Deep Learning Local Optima

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Saddle Points

A critical point \( \| \nabla f(x) \| \rightarrow 0 \)
Saddle Points

A critical point

With Hessian

Local minimum:

$$\| \nabla f(x) \| \rightarrow 0$$

$$\forall_i \lambda_i > 0$$
Saddle Points

A critical point with Hessian:

Local minimum: \( \forall_i \lambda_i > 0 \)

Local maximum: \( \forall_i \lambda_i < 0 \)
Saddle Points

With Hessian

... Saddle point with min-max structure \[ \forall i \quad \lambda_i \neq 0 \]
Saddle Points

With Hessian

... Saddle point with min-max structure \( \forall i \lambda_i \neq 0 \)
Saddle Points

With Hessian

... Saddle point with Monkey structure: \( \exists_i \lambda_i = 0 \)
Saddle Points

With Hessian

... Saddle point with Monkey structure: \( \exists_i \lambda_i = 0 \)
Gradient Descent
Steps proportional to the eigenvalues of the Hessian.
Gradient Descent
Steps proportional to the eigenvalues of the Hessian. Slows down near the saddle point!
Optimization Algorithms around Saddle Points

Newton method

  Takes steps proportional to the inverse of eigenvalues.
Optimization Algorithms around Saddle Point

Newton method

Takes steps proportional to the inverse of eigenvalues. Steps are in the opposite direction if an eigenvalue is negative!

Attracted to saddle points!
Optimization Algorithms around Saddle Point

http://imgur.com/a/Hqolp
Where are we headed?

Deep networks probably have exponentially (in dimension) many saddle points with high errors!
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- Statistical Physics (spin-glass)
- Random Matrix Theory
- Previous theoretical work on neural networks.
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Deep networks probably have exponentially (in dimension) many saddle points with high errors!

Statistical Physics (spin-glass)
Random Matrix Theory
Previous theoretical work on neural networks.

We probably don’t have to worry about bad local minima in large networks.
\[ H = \sum_{i,k=1,N} J_{ik} \sigma_i \sigma_k \]
\[ \sigma \in \{+1, -1\} \]
\[ \Sigma \in \{+1, -1\}^N \]
Random Matrix Theory

In large Gaussian random matrices, the probability of an eigenvalue to be positive or negative is $\frac{1}{2}$.

$$\Pr[\forall i \lambda_i > 0] = \left(\frac{1}{2}\right)^N$$
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$$\Pr[\forall i \lambda_i > 0] = \left(\frac{1}{2}\right)^N$$

The Hessian at a random point on a spin-glass model with Gaussian edges is a Gaussian random matrix.
Existence of saddle points in deep networks is empirically shown.
Existence of saddle points in deep networks is empirically shown. (re)introduce a saddle-free Newton method.
Existence of saddle points in deep networks is empirically shown.
(re)introduce a saddle-free Newton method. Compare on optimization of MLP, RNN, Deep autoencoders.
Existence of saddle points in deep networks.

- $\alpha$: Fraction of the negative eigenvalues of the Hessian at the critical point.
- $\epsilon$: Error at the critical point.

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In Hamiltonian of a Gaussian Spin-Glass model there’s a positive correlation between these two measure.
Existence of saddle points in deep networks.

- $\alpha$: Fraction of the negative eigenvalues of the Hessian at the critical point.
- $\epsilon$: Error at the critical point.

In Hamiltonian of a Gaussian Spin-Glass model there’s a positive correlation between these two measure. Is it the same for neural networks?
Existence of saddle points in deep networks.
(re)introduce a saddle-free Newton method.

- A corrected Newton method where steps are proportional to $\frac{1}{|\lambda_i|}$
(re)introduce a saddle-free Newton method.

● A corrected Newton method where steps are proportional to $\frac{1}{|\lambda_i|}$

● First to justify this heuristic? (mathematically derived)
(re)introduce a saddle-free Newton method.

- A corrected Newton method where steps are proportional to \( \frac{1}{|\lambda_i|} \)
- First to justify this heuristic? (mathematically derived)
- In practice optimize in a lower-dimensional Krylov subspace.
Conclusion

If deep networks are similar to spin-glass models we will need a better plan to optimize.
A justifiable setting from which a saddle-free Newton method could be derived.

Make a connection between deep networks with ReLU and spherical spin-glass models.
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Assuming (i) variable independence (ii) redundancy in network parametrization (iii) uniformity.
Using absolute value loss and hinge loss.
Make a connection between deep networks with ReLU and spherical spin-glass models. Assuming (i) variable independence (ii) redundancy in network parametrization (iii) uniformity. Using absolute value loss and hinge loss. In large networks we probably don’t have to worry about bad local minima.
Make a connection between deep networks with ReLU and spherical spin-glass models.
Assuming (i) variable independence (ii) redundancy in network parametrization (iii) uniformity.
Using absolute value loss and hinge loss.
In large networks we probably don’t have to worry about bad local minima.
Finding a global minimum on the training set is not useful.

Following the same path from the last year.
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In previous work used a low-dimensional Krylov subspace.
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If limit ourselves to diagonal preconditioners can we get a similar conditioning as inverse Hessian with absolute eigenvalues?
Equilibrated Stochastic Gradient Descent!
Following the same path from the last year. In previous work used a low-dimensional Krylov subspace. If limit ourselves to diagonal preconditioners can we get a similar conditioning as inverse Hessian with absolute eigenvalues? Equilibrated Stochastic Gradient Descent! AdaDelta, AdaGrad, RMSProp.
Following the same path from the last year.
In previous work used a low-dimensional Krylov subspace.
If limit ourselves to diagonal preconditioners can we get a similar conditioning as inverse Hessian with absolute eigenvalues?
Equilibrated Stochastic Gradient Descent!
AdaDelta, AdaGrad, RMSProp.
Works as well or better than RMSProp.

Figure 3: Learning curves for deep auto-encoders on a) MNIST and b) CURVES comparing the different preconditioned SGD methods.
Conclusion

The first group argue saddle points are an important problem in deep networks.
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The first group argue saddle points are an important problem in deep networks. The second group show as the network grows you probably do not need to worry about bad local minima. If you get past the saddle points, and settle for a lower index critical point, you’re probably in a good local minima.
Conclusion

The first group introduced an impractical saddle-free Newton method with limited experiments on arguably small networks.
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Conclusion

The second group made a direct connection to spin-glass models with unrealistic assumptions. Input independent activation of ReLUs. Independent inputs for each path on the network.
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Input independent activation of ReLUs.
Independent inputs for each path on the network.

Working on ways to relax these assumptions?
Thank you!