Minimizing Finite Sums

Mohamed Osama Ahmed
13/10/2015

Big-N Problems

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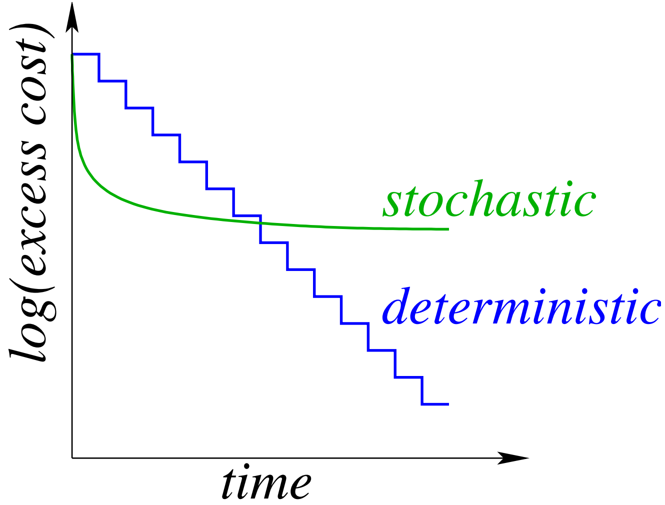
- For minimizing finite sums, can we design a better method?

Motivation for Hybrid Methods



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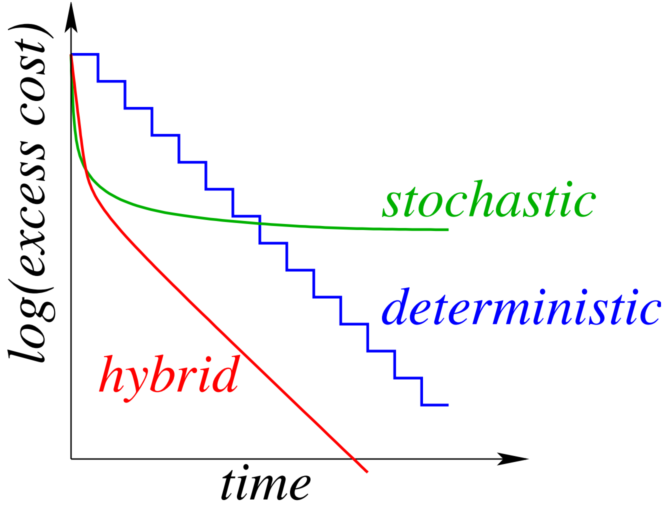

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- A common variant is to use larger sample $\mathcal{B}^t$,

$$\frac{1}{|\mathcal{B}^t|}\sum_{i\in\mathcal{B}^t} \nabla f_i(x^t) \approx \frac{1}{N}\sum_{i=1}^{N} \nabla f_i(x^t).$$

Approach 1: Batching

- The SG method with a sample $\mathcal{B}^t$ uses iterations

$$x^{t+1} = x^t - \frac{\alpha^t}{|\mathcal{B}^t|} \sum_{i \in \mathcal{B}^t} f_i(x^t).$$

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- **Gradient error decreases as sample size $|\mathcal{B}^t|$ increases.**
- Common to gradually increase the sample size $|\mathcal{B}^t|$.
  [Bertsekas & Tsitsiklis, 1996]
- We can choose $|\mathcal{B}^t|$ to achieve a linear convergence rate:
  - Early iterations are cheap like SG iterations.
  - Later iterations can use a Newton-like method.

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  - Stochastic variant of increment average gradient (IAG). [Blatt et al., 2007]
  - Assumes gradients of non-selected examples don't change.
  - Assumption becomes accurate as $||x^{t+1} - x^t|| \to 0$.

Convergence Rate of SAG

- If each f_i' is $L-$continuous and f is strongly-convex, with $\alpha_t = 1/16L$ SAG has

$$\mathbb{E}[f(x^t) - f(x^*)] \leqslant \left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^t C,$$

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- Linear convergence rate but only 1 gradient per iteration.
  - For well-conditioned problems, constant reduction per pass:

$$\left(1 - \frac{1}{8N}\right)^N \leq \exp\left(-\frac{1}{8}\right) = 0.8825.$$

  - For ill-conditioned problems, almost same as deterministic method (but N times faster).

Rate of Convergence Comparison

- Assume that $N = 700000$, $L = 0.25$, $\mu = 1/N$:
  - Gradient method has rate $\left(\frac{L-\mu}{L+\mu}\right)^2 = 0.99998$.
  - Accelerated gradient method has rate $\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.99761$.
  - SAG (N iterations) has rate $\left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^N = 0.88250$.
  - *Fastest possible* first-order method: $\left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2 = 0.99048$.

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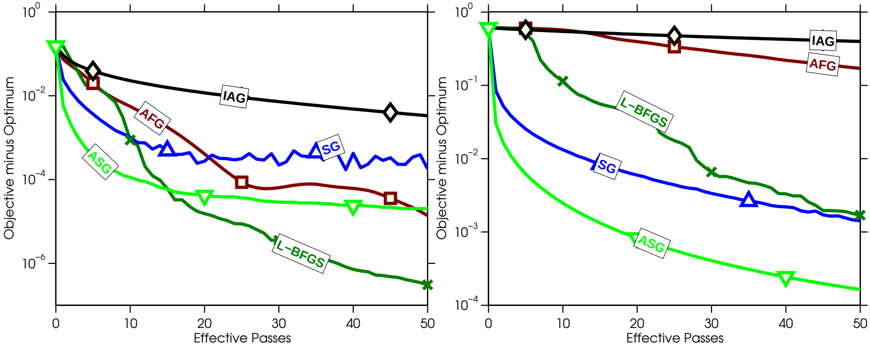
- Number of f_i' evaluations to reach ϵ:
  - Stochastic: $O(\frac{L}{\mu}(1/\epsilon))$.
  - Gradient: $O(N\frac{L}{\mu}\log(1/\epsilon))$.
  - Accelerated: $O(N\sqrt{\frac{L}{\mu}}\log(1/\epsilon))$.
  - SAG: $O(\max\{N, \frac{L}{\mu}\}\log(1/\epsilon))$.

Comparing Deterministic and Stochatic Methods

- quantum ($n = 50000$, $p = 78$) and rcv1 ($n = 697641$, $p = 47236$)



SAG Compared to FG and SG Methods

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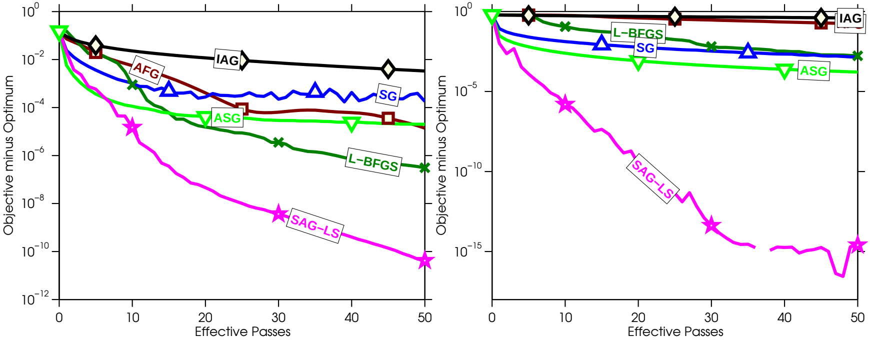

Other Linearly-Convergent Stochastic Methods

- Subsequent stochastic algorithms with linear rates:
  - Stochastic dual coordinate ascent [Shalev-Schwartz & Zhang, 2013]
  - Incremental surrogate optimization [Mairal, 2013].
  - Stochastic variance-reduced gradient (SVRG) [Johnson & Zhang, 2013, Konecny & Richtarik, 2013, Mahdavi et al., 2013, Zhang et al., 2013]
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- SVRG has a much lower memory requirement.
- There are also non-smooth extensions.

SAG Implementation Issues

- Basic SAG algorithm:
  - while(1)
  - Sample i from $\{1, 2, \dots, N\}$.
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  - Adaptive non-uniform sampling [Schmidt et al., 2013].

Reshuffling and Non-Uniform Sampling

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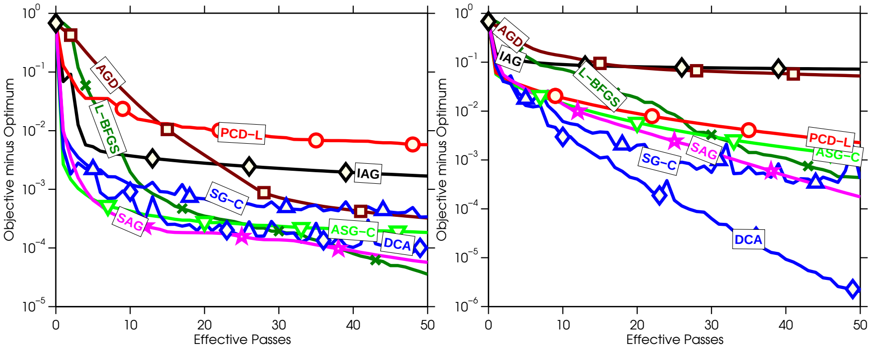
  - Adaptively estimate L_i as you go.
  - Slowly learns to ignore well-classified examples.

SAG with Adaptive Non-Uniform Sampling

- protein ($n = 145751$, $p = 74$) and sido ($n = 12678$, $p = 4932$)



- Datasets where SAG had the worst relative performance.

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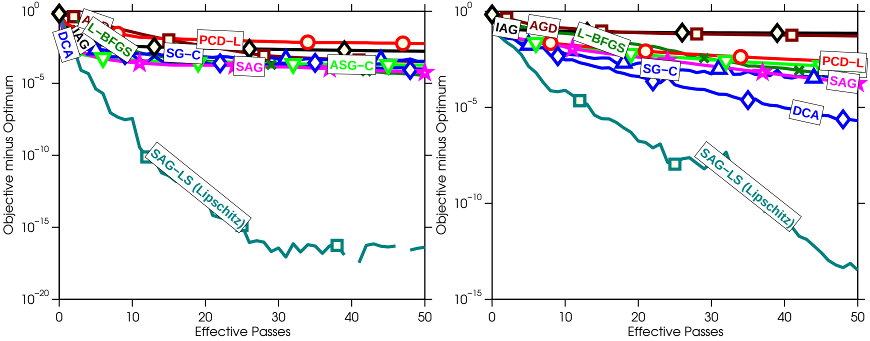

- Adaptive non-uniform sampling helps a lot.

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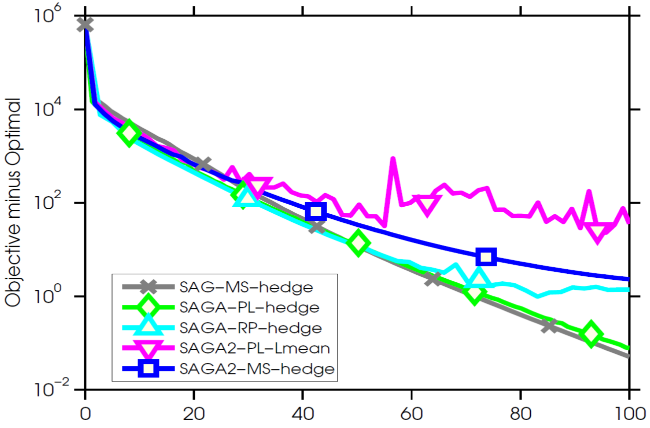
  3. Increase convergence rate.
     (classic SG methods: only changes constant)
- Convergence rate depends on L for mini-batches:
  - $L(\mathcal{B}) \leq L(i)$, possibly by up to $|\mathcal{B}|$.
  - Allows bigger step-size, $\alpha = 1/L(\mathcal{B})$.
  - Place examples in batches to make $L(\mathcal{B})$ small.

Comparing SAG and SAGA

- named-entity recognition tasks (CoNLL-2000)



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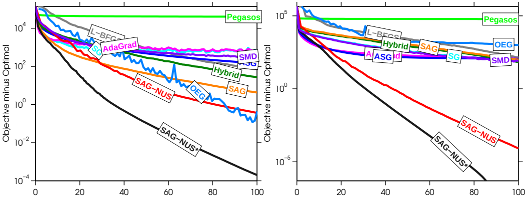

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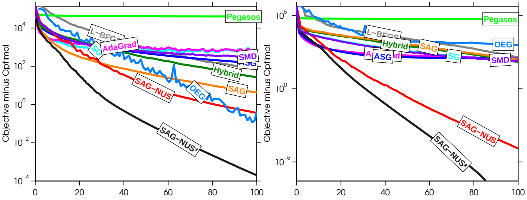

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- If the above don't work, use SVRG...

Stochastic Variance-Reduced Gradient

SVRG algorithm:

- Start with x_0
- for $s = 0, 1, 2 \ldots$
  - $d_s = \frac{1}{N}\sum_{i=1}^{N} f_i'(x_s)$
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  - for $t = 1, 2, \ldots m$
    - Randomly pick $i_t \in \{1, 2, \ldots, N\}$
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Practical issues similar to SAG (acceleration versions, automatic step-size/termination, handles sparsity/regularization, non-uniform sampling, mini-batches).

Conclusions

- Stochastic methods require 1 gradient per iteration but slow convergence.

- Deterministic methods are fast but requires N gradients per iteration.

- SAG, SVRG, and similar methods achieve faster convergence rate with few gradient evaluations