Supplemental Material
Displacement Interpolation Using Lagrangian Mass Transport

This document provides an overview of several notions useful in the understanding of optimal mass transport and displacement interpolation. To improve clarity, it exercises some liberty with definitions and makes some minor simplifying assumptions. For a deeper, more rigorous introduction, we refer the reader to the following books [Bourbaki 2003; Villani 2008; Villani 2003; Gallier 2000]. We will first describe the notion of measure, then recall the Monge-Kantorovich mass transportation problem and finally describe the displacement interpolation for continuous functions.

1 Elements of measure theory

The general formulation of the mass transportation domain involves the use of measures, and some familiarity with this theory can be useful to understand the optimal transportation problem.

A positive measure μ can be seen as a function which assigns a volume $\mu[B]$ (a real positive number) to a set B. It thus has to satisfy a number of properties:

- the measure should give a positive or null value,
- the measure of the union of two disjoint sets is the sum of the measures of each set, and
- the measure of an empty set is 0

The concept of measure is useful in integration theory. Indeed, one can define the integral of a function $s(x)$, $\int_\Omega s(x)d\mu(x)$, as a sum over disjoint subsets, of the measure of each subset times the value of the function s to be integrated over each subset.

Often, one will encounter the Lebesgue measure, noted "dx", assigning the usual length, area or volume to an interval or product of intervals. Another commonly used measure is the Dirac measure, noted "δ_x" assigning a volume of 1 to a set, if and only if the set contains the point x.

One can further build measures by multiplying the Lebesgue measure by a continuous function f such that $d\mu(x) = f(x)dx$. This new measure is said to be *absolutely continuous*.

A useful class of measures is the class of *probability measures*. A measure is a probability measure if and only if the measure of the whole space is 1. One can see that the Dirac measure is a probability measure. An absolutely continuous measure is a probability measure if and only if the integral of f with respect to the Lebesgue measure is 1. In that case, $f(x)$ is called the density, or probability density function (PDF).

Finally, a measure ν is said to be the push-forward measure of another measure μ by a function $T(x)$ if and only if for any measurable set B, $\nu[B] = \mu[T^{-1}(B)]$. If the measures are absolutely continuous, this can be seen as a warping of the first function toward the second. Equivalently, this can be written as: for all non-negative functions ψ, the following relationship holds:

$$\int_X (\psi \circ T) d\mu = \int_Y \psi d\nu$$

This property is denoted as $\nu = T\sharp\mu$.

2 The Monge-Kantorovich problem

The initial goal of Monge in the 18th century was to find a function $T(x)$ - a warp - which reshapes a pile of sand so that it fills several holes in a way which minimizes the cost $c(x, y)$ of moving each particle of sand from the location x to y. This is equivalent to minimizing:

$$\min \int_X c(x, T(x)) d\mu(x) \quad (1)$$

such that:

$$\nu = T\sharp\mu, \quad (2)$$

where the minimum is taken over all possible warps $T(x)$ moving each particle from location x to $T(x)$. This considers that the measures are probability measures. This is known as the Monge problem. When the measures μ and ν are absolutely continuous: $d\mu(x) = f(x)dx$ and $d\nu(y) = g(y)dy$, the constraint in Eq.2 is equivalent to the *change of variable formula*: $f(x) = g(T(x)) \cdot |\det \nabla T(x)|$.

The Monge problem is non-linear and difficult to solve. In addition, it may not have a solution at all: consider for example $\mu = \delta_0$ and $\nu = 0.5\delta_{-1} + 0.5\delta_1$. In this example, it is not possible to "warp" a single Dirac into two half-Diracs: the mass cannot be split into two. However, when the measures are absolutely continuous, and the cost a strictly convex continuous function of the difference $x-y$, it can be shown that the solution exists and is unique.

Later, in the 20th century, Kantorovich proposed a relaxed version of the above problem. The new formulation is linear, and coincides with Monge's problem wherever Monge's problem has a solution. The goal is instead to minimize:

$$\min \int_{X \times Y} c(x, y) d\pi(x, y)$$

over all measures π defined on the product space of the source and target spaces, such that:

$$\pi[A \times Y] = \mu[A]$$

and

$$\pi[X \times B] = \nu[B]$$

for all subsets A of X and B of Y. This means that π has *marginals* μ and ν. If the measures are absolutely continuous, with densities $f(x)$ and $g(y)$, and if π also had a density $P(x, y)$, the constraints would be equivalent to $\int_Y P(x, y)dy = f(x)$ and $\int_X P(x, y)dx = g(y)$. In contrast, under a weak continuity assumption on the cost, this problem always admits at least one solution.

The topology of the spaces in which the subsets are defined (X and Y) does not need much regularity. Polish spaces are often used: a complete separable metric space, without further regularity (not necessarily a differentiable manifold).

When μ and ν are described as a sum of weighted Dirac masses placed at several locations, then this continuous problem collapses to the Hitchcock-Koopmans formulation described in the paper (Equations (1a-c)).

3 Displacement interpolation

In this section, we describe the motivation behind the notion of displacement interpolation. Here, we have more strict requirements on the regularity of the spaces. We will use smooth Riemmanian manifolds, although the curvature of the manifold can be arbitrary. For more information on these notions, we refer the reader to the book by Gallier [2000].

We now consider the particles of sand to have some dynamics. In particular, they follow the least action principle: the cost of moving a particle from location x to y is the minimum over all the possible paths γ starting at x (at t=0) and leading to y (at t=1) of a function $\mathcal{A}(\gamma)$ which depends on the path taken: $c(x, y) = \inf_\gamma \mathcal{A}(\gamma)$ for all curves γ such that $\gamma_0 = x$ and $\gamma_1 = y$. $\mathcal{A}$ is the action functional, and each particle follows the path which minimizes this cost.

This makes sense if one chooses an action $\mathcal{A}(\gamma)$ defined as the integral of a Lagrangian over the path γ: $\mathcal{A}(\gamma) = \int_\gamma L(x, \dot{x}, t)dt$. In addition, from differential geometry, we know that if $L(x, \dot{x}, t) = |\dot{x}|^p$ for any $p \geq 1$ then the cost of moving a particle from x to y – the infimum of the action functional $\mathcal{A}(\gamma)$ over all paths – is equal to the geodesic distance raised to the power p, and the path followed by the particle will be a geodesic path on the manifold. In the Euclidean space $\mathbb{R}^n$, theses paths are just straight lines and the geodesic distance is the standard L_2 distance. For $p = 1$ the parameterization of the curve is arbitrary, but for $p > 1$, the particles have constant speed along the path. In our paper, we restricted ourselves to these costs and these particular Lagrangians.

Equipped with these properties, one can define the displacement interpolation as the interpolation obtained by moving each particle along their curve, with constant speed if $p > 1$, after solving the Monge-Kantorovich mass transportation problem.

References

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