configuration space, see for example [GA91]. Furthermore, the “hardness” of a compound task (corresponding to a trajectory in configuration space) can be measured by the integration of a local measure over time.

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Several local measures have been proposed in the past. Yoshikawa [Yos85] proposed the scalar  $\sqrt{\det \mathbf{J}\mathbf{J}^T}$ , where  $\mathbf{J}$  is the Jacobian matrix of the manipulator. This measures how much the end-effector moves for a given infinitesimal movement of joint angles, averaged over all directions. It is the generalization to redundant manipulators of the determinant of the Jacobian. The related matrix  $(\mathbf{J}\mathbf{J}^T)^{-1}$ , when sandwiched between the workspace direction vector and its transpose, measures the analogous manipulability in a particular direction. The inverse,  $\mathbf{J}\mathbf{J}^T$ , then measures “pushability” in a given direction [Chi88].

From the singular value decomposition representation [KL80, Nak91] some closely related measures can be derived. Salisbury and Craig [SC82] proposed to use the condition number of  $\mathbf{J}$ , which is the ratio of the largest and smallest singular values of  $\mathbf{J}$ . Klein and Blaho [KB87] use the minimum singular value as a measure. The distance of the joint angles from their central positions was used by Liegeois [Lie77]. Asada [Asa83] computed the effective inertia matrix for a non-redundant manipulator and proposed to minimize the inhomogeneity of the moment of inertia.

Angeles and Ma [MA90, MA93] introduced the dynamical conditioning index, which is a measure for the anisotropy of the moment of inertia of the arm, which was used to design isotropic manipulators. The kinematic isotropy was investigated in [ALC92].

In this paper we introduce a unified approach to the construction of local performance measures, applicable to any kind of manipulating device. All the measures mentioned above arise naturally in our framework and we also introduce several new measures. Care has to be taken to ensure that a constructed measure corresponds to a physical property of the device and is not just a mathematical construct. In our method this is achieved by imposing invariance under general coordinate transformations in configuration space, i.e. we consider measures that do not depend on the actual coordinates (such as the joint angles for revolute joint arms for example) used to describe the posture of the manipulator. This criterion is sufficient, but not necessary in general, as the coordinates used usually have a direct physical meaning.

Our approach is to define various metrics on the configuration space, i.e. to define a “distance” between configurations, by a metric tensor. From the forward kinematics of the system we then construct an “induced” metric tensor on the work space and, for redundant manipulators, on the self-motion manifolds (the subspaces of configuration space that map to the same point in the work space). This allows us to use standard differential geometry (see, for example, [DP90]) to construct geometrical tensor fields which we interpret and use as performance measures.

The physical interpretation of these measures will depend on the initial choice of the metric on configuration space. For example, the Euclidean metric leads to kinematic measures, while using the inertia matrix as the metric tensor leads to dynamic measures. Besides reconstructing well-known measures in a unified framework, we also generate several new ones such as a measure of the non-linearity of the arm and a class of “redundancy” measures, that measure the ability of the arm to move to a different posture with the end-effector remaining fixed.

This approach is applicable to general manipulators. For the purpose of illustration we apply the formalism to planar positioning manipulators with 2 links (non-redundant) and 3 links (redundant), with some specific choices for the configuration space metrics.

The paper is organized as follows. Section 2 provides the notation and definitions used in the remainder of the paper. The construction of the measures is presented in Section 3. Section 4 deals with the interpretation of the measures for some specific choices of the configuration space metrics and we apply them to an analysis of the two and three link planar arms. We plot the values of the measures and compute optimal postures of a three link arm, using the measures.

## 2 Conventions and Notation

The configuration space is denoted by  $\mathcal{C}$ , and its dimension is  $n$ . It is typically the set of all manipulator postures. The task is described in the work space  $\mathcal{W}$ , which is typically the set of end-effector positions of the manipulator,  $\mathcal{E} = \mathbb{R}^3 \otimes SO(3)$ . We denote its dimension by  $m$ . The relation between the configurations space  $\mathcal{C}$  and the work space  $\mathcal{W}$  is described by the mapping

$$\kappa : \mathcal{C} \mapsto \mathcal{W}$$

which associates a point in  $\mathcal{W}$  with every point in  $\mathcal{C}$ . For typical robot arms with a manipulating device attached to the end of the arm, this mapping is the forward kinematics of the system. With  $\mathcal{C}$  and  $\mathcal{W}$  we associate 1-to-1 mappings  $\mathcal{C} \mapsto \mathbb{R}^n$  and  $\mathcal{W} \mapsto \mathbb{R}^m$ , which define charts or coordinate systems on the manifolds. The coordinate system on a manifold is considered here as an arbitrary labeling of the points in the manifold. A point in  $\mathcal{C}$  is denoted by the coordinates  $\mathbf{q} = (q^1, \dots, q^n)$  and a point in  $\mathcal{W}$  by  $\mathbf{x} = (x^1, \dots, x^m)$ .

The preimage of a point  $\mathbf{x}$  in  $\mathcal{W}$  is the self-motion manifold  $\mathcal{N}_{\mathbf{x}} \subset \mathcal{C}$ , which is the set of all postures that map to this point in  $\mathcal{W}$ . The tangent space to  $\mathcal{N}$  at  $\mathbf{q}$  is the null-space of the Jacobian of the mapping  $\kappa$ . Where it is clear what  $\mathbf{x}$  is being referred to, we will drop the subscript  $\mathbf{x}$ .

Tensor indices in  $\mathcal{C}$  are denoted by lowercase latin letters ranging from 1 to  $n$ , in  $\mathcal{W}$  by greek letters ranging from 1 to  $m$  and in  $\mathcal{N}$  by capital latin letters ranging from 1 to  $n - m$ .<sup>1</sup>

The components of tensors in  $\mathcal{C}$  are w.r.t. the coordinate basis  $\mathbf{e}_i = \frac{\partial}{\partial q^i}$  and the dual 1-form basis  $\omega^i = dq^i$ . We use the Einstein summation convention for tensor indices throughout. That is, any repeated tensor index is implicitly summed over its appropriate range. See [DP90] (or any book on differential geometry) for a detailed description of the tensor formalism.

A metric tensor on  $\mathcal{C}$  is written as  $\mathbf{h}$  with components  $h_{ij}$ . The (length)<sup>2</sup> of an infinitesimal vector  $dq^i \mathbf{e}_i$  is given by the “line-element”

$$ds^2 = h_{ij} dq^i dq^j.$$

$h_{ij}$  is the covariant (lower indices) form of the metric tensor. The inverse of  $\mathbf{h}$  has components  $h^{ij}$  and satisfies

$$h_{ij} h^{jk} = \delta_i^k$$

where  $\delta$  is given by

$$\delta_j^i = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

On  $\mathcal{W}$  we have a “task” metric tensor  $\eta$  with components  $\eta_{\mu\nu}$  such that

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

is the square of the “task” length of the infinitesimal vector with components  $dx^\mu$ . For a typical robot arm, this will be some combination of the spatial distance between points and their difference in orientation in the work space. Note that this metric is non-trivial in general as  $\mathcal{W}$  is generally non-Euclidean. For example if  $\mathcal{W}$  is the set of positions and orientations of the end-effector of a three dimensional manipulator  $\mathcal{W} = \mathbb{R}^3 \otimes SO(3)$ , which is not Euclidean.

Besides the task metric we construct an “induced” metric on  $\mathcal{W}$  (induced by the metric  $\mathbf{h}$  on  $\mathcal{C}$ ), written as  $\mathbf{g}$ , with components  $g_{\mu\nu}$ . We shall call this the “performance” metric on  $\mathcal{W}$ .

Finally, the induced metric on  $\mathcal{N}$  is written as  $\mathbf{n}$ , with components  $n_{AB}$ .

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<sup>1</sup>Generically, i.e. if the mapping  $\kappa$  is non-singular. See also [PL92].

### 3 General Formalism

#### 3.1 Outline

The construction of the measures proceeds as follows.

- Define an appropriate configuration space  $\mathcal{C}$  and a metric tensor  $\mathbf{h}$  on  $\mathcal{C}$ . Typically,  $\mathcal{C}$  is the joint space  $\mathcal{J}$  of the arm, but it can be much more general. For example, if we consider measures on the self-motion manifold  $\mathcal{N}_{\mathbf{x}}$  associated with a point  $\mathbf{x}$  in  $\mathcal{W}$ , we define  $\mathcal{N}_{\mathbf{x}}$  as the configuration space manifold.
- Define a task space  $\mathcal{W}$  and a metric  $\eta$  on  $\mathcal{W}$ . For a typical robot arm  $\mathcal{W} = \mathcal{E} = \mathfrak{R}^3 \otimes SO(3)$  and  $\eta$  measures the difference in position and orientation of the end-effector of the manipulator.
- Define a mapping  $\kappa : \mathcal{C} \mapsto \mathcal{W}$ . If  $\mathcal{C}$  is the joint space,  $\mathcal{W} = \mathcal{E}$  and  $\kappa$  is the forward kinematics of the system. An important case is  $\mathcal{W} = \mathcal{C}$ , i.e. we require a task in  $\mathcal{C}$ , but with its own metric  $\eta$ .
- From  $\kappa$  and  $\mathbf{h}$  we construct an induced metric on  $\mathcal{W}$ , the performance metric  $\mathbf{g}$ . This is a tensor valued performance field. A directional performance field is obtained by double contraction with a vector in  $\mathcal{W}$ .
- From  $\mathbf{g}$  and  $\eta$  we construct the following measures. Note that, in general, a measure is defined by the tuple  $\{\mathcal{C}, \mathcal{W}, \kappa, \mathbf{h}, \eta\}$ .
  - A measure  $R$  for the non-linearity of the motion of the manipulator. This is the double contraction of the Riemann tensor of the manifold. This measure depends on  $\mathcal{C}$  and  $\mathbf{h}$  only.
  - A relative directional performance field  $U_{\mathbf{u}}$ . This field depends on a direction specified by the vector  $\mathbf{u}$  in  $\mathcal{W}$ .
  - A relative average performance field, which we call the generalized Yoshikawa measure  $Y$ .
  - A measure of the performance anisotropy of  $\mathbf{g}$ , the anisotropy measure  $A$ . (This measure actually does not depend on the task metric  $\eta$ .)

Measures on the self-motion manifold  $\mathcal{N}_{\mathbf{x}}$  are constructed by defining a configuration space  $\mathcal{C}_2 = \mathcal{N}_{\mathbf{x}}$  and a taskspace  $\mathcal{W}_2 = \mathcal{C}_2$  with a trivial mapping  $\kappa_2$  (the identity map). We take the metric on  $\mathcal{C}_2$  to be just  $\mathbf{h}$  restricted on  $\mathcal{C}_2$ . The metric on  $\mathcal{W}_2 = \mathcal{C}_2$  is taken to be the restriction of some “secondary” metric  $\hat{\mathbf{h}}$  defined on  $\mathcal{C}$  to  $\mathcal{C}_2$ . We can now apply the above construction on  $\mathcal{C}_2$ ,  $\mathcal{W}_2$  and associated metrics and generate a set of measures on  $\mathcal{N}_{\mathbf{x}}$ . By considering all possible points  $\mathbf{x}$  we then arrive at a set of measures defined on the whole configuration manifold  $\mathcal{C}$ . We call these measures the “redundancy” measures.

#### 3.2 The Performance Metric

The map  $\kappa : \mathcal{C} \mapsto \mathcal{W}$  (e.g., the forward kinematics) is defined by

$$x^\mu = \kappa^\mu(\mathbf{q}) \tag{1}$$

and the Jacobian  $\mathbf{J}$  of the map is written as

$$J_i^\mu(\mathbf{q}) = \frac{\partial \kappa^\mu(\mathbf{q})}{\partial q^i}$$

The construction of an induced metric on  $\mathcal{W}$  proceeds as follows. Let the manipulator be in posture  $\mathbf{q}$ , with the end-effector at  $\mathbf{x}$ . We define the induced distance  $d_{\mathbf{x}}(\mathbf{y})$  from  $\mathbf{x}$  to  $\mathbf{y}$  as the length of the shortest path in  $\mathcal{C}$  (with respect to the chosen metric  $\mathbf{h}$ ) that moves the end-effector from  $\mathbf{x}$  to  $\mathbf{y}$ .

The metric tensor on  $\mathcal{W}$  can now be derived by considering the shortest path in  $\mathcal{C}$ , from  $\mathbf{q}$  to  $\mathbf{q} + d\mathbf{q}$ , that moves the end-effector from  $\mathbf{x}$  to  $\mathbf{x} + d\mathbf{x}$ . From eq. (1) we obtain

$$d\mathbf{x} = \mathbf{J}d\mathbf{q}. \quad (2)$$

The minimum norm solution of eq. (2) is given by

$$d\mathbf{q} = \mathbf{J}^+ d\mathbf{x} \quad (3)$$

where the pseudoinverse  $\mathbf{J}^+$  is given by

$$\mathbf{J}^+ = \mathbf{h}^{-1} \mathbf{J}^T (\mathbf{J} \mathbf{h}^{-1} \mathbf{J}^T)^{-1}.$$

So the induced norm of  $d\mathbf{x}$  is given by

$$ds^2 = h_{ij} dq^i dq^j = d\mathbf{x}^T \mathbf{J}^{+T} \mathbf{h} \mathbf{J}^+ d\mathbf{x}$$

and we define

$$\mathbf{g} = \mathbf{J}^{+T} \mathbf{h} \mathbf{J}^+$$

as the covariant metric tensor on  $\mathcal{W}$ . The field  $\mathbf{g}$  is a tensor valued performance field.

$\mathbf{g}$  has a simpler contravariant form

$$g^{\mu\nu} = J_i^\mu h^{ij} J_j^\nu$$

or, in matrix notation (as  $g^{\mu\nu}$  are the components of  $\mathbf{g}^{-1}$ ),

$$\mathbf{g}^{-1} = \mathbf{J} \mathbf{h}^{-1} \mathbf{J}^T.$$

The minimum norm solution to eq. (2) can now be written as

$$dq^i = J_\mu^i dx^\mu$$

where  $J_\mu^i$  are the components of  $\mathbf{J}^+$  and are obtained from  $J_i^\mu$  by raising and lowering its indices with the appropriate metric:

$$J_\mu^i = g_{\mu\nu} h^{ij} J_j^\nu.$$

This provides an elegant tensor representation of the pseudoinverse of the Jacobian.

Note that  $g_{\mu\nu}$  does not carry any  $\mathcal{C}$  space indices and is therefore invariant under general coordinate transformations in  $\mathcal{C}$ . It transforms as a tensor under general coordinate transformations in  $\mathcal{W}$ . Scalar directional performance fields can be obtained from  $g_{\mu\nu}$  by contraction with an appropriate vector  $\mathbf{u}$  in  $\mathcal{W}$ , i.e.  $g_{\mu\nu} u^\mu u^\nu$ .

### 3.3 Work Space Measures

The scalar

$$U_{\mathbf{u}} = \frac{\eta_{\mu\nu} u^\mu u^\nu}{g_{\mu\nu} u^\mu u^\nu} \quad (4)$$

with  $\eta_{\mu\nu}$  the task metric on  $\mathcal{W}$ , measures task space length per  $\mathcal{C}$  space length for movements in direction  $\mathbf{u}$ . It is a directional performance field.

From  $\mathbf{g}$  and  $\eta$  we derive a scalar measure that we call the generalized Yoshikawa measure. Consider an infinitesimal parallelepiped  $dx^1 \dots dx^m$  located at the point  $\mathbf{x}$  in  $\mathcal{W}$  and suppose the arm is in configuration  $\mathbf{q}$ . The  $m$ -volume of the smallest region in  $\mathcal{C}$  that will cover the parallelepiped in  $\mathcal{W}$  is given by

$$\sqrt{\det(\mathbf{g})} dx^1 \dots dx^m.$$

The task volume of this region is

$$\sqrt{\det(\eta)} dx^1 \dots dx^m.$$

We define the generalized Yoshikawa measure as the  $\mathcal{W}$  space volume per  $\mathcal{C}$  space volume of the parallelepiped,

$$Y = \sqrt{\frac{\det(\eta)}{\det(\mathbf{g})}} \quad (5)$$

It is a measure of the performance of the manipulator, averaged over all directions. Note that  $Y$  is invariant under general coordinate transformations in  $\mathcal{C}$  as well as in  $\mathcal{W}$  space and therefore measures a true physical property of the system, independent of the coordinate systems.<sup>2</sup> If  $\mathbf{h}$  is the Euclidean metric in “joint-angle” coordinates and the task metric on  $\mathcal{W}$  is Euclidean, it reduces to the original Yoshikawa measure  $\sqrt{\det(\mathbf{J}\mathbf{J}^T)}$  [Yos85].

A second scalar derived from the metric tensor  $\mathbf{g}$  is the anisotropy, defined as

$$A = 1 - \frac{\mu_-}{\mu_+} \quad (6)$$

where  $\mu_+$  and  $\mu_-$  are the largest and smallest eigenvalues of the metric in contravariant form, i.e.  $\mathbf{g}^{-1}$ . The contravariant form is chosen for the definition as the metric  $\mathbf{g}$  has singularities. (If the manipulator is in a singular configuration such that it cannot move in some direction in  $\mathcal{W}$ , the metric tensor  $\mathbf{g}$  will become infinite in that direction.) The measure given in eq. (6) measures the anisotropy. If isotropy is desired for a task, this measure should be minimized. For positive definite metrics  $\mathbf{h}$  we have  $0 \leq A \leq 1$  and isotropy is obtained if  $A = 0$ . The anisotropy is invariant under general coordinate transformations in  $\mathcal{C}$ , but not invariant under general coordinate transformations in  $\mathcal{W}$ . However it is invariant under orthogonal transformations (rotations and reflections) in  $\mathcal{W}$

$$\mathbf{x} \rightarrow \mathbf{T}\mathbf{x}$$

where  $\mathbf{T}$  satisfies  $\mathbf{T}\mathbf{T}^T = \mathbf{I}$ , with  $\mathbf{I}$  the identity matrix, and under homogeneous scaling transformations

$$\mathbf{x} \rightarrow s\mathbf{x}$$

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<sup>2</sup>The field  $\sqrt{\det(\mathbf{g})}$  itself is not invariant under general coordinate transformations in  $\mathcal{W}$ . It is called a scalar density.

where  $s$  is a non-zero number.

Isotropy can be an important property for certain tasks. Manipulators have been designed for which isotropic points exist in the work space [MA93].

Other  $\mathcal{W}$  space measures can be derived as needed from the above by covariant differentiation and contraction. For example the local variation of the generalized Yoshikawa measure could be defined by

$$g^{\mu\nu} J_{\mu}^i J_{\nu}^j \frac{\partial^2 Y}{\partial q^i \partial q^j}$$

### 3.4 Non-linearity Measures

The non-linearity of the manipulator with respect to the metric on  $\mathcal{C}$  is measured by the Riemann tensor, which is a rank 4 tensor field on  $\mathcal{C}$ , with components  $R^i_{jkl}$ . In a coordinate basis it is given by

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial q^k} - \frac{\partial \Gamma^i_{jk}}{\partial q^l} + \Gamma^i_{mk} \Gamma^m_{jl} - \Gamma^i_{ml} \Gamma^m_{jk}$$

where the Christoffel symbols  $\Gamma$  are given by

$$\Gamma_{ijk} = \frac{1}{2} \left( \frac{\partial h_{ij}}{\partial q^k} + \frac{\partial h_{ik}}{\partial q^j} - \frac{\partial h_{jk}}{\partial q^i} \right)$$

and the  $\Gamma^i_{jk}$  are obtained by raising the first index, e.g.

$$\Gamma^i_{jk} = h^{il} \Gamma_{ljk}.$$

The Riemann tensor has the following interpretation. Suppose we have two nearby points in configuration space with separation vector  $\mathbf{u}$  and these points move in the same direction (i.e. on parallel trajectories), described by a velocity vector  $\mathbf{v}$ , on geodesic trajectories. If the space is non-Euclidean, the trajectories will either diverge or converge (at least for some choice of  $\mathbf{u}$  and  $\mathbf{v}$ ) and the relative acceleration between the points is given by

$$a^i = R^i_{jkl} v^j u^k v^l$$

Therefore, this tensor measures non-linearity of the arm in a particular direction. The Riemann tensor has  $\frac{n^2}{12}(n^2 - 1)$  independent components.

The associated curvature scalar,

$$R = h^{ij} R^k_{ikj} \quad (7)$$

gives an overall measure of the non-linearity. The curvature scalar is a generalization of the Gaussian curvature of a surface. The curvature  $R$  is a scalar field, i.e. it is invariant under general coordinate transformations in  $\mathcal{C}$  and therefore measures an intrinsic property of the system. Note that this measure does not depend on the forward kinematics.

This measure (or  $R^2$ ) could be used to estimate the goodness of linear approximations used in controlling the manipulator. If  $R = 0$  at some point, configuration space is flat at this point. An interesting observation is that if there are points in  $\mathcal{C}$  with  $R > 0$  and with  $R < 0$ , there are  $(n - 1)$ -dimensional subspaces of  $\mathcal{C}$  with vanishing curvature. If  $m < n$  these subspaces will map onto  $m$ -dimensional regions in  $\mathcal{W}$ , so these are sections in  $\mathcal{W}$  that can be reached with an optimal posture with respect to the curvature measure.

A curvature tensor and its contraction can also be defined on  $\mathcal{W}$ , using the performance metric  $\mathbf{g}$ . This would measure the non-linearity of the motion of the end-effector. Note that for a non-redundant manipulator, i.e.  $m = n$ , we can view the the forward kinematics as given by eq. (1) as a change of coordinate system on  $\mathcal{C}$  and the curvature scalar  $R$  will therefore be the same whether constructed from  $h_{ij}$  on  $\mathcal{C}$  or from  $g_{\mu\nu}$  on  $\mathcal{W}$ .

### 3.5 Redundancy Measures

Another class of measures is related to the ability of the manipulator to move in the self-motion manifold  $\mathcal{N}$  associated with every point in  $\mathcal{W}$ .

Consider a redundant robot arm that performs some assembly task inside some enclosed region by reaching inside, through a small opening. It will be “easier” to reconfigure the arm (change its posture without moving the end-effector) if the segment of the arm that enters through the opening does not move very much during the reconfiguration. A possible measure for the “goodness” of the posture for a task like this is the ratio of the  $\mathcal{C}$  space distance and the displacement of the segment that reaches through the hole (as measured in the plane of the surface with the opening).

The redundancy measures are obtained by taking a “secondary” set  $\{\mathcal{C}, \mathcal{W}, \kappa, \mathbf{h}, \eta\}$  with  $\mathcal{C}_2 = \mathcal{N}_{\mathbf{x}}$  of a generic point in  $\mathcal{W}$  and taking  $\mathcal{W}_2 = \mathcal{C}_2$  (so  $\kappa_2$  is the identity map). The metric on  $\mathcal{C}_2$  is just  $\mathbf{h}$  restricted to  $\mathcal{N}_{\mathbf{x}}$ . We denote it by  $\mathbf{n}$ . The task metric  $\eta_2$  on  $\mathcal{W}_2$  is obtained by restricting a task metric  $\hat{\mathbf{h}}$  to  $\mathcal{N}_{\mathbf{x}}$ .

An infinitesimal displacement  $d\mathbf{q}$  in  $\mathcal{N}_{\mathbf{x}}$  can be written as

$$dq^i = P_j^i d\psi^j$$

for arbitrary  $d\psi^j$ . The projection operator  $\mathbf{P}$  is given by

$$P_j^i = \delta_j^i - J_\mu^i J_j^\mu$$

The Jacobian  $J_i^\mu$  defines  $n - m$  independent vectors  $\mathbf{w}_A$ , ( $A = 1, \dots, n - m$ ) that generate motions in  $\mathcal{N}_{\mathbf{x}}$  (except at singular points). The  $\mathbf{w}_A$  satisfy

$$J_i^\mu w_A^i = 0. \quad (8)$$

An arbitrary vector in  $\mathcal{N}_{\mathbf{x}}$  can be written as

$$v^i = w_A^i v^A$$

Its  $\mathbf{h}$ -length is

$$h_{ij} v^i v^j = h_{ij} w_A^i w_B^j v^A v^B$$

and we obtain the components of the metric tensor  $\mathbf{n}$  on  $\mathcal{C}_2$  as

$$n_{AB} = h_{ij} w_A^i w_B^j$$

Similarly, the task metric  $\eta_2$  is obtained by restricting  $\hat{\mathbf{h}}$  to  $\mathcal{N}_{\mathbf{x}}$ , i.e.

$$\eta_{2AB} = \hat{h}_{ij} w_A^i w_B^j$$



As the mapping  $\kappa_2$  is just the identity, the Jacobian as just the identity matrix and the induced performance metric  $\mathbf{g}_2$  on  $\mathcal{W}_2$  is just the metric  $\mathbf{n}$ . We now can construct the measures for the secondary workspace  $\mathcal{W}_2$ , which we call the “redundancy” measures.

The precise interpretation of the redundancy will depend on the secondary task metric  $\hat{\mathbf{h}}$  chosen, which is very task-specific in general. In examples below, we make some choices for  $\hat{\mathbf{h}}$  and interpret the secondary generalized Yoshikawa measures. Other redundancy measures, such as the anisotropy or non-linearity (i.e. the curvature scalar) can only be defined if the dimension of  $\mathcal{N}_{\mathbf{x}}$  is at least 2).

Note that in constructing the redundancy measures for a robot arm we are effectively considering a linkage with a closed loop, constructed out of the original arm by fixing the end-effector. A more general set of redundancy measures (which are just the work space measures of the closed loop linkage) can be obtained by considering non-trivial mappings  $\kappa_2$ , to a workspace  $\mathcal{W}_2$  of possibly lower dimension than  $\mathcal{N}_{\mathbf{x}}$ .

## 4 Interpretation and Applications

In this section we shall construct and interpret the measures for the two and three link planar manipulators. We shall consider two metrics on  $\mathcal{C}$ , the Euclidean metric which measures (distance)<sup>2</sup> as the sum of the squares of the joint angle differences, and the inertia tensor, i.e. we take

$$\mathbf{h} = \mathbf{H} \tag{9}$$

where  $\mathbf{H}$  is the inertia matrix of the arm. We shall refer to the Euclidean metric as the kinematic metric and to the metric defined by eq. (9) as the inertial metric.

Another possible metric is the joint-compliance matrix, which measures the “stiffness” of the arm (e.g., [PL91]). This can be considered as a generalization of the Euclidean metric with different weights for joint distances. We note that the kinematic measures are the same as the measures constructed from a compliance matrix with equal joint compliances.

The two link arm consists of two links of length  $l_1$  and  $l_2$ . We take all mass to be concentrated in the middle joint and in the tip of the second link, with masses  $m_1$  and  $m_2$  respectively. The configuration of the arm is described by the two joint angles  $q^1$  and  $q^2$ , with  $-\pi < q^i \leq \pi$ .

The three link arm consists of three links of length  $l_i$  ( $i = 1, 2, 3$ ) which we take to be thin rods of uniform mass density  $m_i$ . The configuration of the arm is described by the three joint angles  $q^i$ , with  $-\pi < q^i \leq \pi$ .

Note that the non-directional measures do not depend on the first joint angle  $q^1$ , hence it is not shown in plots of the values of the measures.

Unless given explicitly in a formula, we take all lengths to be equal to one and all masses to be equal to 0.5 in our calculations.

### 4.1 Kinematic Measures

In this case the induced metric on  $\mathcal{W}$ ,  $\mathbf{g}$ , leads to Yoshikawa’s [Yos85] manipulating force ellipses, defined as the set of all forces  $f_\mu$  satisfying

$$g^{\mu\nu} f_\mu f_\nu \leq 1$$

and the associated manipulability ellipses, defined as the set of all end-effector velocities  $v^\mu$  that satisfy

$$g_{\mu\nu} v^\mu v^\nu \leq 1$$

For a desired end-effector force  $f_\mu$ , the measure

$$g^{\mu\nu} f_\mu f_\nu$$

equals the Euclidean norm of the generalized joint torque needed to apply a prescribed force  $\mathbf{f}$  with the end-effector. Denoting the joint torque by  $\tau_i$ , the condition of static equilibrium is

$$\tau_i = f_\mu J_i^\mu$$

and it follows that

$$\sum_i \tau_i^2 = h^{ij} \tau_i \tau_j = f_\mu f_\nu J_i^\mu J_j^\nu h^{ij} = f_\mu f_\nu g^{\mu\nu}$$

In figs. 1-4 we show the best and worst postures of the three link arm for applying a vertical and horizontal force with the end-effector at a distance of 2.0 from the base of the arm.

The generalized Yoshikawa measure, as defined in eq. (5), reduces to the Yoshikawa measure [Yos85] if the factor  $\sqrt{\det(\eta)}$  is constant and equal to one as is the case for the planar positioning manipulators we are considering. For easy reference we reproduce here some of the results from [Yos85] for the planar manipulators. For the two link arm we have

$$Y = l_1 l_2 |\sin(q^2)|$$

and for the three link arm we have

$$\begin{aligned} Y^2 = & \frac{l_1^2 l_2^2}{2} + l_1^2 l_3^2 + l_2^2 l_3^2 + l_1 l_2 l_3^2 \cos(q^2) - \frac{l_1^2 l_2^2 \cos(2q^2)}{2} + \\ & l_1^2 l_2 l_3 \cos(q^3) - l_2^2 l_3^2 \cos(2q^3) - l_1^2 l_3^2 \cos(2(q^2 + q^3)) - \\ & l_1^2 l_2 l_3 \cos(2q^2 + q^3) - l_1 l_2 l_3^2 \cos(q^2 + 2q^3) \end{aligned}$$

In fig. 5 we have plotted  $Y$  as a function of the joint angles  $q^2$  and  $q^3$ . Fig. 6 shows the optimal postures for reaching a point at a distance of  $d = \{0.5, 1.0, 1.5, 2.0, 2.5\}$  from the base of the arm and fig. 7 shows the values of  $Y$  at these optimal postures as a function of the reach  $d$ .

The anisotropy measure  $A$ , as defined in eq. (6), measures the deviation from kinematic isotropy. It is zero if the manipulability and force ellipses become circles. In fig. 8 we plot  $A$  as a function of the second joint angle  $q^2$  for the two link arm. Figs. 9-11 show the graph of  $A$  for the three link arm, its optimal postures and the values of  $A$  for these postures. There is a completely isotropic posture ( $A = 0$ ) at a reach of  $d = 1$ .

Note that if the arm moves inward from a stretched position, there is a discontinuity in the optimal arm posture between  $d = 1.0$  and  $d = 0.5$ . A more detailed investigation (a more dense set of postures) shows that the *critical distance*  $d_c$  is located at  $d_c \approx 0.5$ . The discontinuity arises as follows. For a given position  $\mathbf{x}$  of the end-effector, the possible postures form the self-motion manifold  $\mathcal{N}_{\mathbf{x}}$ . The optimal posture is found as the global minimum (or maximum) of the measure

on  $\mathcal{N}_{\mathbf{x}}$ . If for a certain  $\mathbf{x}$  there are *two* global minima <sup>3</sup>, an infinitesimal displacement  $d\mathbf{x}$  of  $\mathbf{x}$  can resolve the degeneracy. It will depend on the direction of  $d\mathbf{x}$  in general which of the two minima will become the new global minimum. It follows that a continuous motion that optimizes the measure through  $\mathbf{x}$  is not possible in general. An interesting observation is that since  $\mathcal{N}_2$  is not necessarily connected, a motion may be required from one connected component to the other, which is not possible. In that case there is *no* self-motion possible at all that optimizes the measure. For the three link arm, the connected components of  $\mathcal{N}_{\mathbf{x}}$ , which appear for  $d < 1$ , are related by the symmetry  $q^i \rightarrow -q^i$ . As the measures we consider all observe this symmetry, there will always be two symmetry-related global minima. By picking the appropriate one, the arm is thus never forced to move from one connected component to the other, so that self-motion optimizing the measure is guaranteed to be possible. This is true for the  $\mathcal{W}$  space measures, but not for the redundancy measures in general, for the task space and therefore the task metric  $\hat{\mathbf{h}}$  need not have the same symmetries as the manipulator (though it is in all our examples).

The Riemann tensor and therefore the curvature measure  $R$ , as defined in eq. (7), vanishes for the kinematic metric on  $\mathcal{C}$ , as the  $h_{ij}$  are constant.

## 4.2 Inertial Measures

The generalized inertia matrix  $\mathbf{H}$  of the system is given by

$$H_{ij} = \frac{\partial E_k(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{q}^i \partial \dot{q}^j}$$

where  $E_k$  is the kinetic energy of the arm. We take the  $\mathcal{C}$  space metric as

$$h_{ij} = H_{ij}$$

The line-element

$$ds^2 = h_{ij} dq^i dq^j$$

measures the displaced mass over the distance  $d\mathbf{q}$  in  $\mathcal{C}$ . If the arm moves by  $d\mathbf{q}$  in a time  $dt$  with constant velocity, the kinetic energy of this motion is given by

$$E_k = \frac{ds^2}{dt^2}$$

The geodesics in  $\mathcal{C}$  are now precisely the motions of the arm in the absence of torques, friction and gravity.

The induced metric  $\mathbf{g}$  can be interpreted as an effective inertia matrix on  $\mathcal{W}$ . That is, if the end-effector moves with velocity  $\dot{\mathbf{x}}$  in such a way as to minimize the total kinetic energy of the arm, this kinetic energy is given by

$$E_k = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

The tensor  $g_{\mu\nu}$  can be measured, in principle, by applying forces to the end-effector, with the arm at rest, and recording the resulting accelerations of the end-effector. For if a force  $f_\mu$  is applied to the end-effector, with the arm at rest, the  $\mathcal{W}$  space acceleration  $\ddot{x}^\mu$  satisfies

$$\ddot{x}^\mu = J_i^\mu \ddot{q}^i + \frac{\partial J_i^\mu}{\partial q^j} \dot{q}^i \dot{q}^j = J_i^\mu \ddot{q}^i. \quad (10)$$

---

<sup>3</sup>We refer here to “accidentally” degenerate minima that are not related to each other by a symmetry of the system.

The equations of motion give

$$\frac{d}{dt}(h_{ij}\dot{q}^j) = h_{ij}\ddot{q}^j = f_\mu J_i^\mu. \quad (11)$$

Combining (10) and (11) gives

$$\ddot{x}^\mu = h^{ij} J_i^\mu J_j^\nu f_\nu$$

or

$$f_\mu = g_{\mu\nu}\ddot{x}^\nu.$$

Contracting  $g_{\mu\nu}$  with a velocity vector  $v^\mu$  gives the required kinetic energy of the arm to move the end-effector with this velocity. In figs. 12-15 we show the best and worst postures for achieving a high vertical and horizontal velocity of the end-effector (“fly-swatting”) at a distance  $d = 2.0$  from the base of the arm.

The generalized Yoshikawa measure (5) is now the inverse of the square root of the determinant of the effective inertia matrix; it measures the “lightness” of the arm. For the two link arm it is given by

$$\frac{|\sin(q^2)|}{\sqrt{m_2(m_1 + m_2\sin^2(q^2))}}.$$

In fig. 16 we plot this measure for the three link arm. Due to the relatively flat ridges at  $q^3 = \pm\pi/2$ , there is a large range of postures with good values of this measure as can be seen in figs. 17-18. There is a critical distance  $d_c \approx 1.5$  where there is a discontinuity in posture as a function of the reach.

In fig. 19 we show the inertial anisotropy, as defined in eq. (6) for the two link arm. Figs. 20-22 show this measure for the three link arm. It also has two valleys at  $q^3 = \pm\pi/2$ , indicating the existence of a range of “good” postures for inertial isotropy. As the shape of the valleys is different than that of the the ridges for  $Y$ , the optimal postures are quite different though. Note that there is no isotropic point in  $\mathcal{W}$ . The critical distance is at  $d_c \approx 1.7$ .

The curvature scalar  $R$  as given in eq.(7) is now a measure of the geodesic deviation of orbits in  $\mathcal{C}$  space and is a measure of the non-linearity of the motion of the arm. For the two link arm it is given by

$$R = \frac{2m_1 \cos(q^2)}{l_1 l_2 (m_1 + m_2 \sin^2(q^2))^2}.$$

The curvature scalar is plotted for the three link arm in fig. 23. In fig. 24 we show  $R^2$  which is the actual measure to minimize for maximal linearity of motion. As  $R$  is positive near the origin of the  $(q^2, q^3)$  plane and negative at the edges,  $R^2$  is zero on a two-dimensional surface in  $\mathcal{C}$ , where  $R$  changes sign. So there is a two-dimensional region in  $\mathcal{W}$  that can be reached with “linear” postures.  $R$  has local extrema at the singular points  $(0,0)$  ( $R = 38.1$ ),  $(0,\pi)$  ( $R = -21.5$ ) and  $(\pi,\pi)$  ( $R = -16.3$ ), but not at  $(\pi,0)$ . It is interesting to note that this gives a classification of the singular points according to their “non-linearity”, as the values of the extrema are distinct. The optimal postures and values of  $R$  are shown in figs. 25-26. The critical distance is  $d_c \approx 1.5$ .

### 4.3 Redundancy Measures

Let us choose the inertial metric as the metric  $\mathbf{h}$  on  $\mathcal{C}$ , and the kinematic metric as the “secondary” task metric  $\hat{\mathbf{h}}$ . The redundant generalized Yoshikawa measure measures the Euclidean distance per

mass moved in  $\mathcal{C}$  for self-motions. That is, if  $Y$  is small, a “large amount of matter” has to be moved to change the posture of the arm.  $Y$  should thus be maximized for optimal mobility in this sense. As the inverse  $D = 1/Y$  behaves better than  $Y$  itself we have shown data for  $D$  instead of  $Y$  itself.

The data for this measure for the three link arm is given in figs. 27-29. The optimal values of  $D$  as a function of  $d$  (i.e. the distance of the end-effector from the base of the arm) is relatively constant for  $d > 1.0$  but increases fast below that value. In fig. 30 we have plotted  $d$  as a function of  $(q^2, q^3)$ , from which we see that at small  $d$  the arm will come near the large peaks in  $D$ , which explains this behavior. The critical distance is at  $d_c \approx 1.1$ .

We would like to interpret this measure as a measure of how much the arm “swings” when reconfiguring. A better measure for this would be the area that the arm sweeps out per Euclidean distance in  $\mathcal{C}$ . Unfortunately, the “swept-area” (or volume for three-dimensional manipulators) metric cannot be described by a quadratic form, so this type of measure falls outside our framework. For comparison we have computed this “swept-area” version of the redundancy measure  $D$ . It is given in figs. 31-33. We observe that this measure behaves qualitatively similar, which supports the interpretation of  $D$  given above.

A different class of redundant generalized Yoshikawa measures can be obtained by choosing the “secondary” task metric on  $\mathcal{C}$  as

$$\hat{h}_{ij}^{(e)} = \begin{cases} 1, & \text{if } i = j = e, \\ 0, & \text{otherwise.} \end{cases}$$

This metric just measures the movement of one particular joint,  $e$  and the corresponding redundancy measure  $D = 1/Y$  will measure the amount of mass moved per movement of joint  $e$ . This could be useful if we want to find an arm position that allows reconfiguration with minimal movement of a particular joint. We call these measures (there are  $n$  of them for an  $n$  link arm) the joint redundancy measures

$$E^{(e)} = Y^{(e)}$$

$E^{(e)}$  measures how much joint  $e$  has to move to move a specified amount of mass. It should be minimized if we want to reconfigure the arm with as little movement as possible of this particular joint.

Plots, optimal postures and optimal values of the joint redundancy measures are given in figs. 34-42. The critical distances are  $d_c \approx 1.1$ ,  $d_c \approx 0.9$  and  $d_c \approx 1.0$  for  $E^{(1)}$ ,  $E^{(2)}$  and  $E^{(3)}$  respectively.

The last example of a secondary generalized Yoshikawa measures can be obtained by choosing the “secondary” metric on  $\mathcal{C}$  according to

$$ds^2 = \text{Euclidean displacement of a test point on a link.}$$

For concreteness, we take the test point to be the mid-point of link two. This metric measures the movement of the center of the middle link and the corresponding redundancy measure  $D = 1/Y$  will measure the amount of mass moved per movement of this link. This could be useful if we want to find an arm position that allows reconfiguration with minimal movement of this link. We call this measure the link redundancy measure

$$L = Y$$

$L$  measures how much the middle link has to move in order to move a specified amount of mass. It should be minimized if we want to reconfigure the arm with as little movement as possible of this particular link.

Plots, optimal postures and optimal values of the link redundancy measure are given in figs. 43-45. The critical distance is  $d_c \approx 1.5$ .

## 5 Conclusions

We have presented a unified formalism for the construction of performance measures for robot manipulators. The formalism is useful since it enables the systematic construction of invariant measures appropriate for a given application. Well known measures such as the Yoshikawa measure arise naturally in our framework. In addition, we have constructed new performance measures using this formalism, to measure non-linearity and redundancy (or self-motion). Finally, we have computed these measures for two example manipulators, one redundant and one non-redundant, and we have interpreted the results.

An interesting property of all measures except the Yoshikawa measure is that there is no smooth motion possible from a stretched position (with the end-effector far from the base) to a position with the end-effector close to the base, that optimizes the measure at each point on the trajectory. And for redundancy measures that break the symmetry relating the connected components of the self-motion manifold, it is possible to have end-effector trajectories for which *no* such motion is possible at all.

The curvature scalar measure  $R$  that we introduced seems particularly interesting for redundant manipulators, as it identifies an  $m$ -dimensional region in work space where the measure vanishes and the arm behaves locally “linear”. This measure also identifies singular points and classifies them according to their curvature. We note that a singular configuration with a large positive curvature is stable in the sense that geodesics in  $\mathcal{C}$  attract each other (as the manifold has positive curvature). Unstable configurations arise for negative  $R$ , where nearby orbits repel each other and diverge.

The measures that we have constructed are by no means the only ones. For example, one could also choose the joint-compliance matrix [PL91] as the metric on  $\mathcal{C}$ , with a set of associated measures. And the secondary metric on  $\mathcal{C}$  for the redundancy measure could be any task specific function. We believe this framework provides a good means for the analysis, design and control of robot manipulators.

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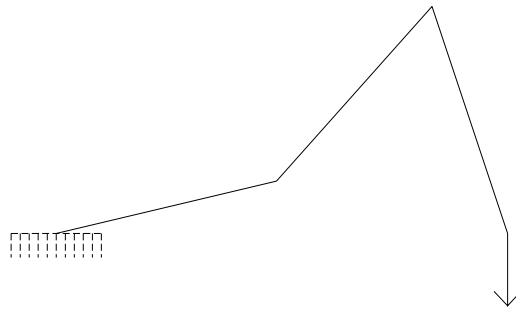


Figure 1: Good posture for applying vertical force



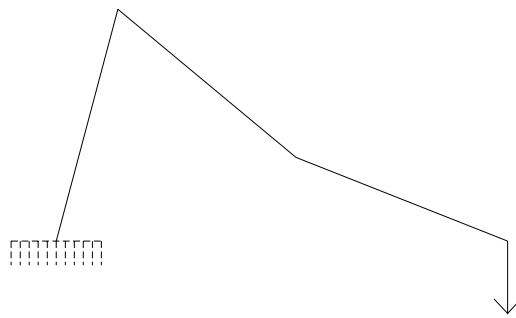


Figure 2: Bad posture for applying vertical force

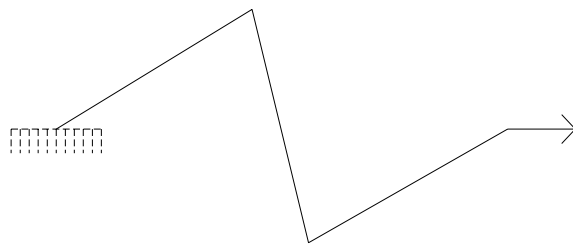


Figure 3: Good posture for applying horizontal force

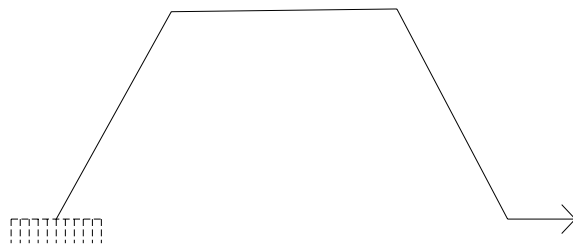


Figure 4: Bad posture for applying horizontal force

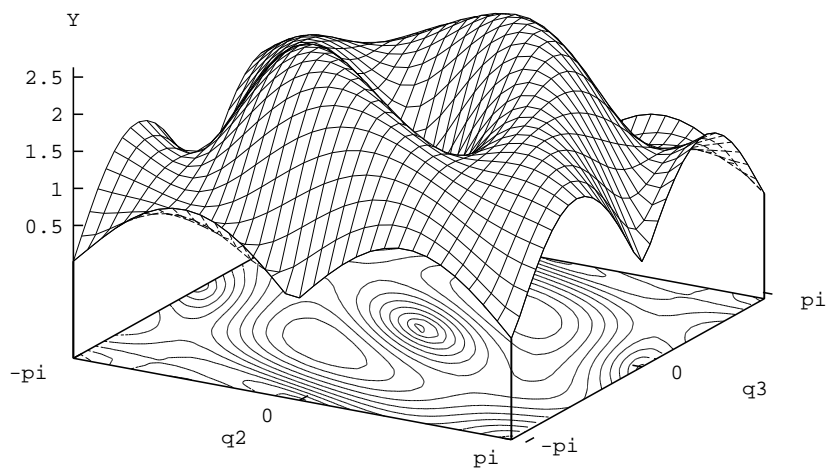


Figure 5: Yoshikawa measure for 3-joint redundant arm

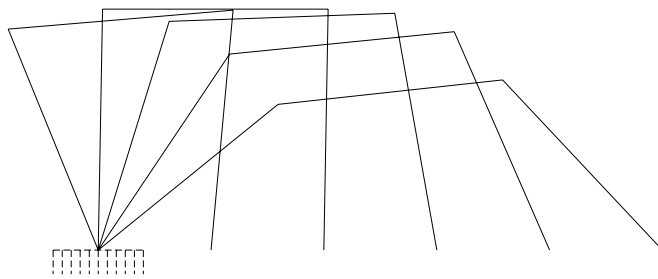


Figure 6: Optimal postures for Yoshikawa measure for given end-effector position

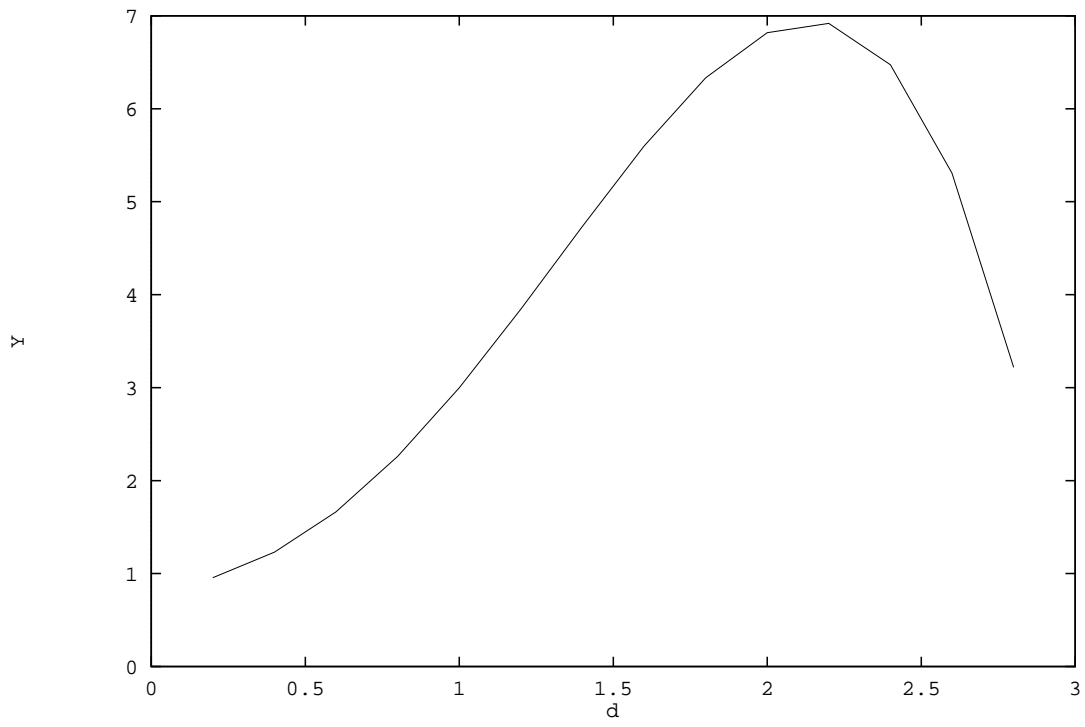


Figure 7: Optimal values for Yoshikawa measure

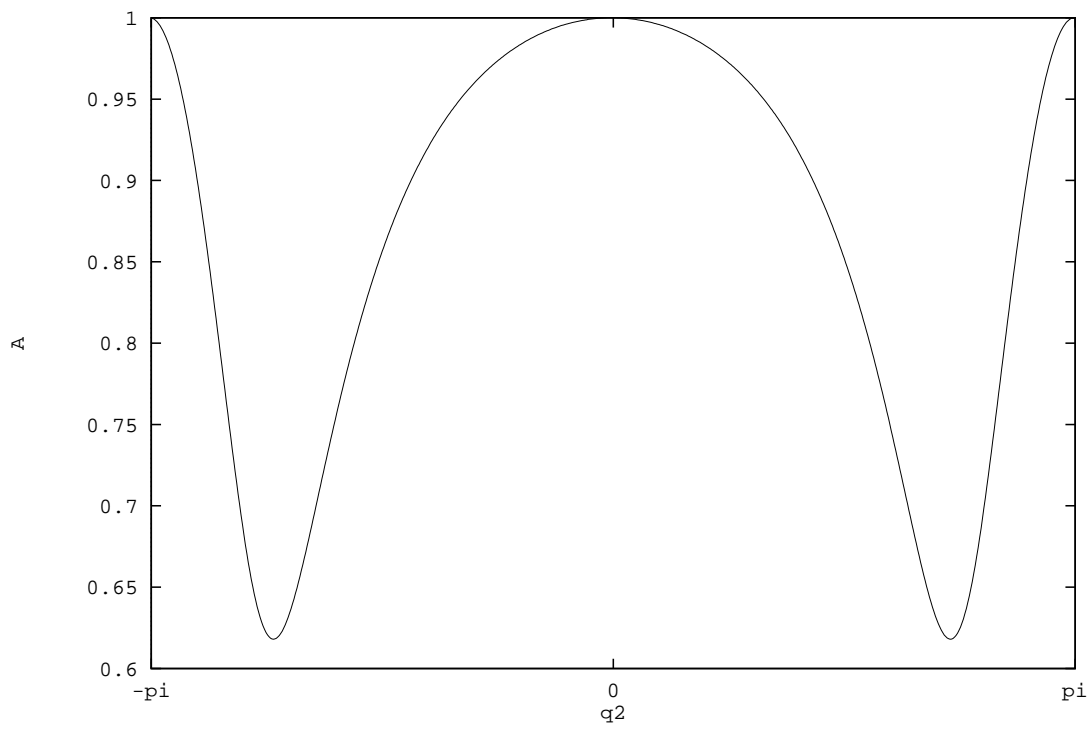


Figure 8: Kinematic anisotropy measure for two link arm

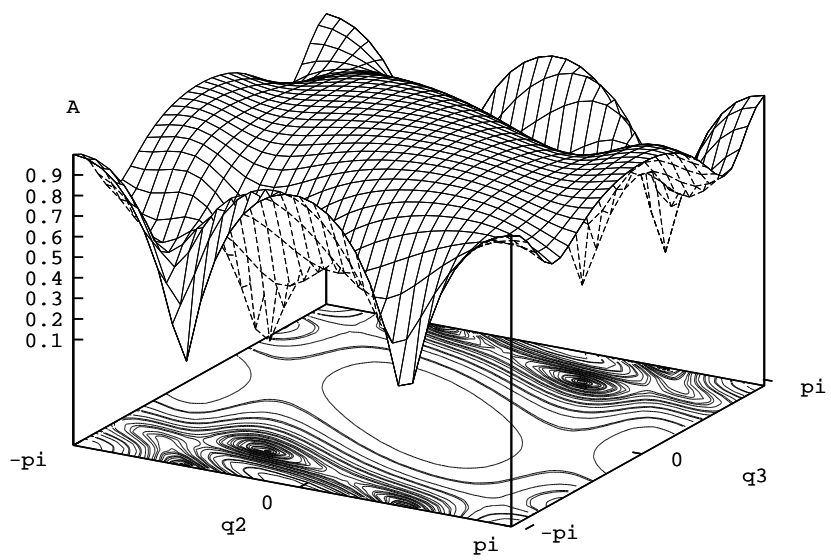


Figure 9: Kinematic anisotropy measure  $A$  for 3-joint redundant arm



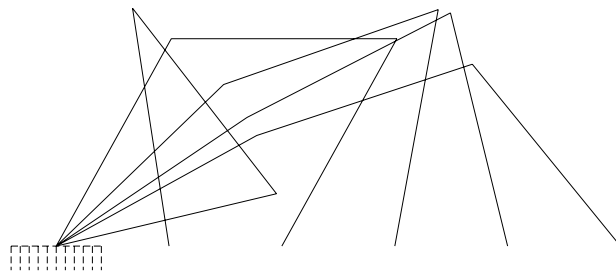


Figure 10: Optimal postures for  $A$

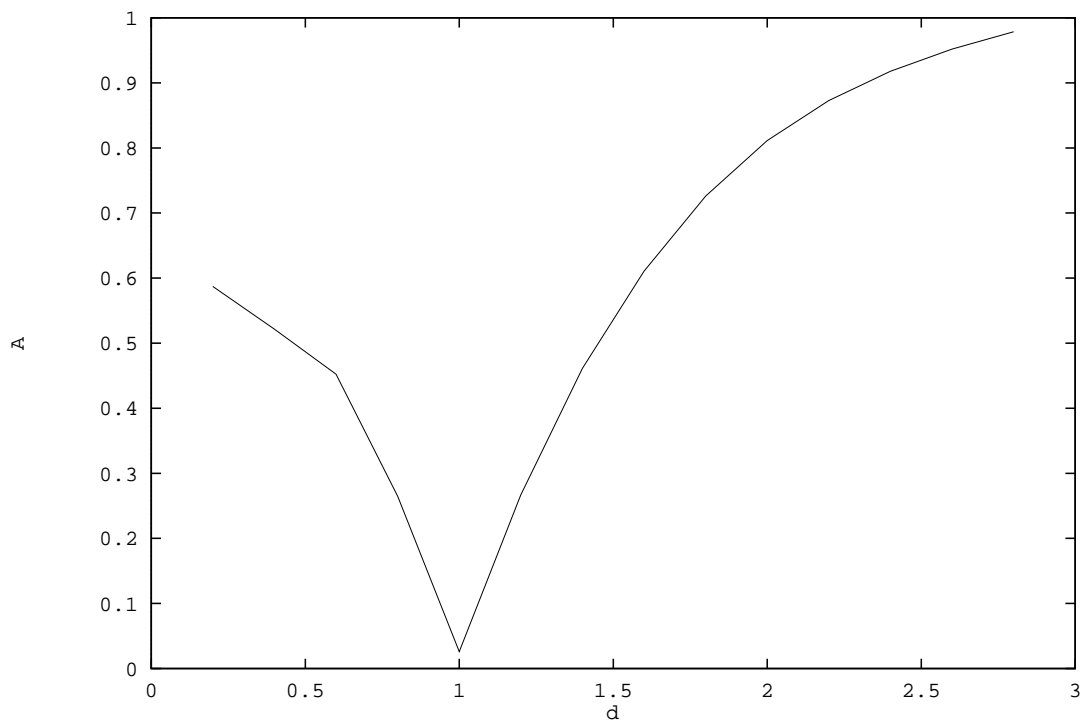


Figure 11: Optimal values for  $A$

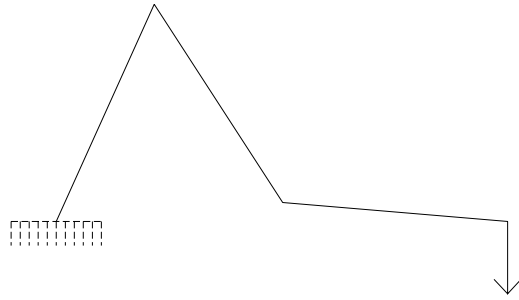


Figure 12: Good posture for vertical swatting

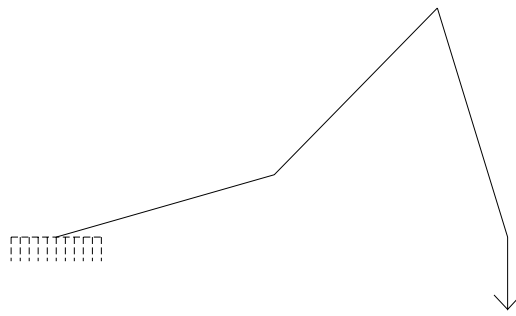


Figure 13: Bad posture for vertical swatting

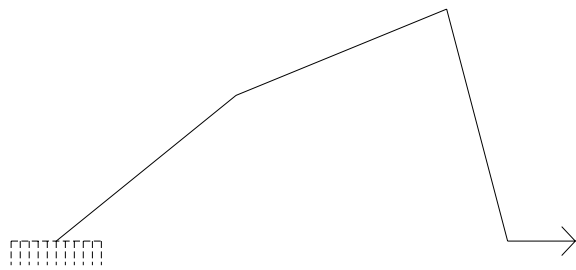


Figure 14: Good posture for horizontal swatting

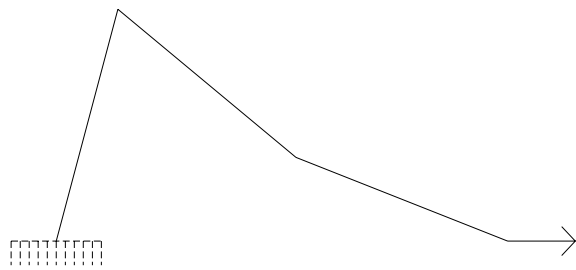


Figure 15: Bad posture for horizontal swatting

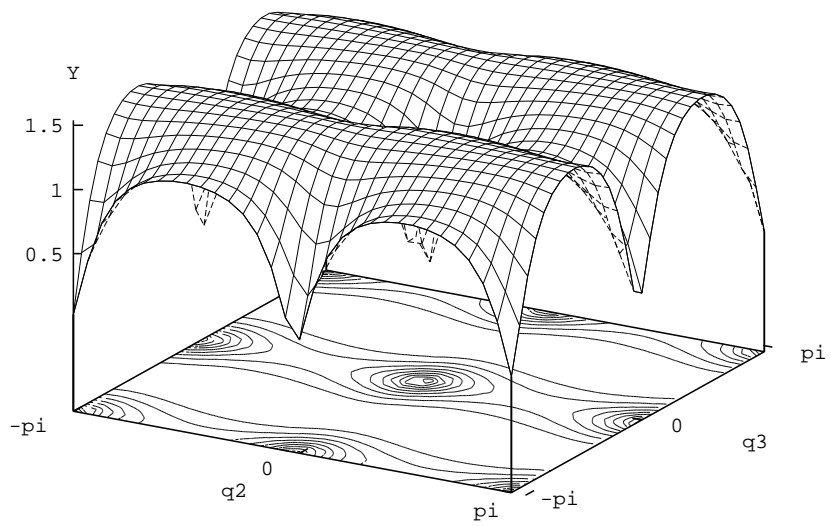


Figure 16: Inertial Yoshikawa measure  $Y$  for 3-joint redundant arm

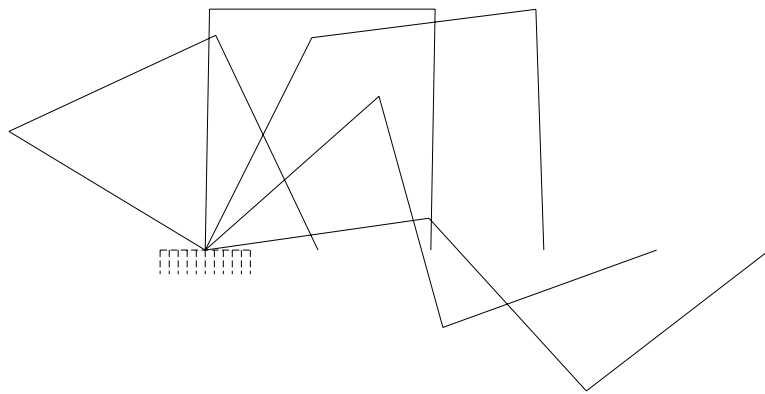


Figure 17: Optimal postures for  $Y$



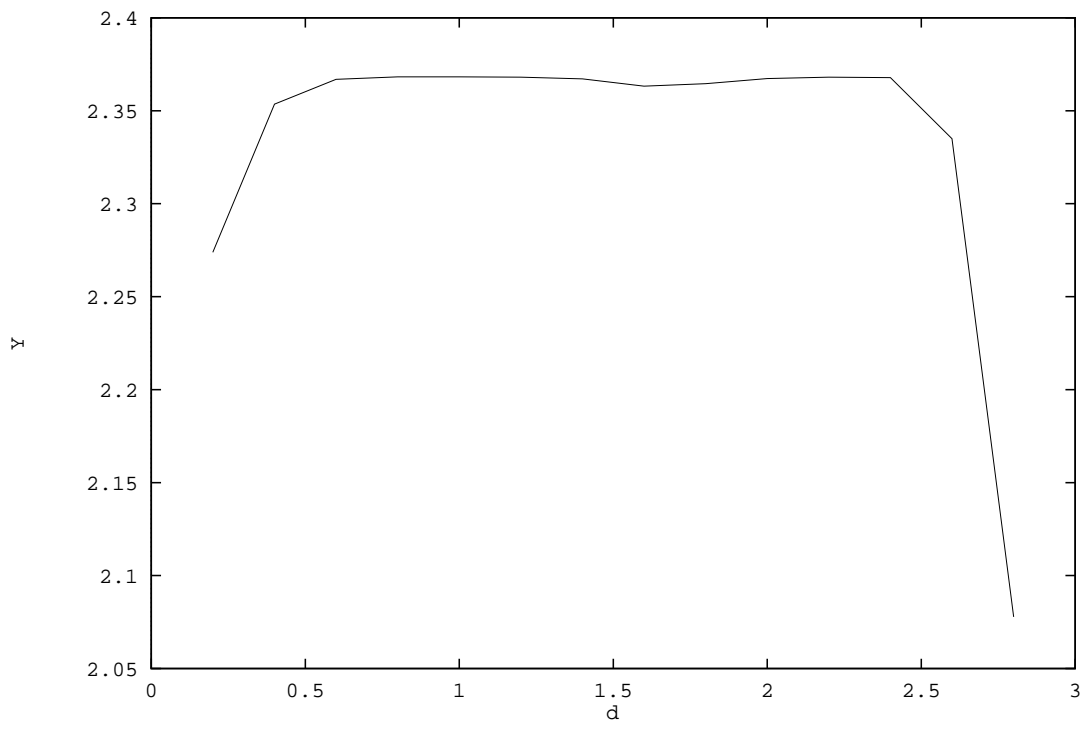


Figure 18: Optimal values for  $Y$

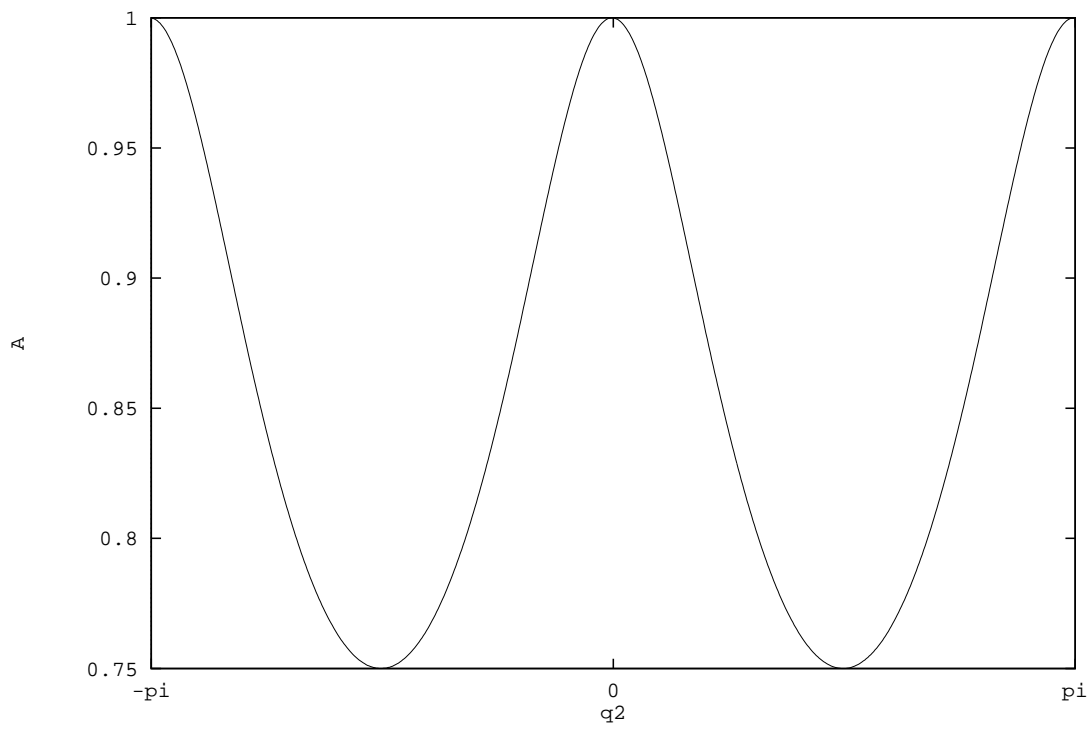


Figure 19: Inertial anisotropy measure for 2-joint arm

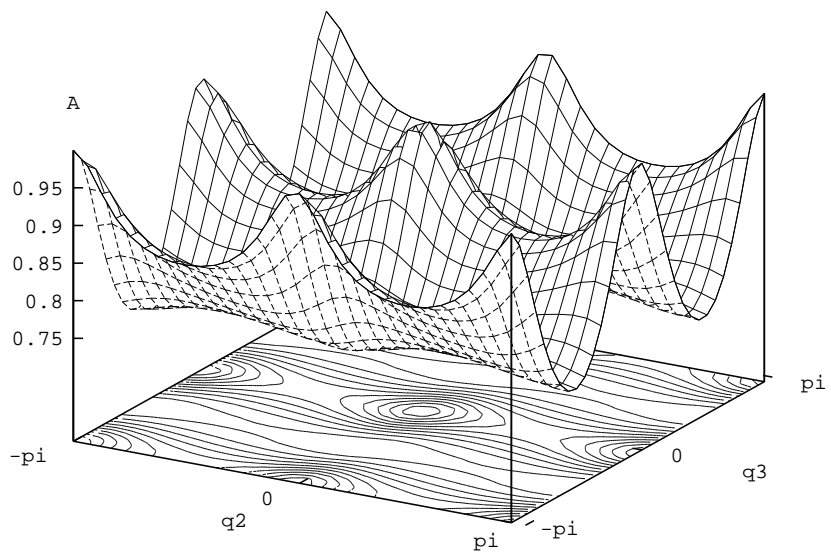


Figure 20: Inertial anisotropy measure for 3-joint redundant arm

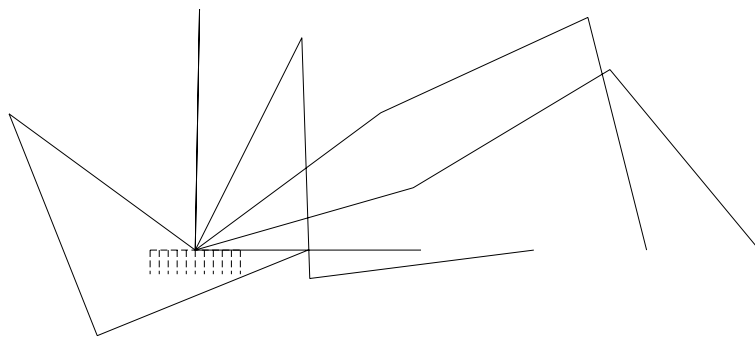


Figure 21: Optimal postures for  $A$

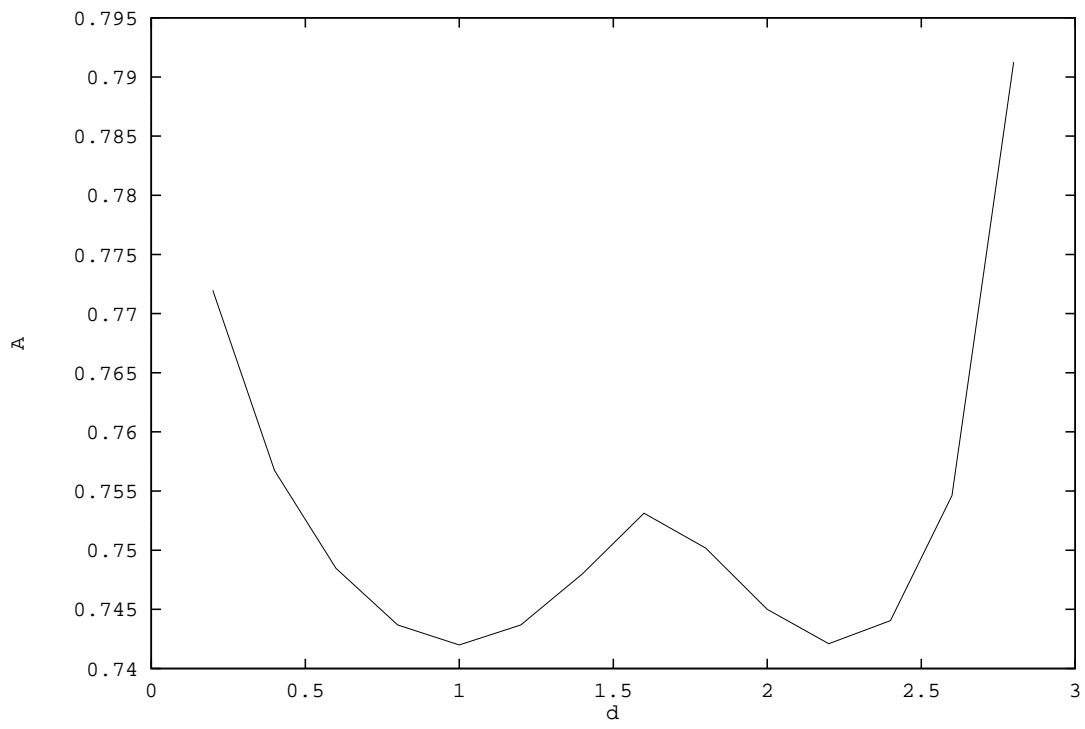


Figure 22: Optimal values for  $A$

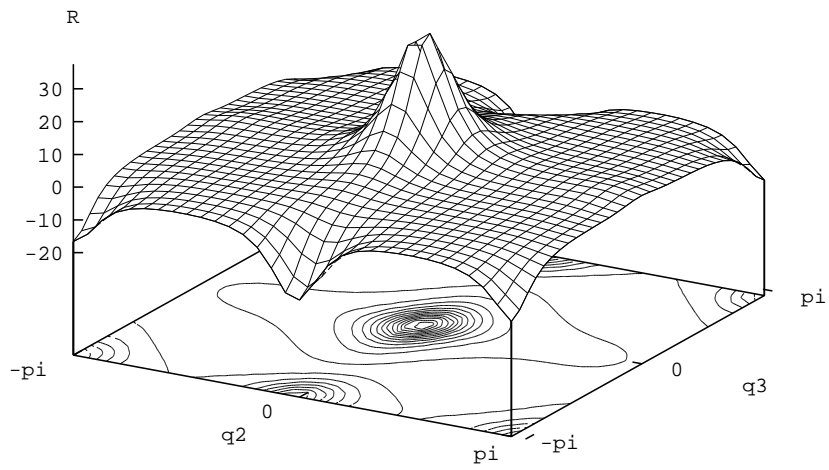


Figure 23: Curvature scalar  $R$  for 3-joint redundant arm

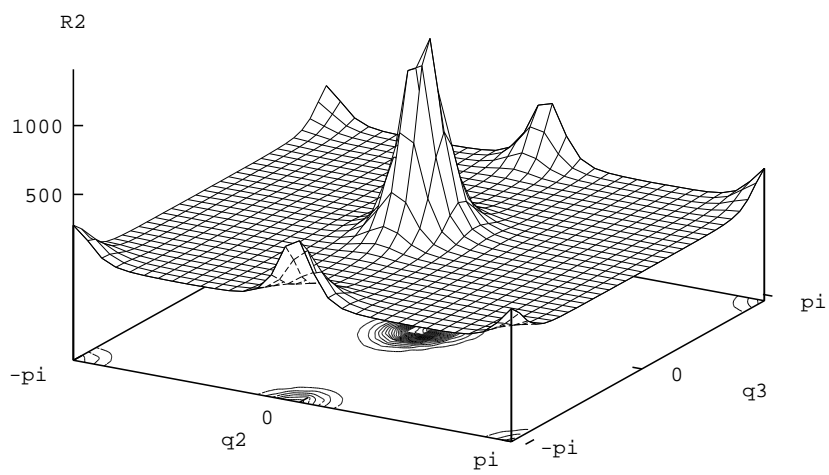


Figure 24:  $R^2$  for 3-joint redundant arm

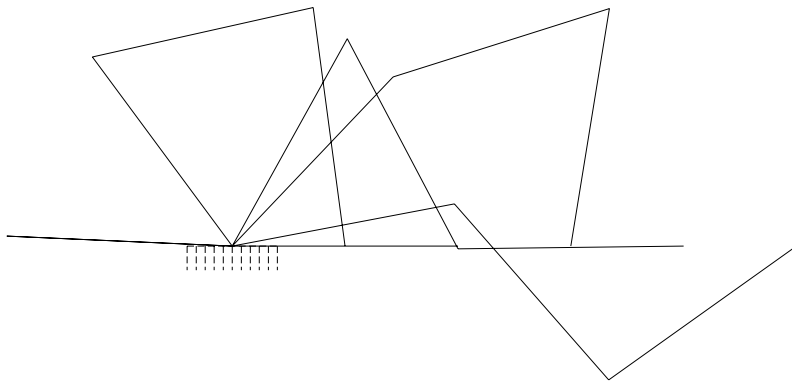


Figure 25: Optimal postures for  $R$



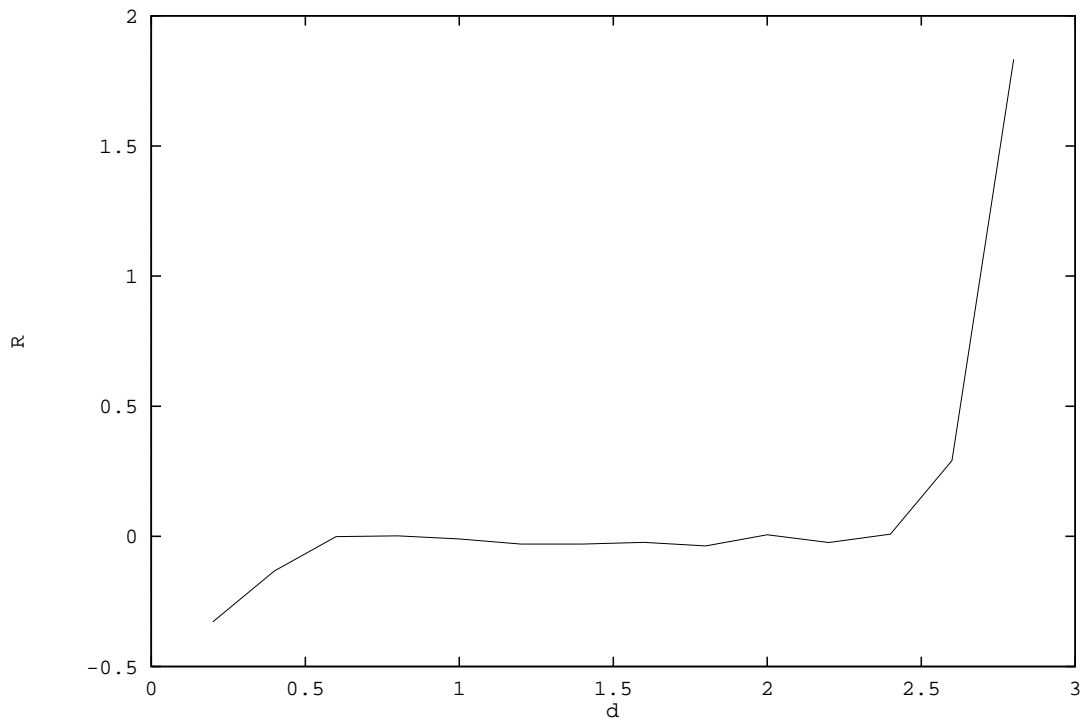


Figure 26: Optimal values for  $R$

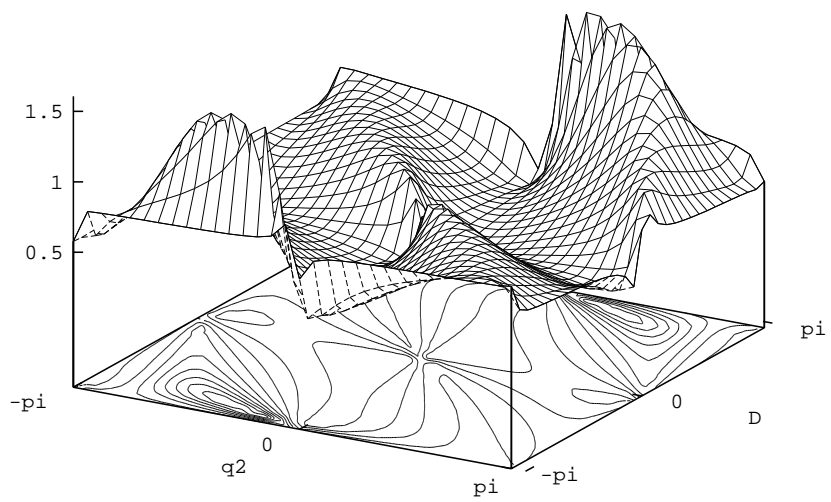


Figure 27: Redundancy measure  $D$  for 3-joint redundant arm

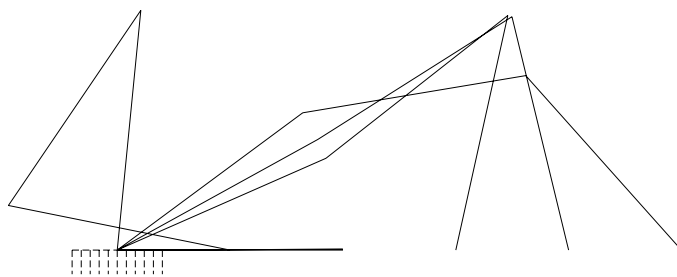


Figure 28: Optimal postures for  $D$

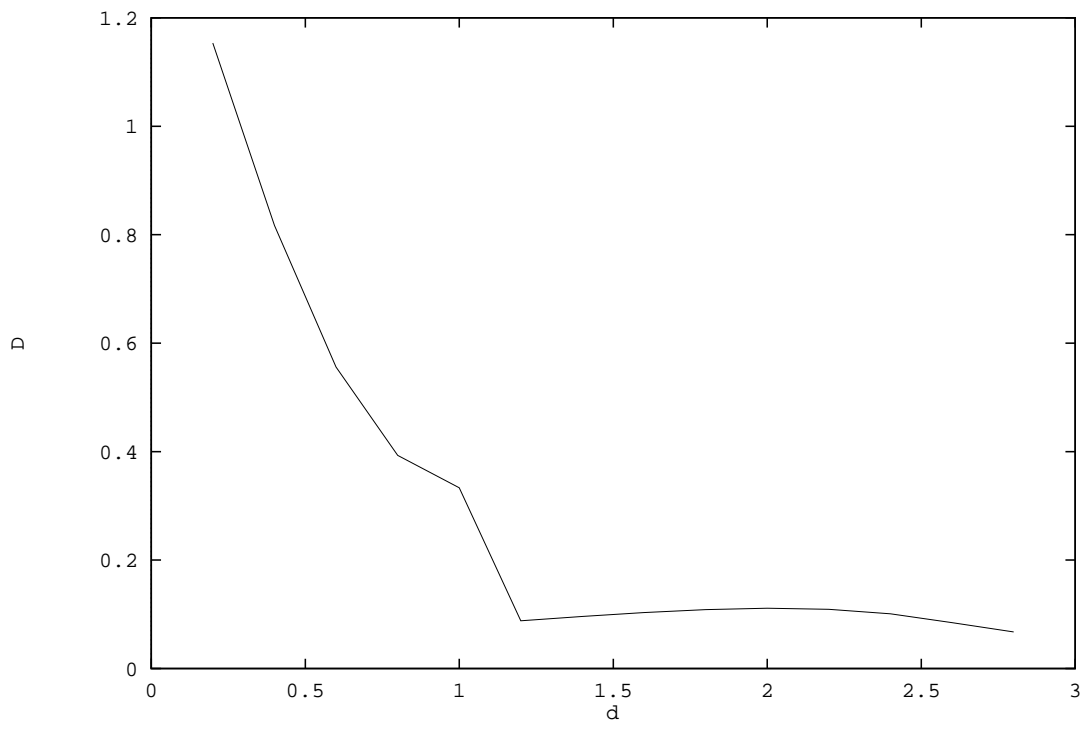


Figure 29: Optimal values for  $D$

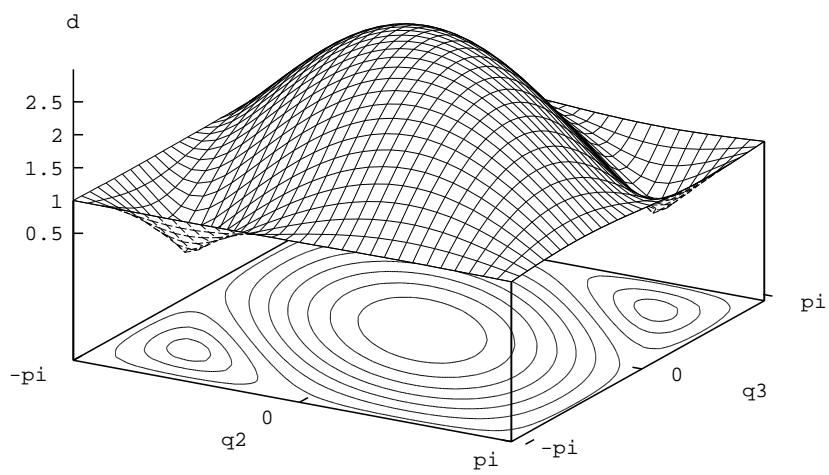


Figure 30: Distance of end-effector from base  $d$

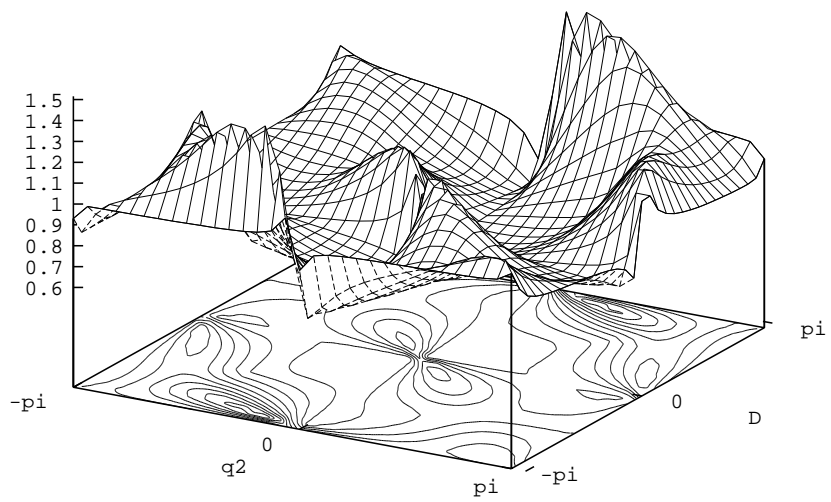


Figure 31: Area redundancy measure  $D$  for 3-joint redundant arm

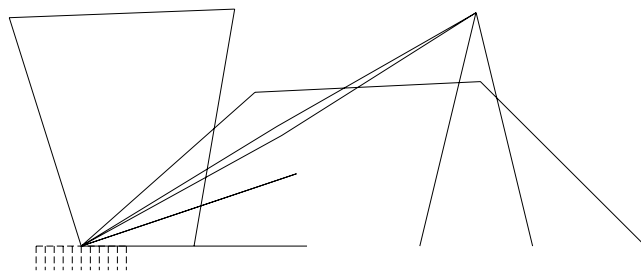


Figure 32: Optimal postures for  $D$

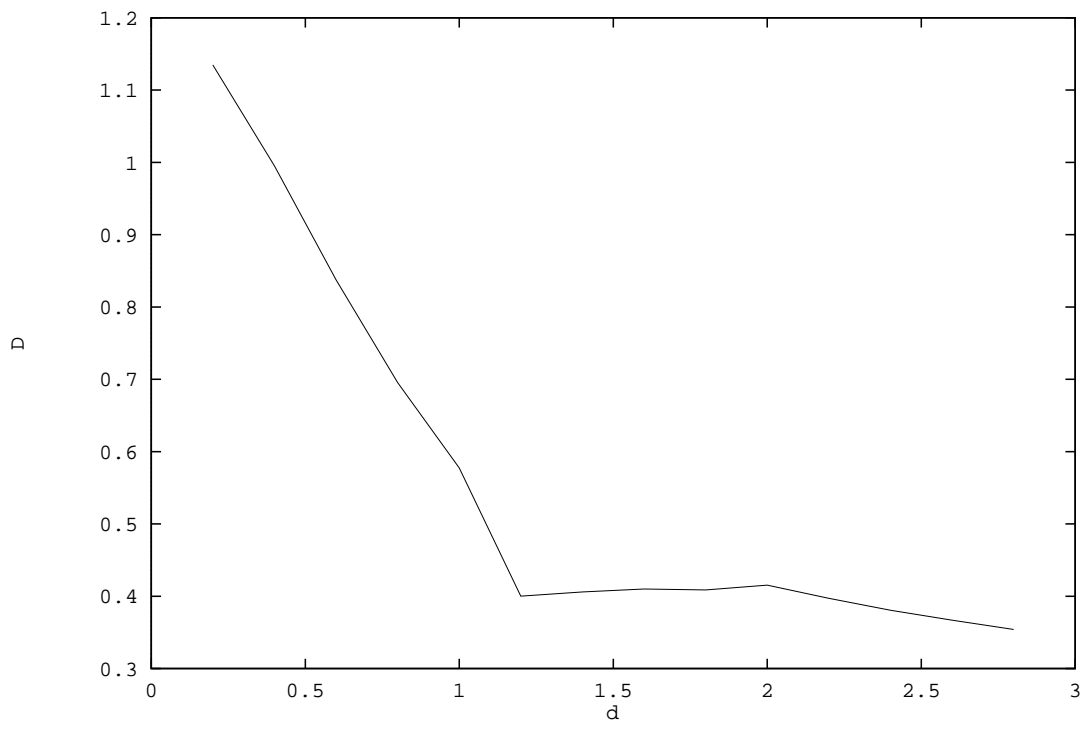


Figure 33: Optimal values for  $D$



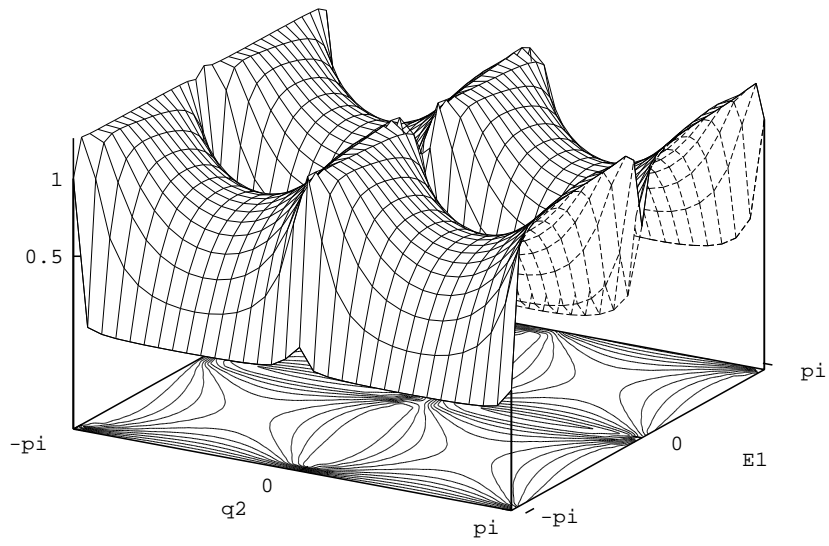


Figure 34: Joint redundancy measure  $E^{(1)}$  for 3-joint redundant arm

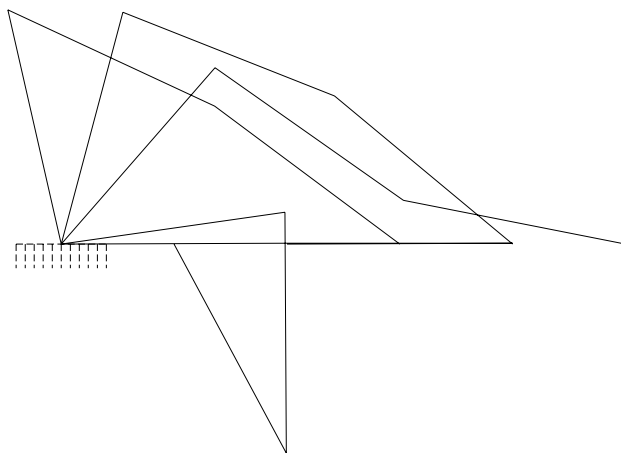


Figure 35: Optimal postures for  $E^{(1)}$

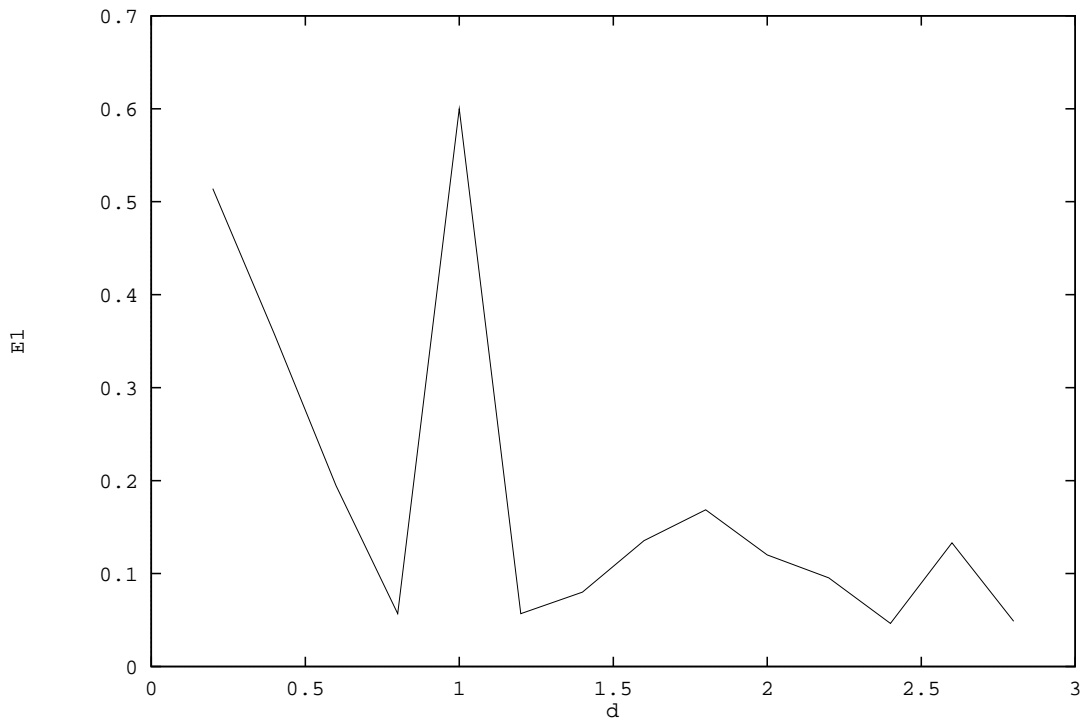


Figure 36: Optimal values for  $E^{(1)}$

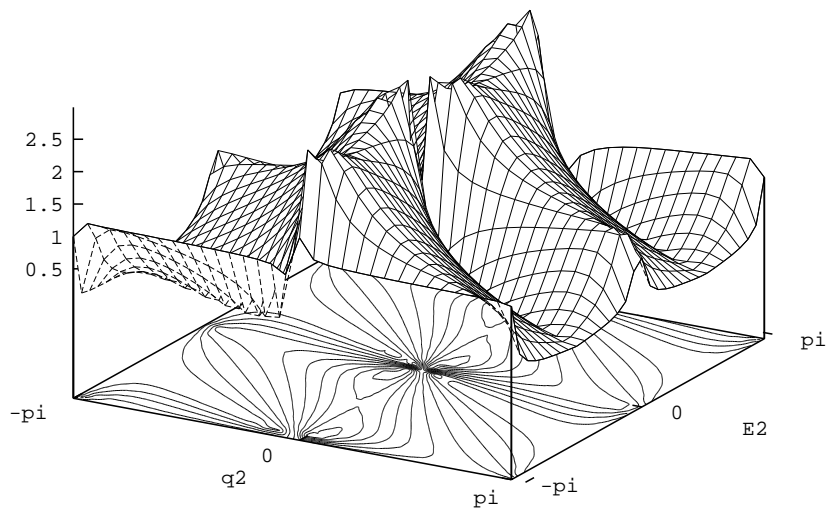


Figure 37: Joint redundancy measure  $E^{(2)}$  for 3-joint redundant arm

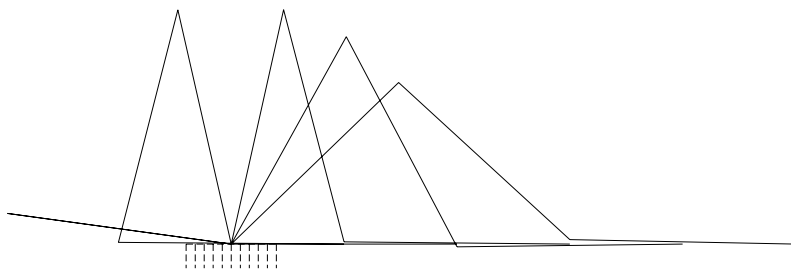


Figure 38: Optimal postures for  $E^{(2)}$

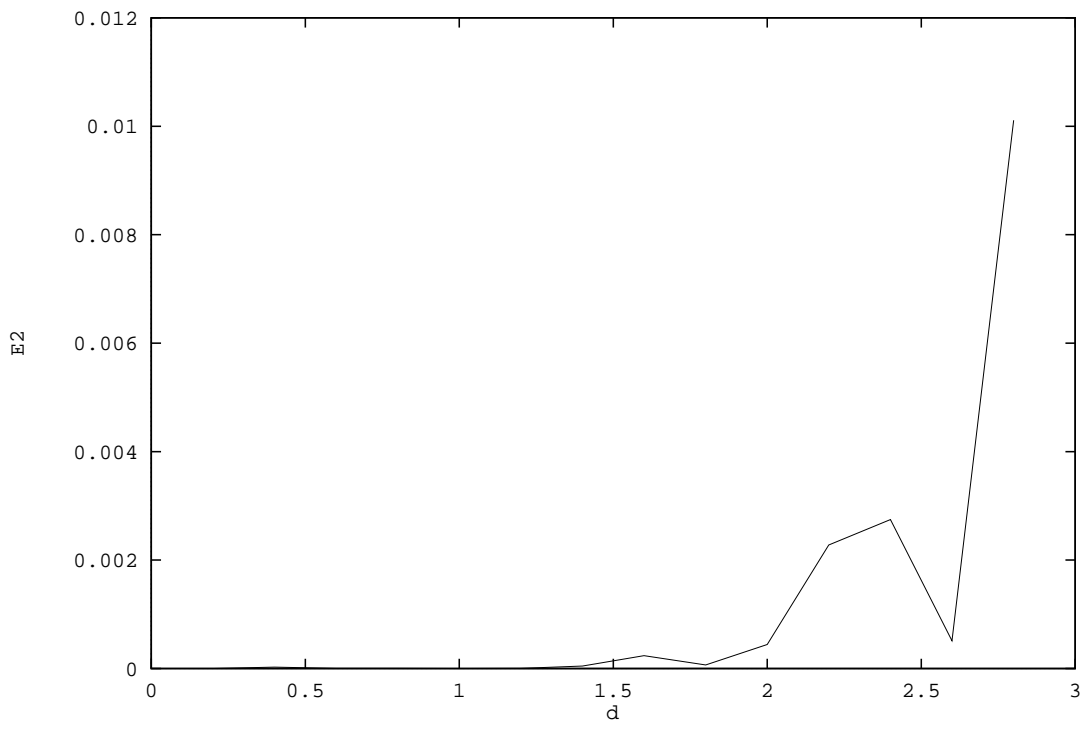


Figure 39: Optimal values for  $E^{(2)}$

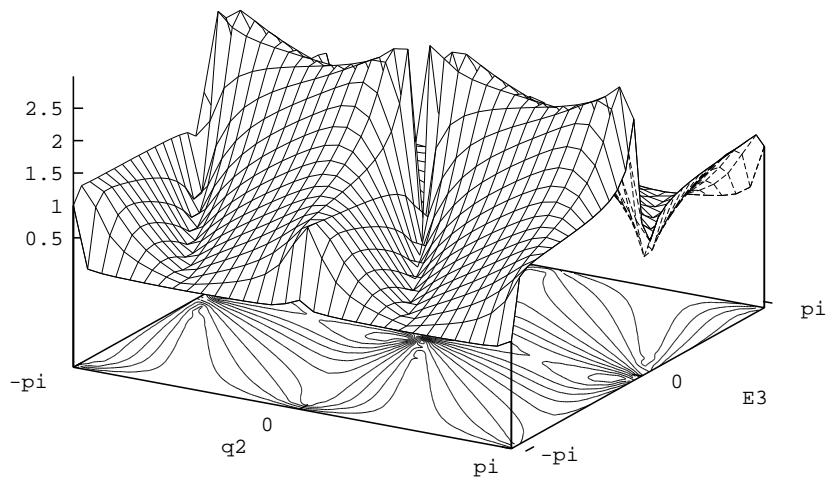


Figure 40: Joint redundancy measure  $E^{(3)}$  for 3-joint redundant arm

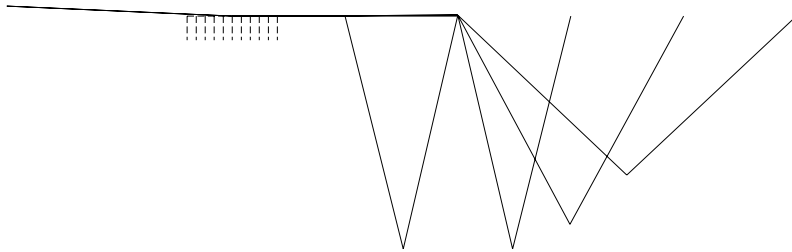


Figure 41: Optimal postures for  $E^{(3)}$



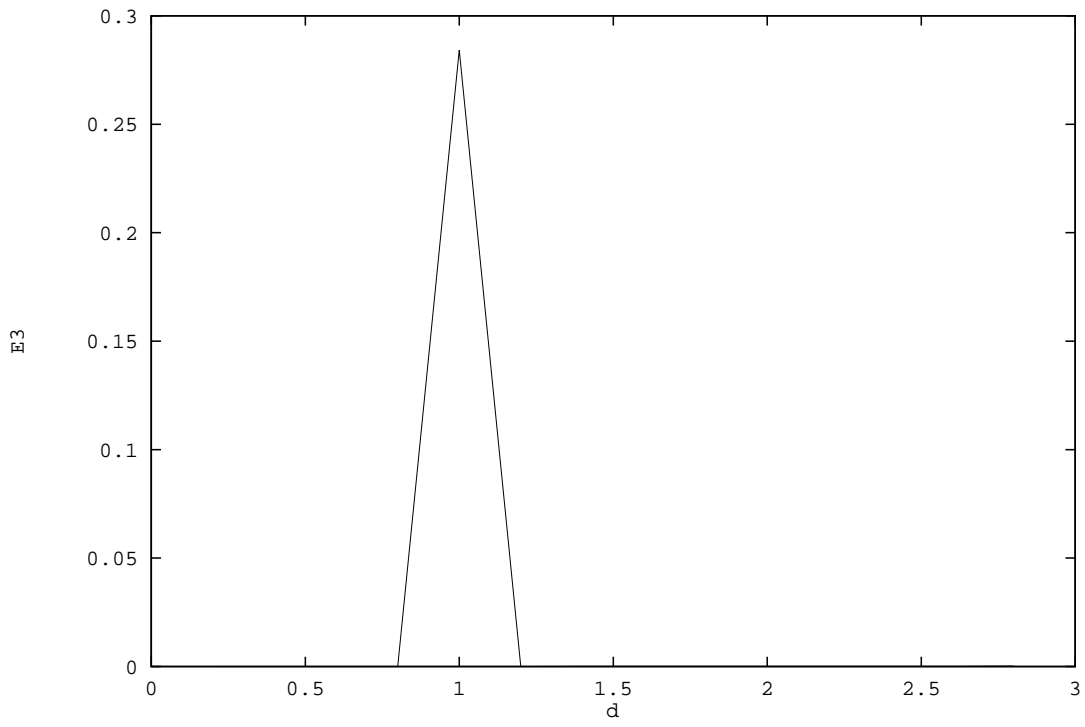


Figure 42: Optimal values for  $E^{(3)}$

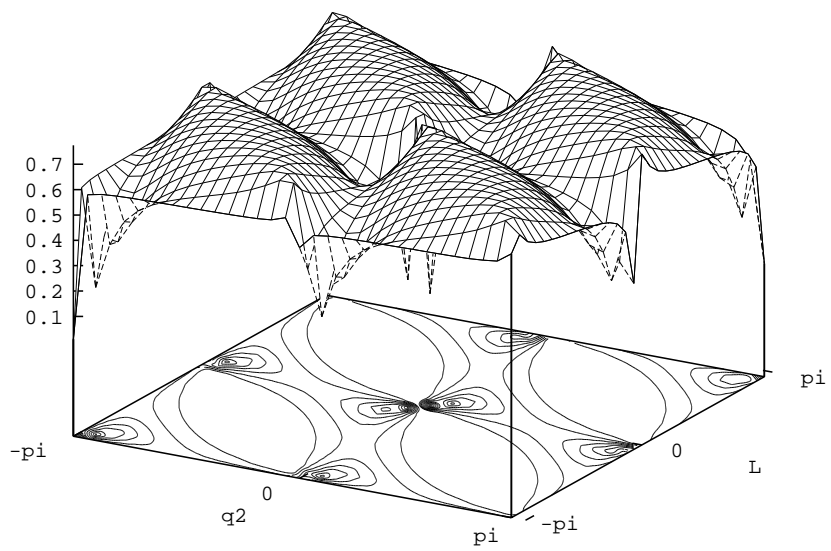


Figure 43: Link redundancy measure  $L$  for 3-joint redundant arm

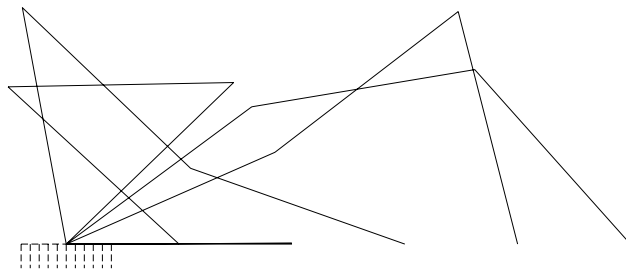


Figure 44: Optimal postures for  $L$

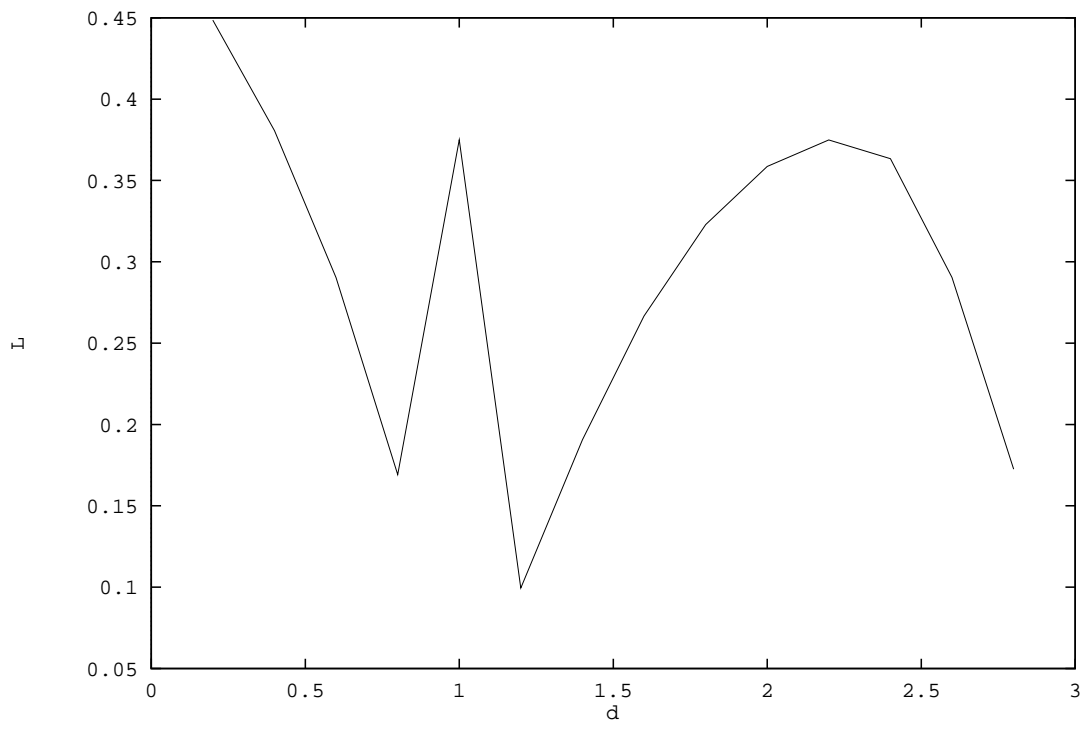


Figure 45: Optimal values for  $L$