Semi-Markov/Graph Cuts

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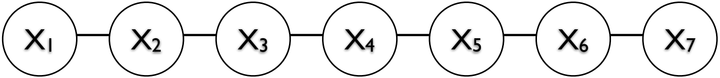
A Quick Review

- For a general chain-structured UGM we have:

$$p(x_1, x_2, \dots, x_n) \propto \prod_{i=1}^{n} \phi_i(x_i) \prod_{i=2}^{n} \phi_{i,i-1}(x_i, x_{i-1}),$$



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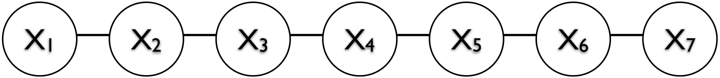

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- Local Markov property in general UGMs:
  - given neighbours, conditional independence of other nodes.

(Marginal independence corresponds to reachability.)

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- For chain-structured UGMs we learned the Viterbi decoding algorithm.
  - Forward phase:

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  - Backward phase: backtrack through argmax values.
  - Solves the decoding problem in $O(ns^2)$ instead of $O(s^n)$.

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  - Semi-Markov chain-structured UGMs.
  - Binary and attractive state UGMs.

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- A subsequence of nodes can have the same state.
  - You can encourage smoothness.
- Useful when you wish to keep track of how long you have been staying on the same state.

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  - The potential of making a transition from s' to s after l steps.
- You can encourage staying in certain states for a period of time.

Decoding the Semi-Markov chain-structured UGMs

- Let us look at the Viterbi decoding again:

$$V_{1,s} = \phi_1(s), \quad V_{i,s} = \phi_i(s) \cdot \max_{s'}\{\phi_{i,i-1}(s, s') \cdot V_{i-1,s'}\},$$

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- Depending on the application we can bound the maximum possible value of l to be L.
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- **Note** that it is *different* from having an order-L Markov chain (why?).

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- Questions?

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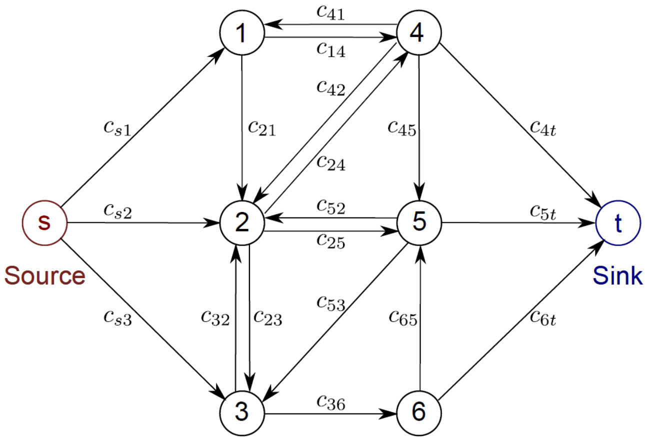
- The following material is borrowed from Simon Prince's (@UCL) slides. Available at computervisionmodels.com

The Max-Flow problem



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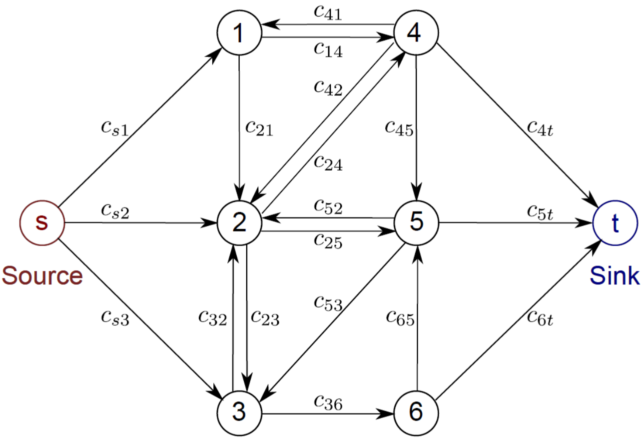

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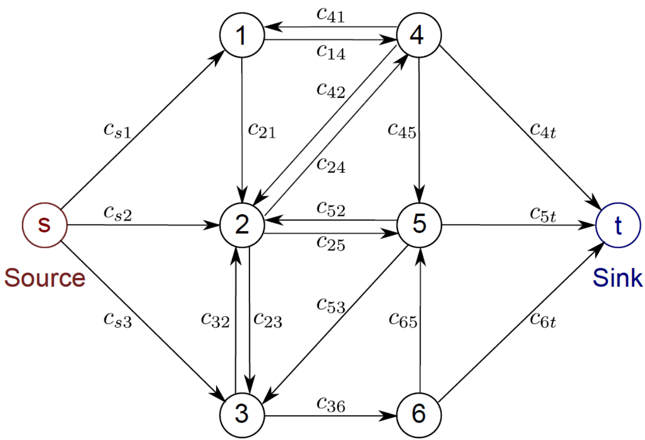

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- Cannot exceed the (non-negative) capacities C_{ij} associated with each edge.

Saturation

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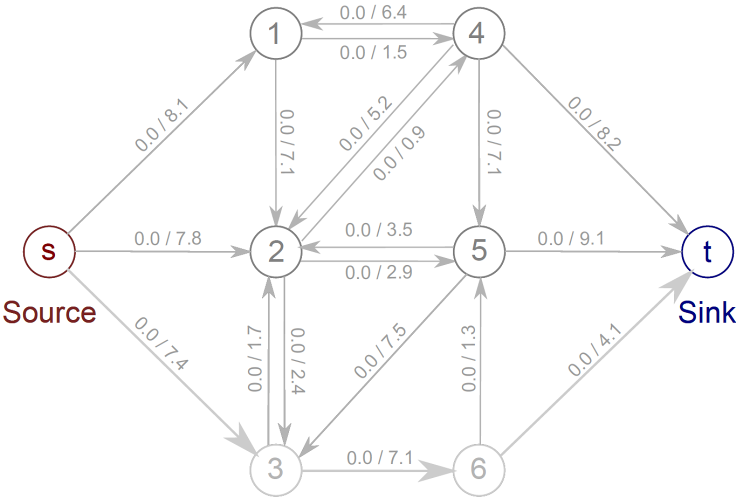
  - The set of saturated edges hence separate the source and sink.
  - This set is simultaneously the min-cut and the max-flow.

An example



- Two numbers are: current flow/ total capacity

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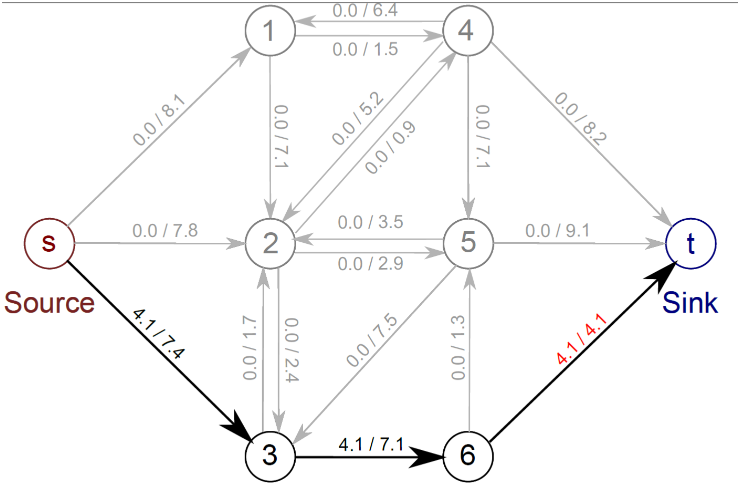

- Chose any path from source to sink with spare capacity and push as much flow as possible.

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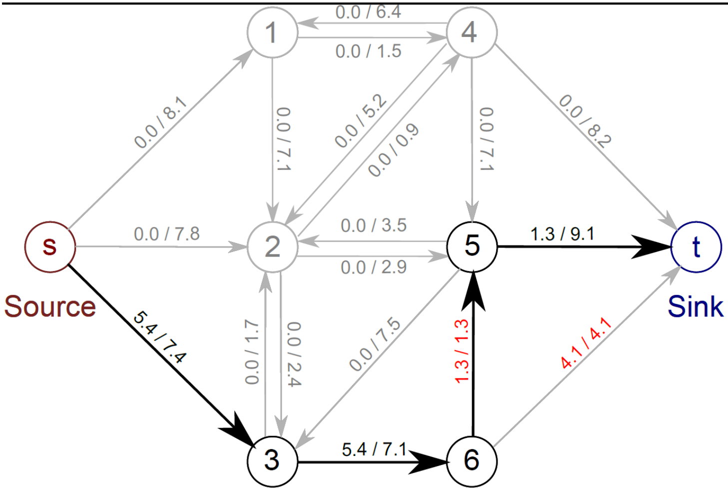

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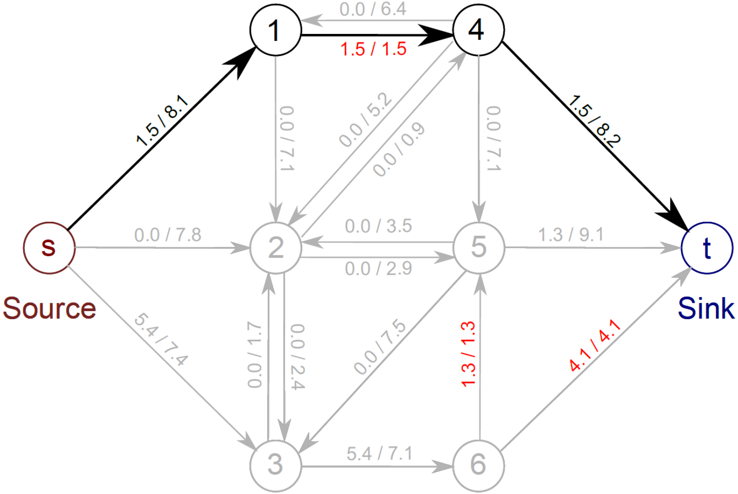
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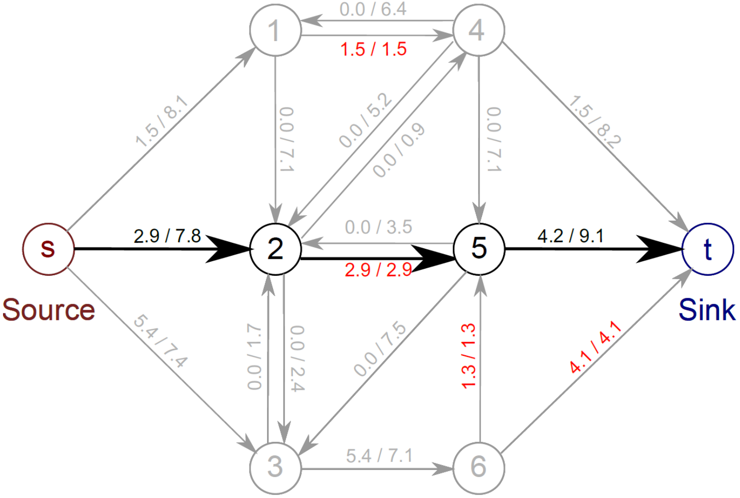

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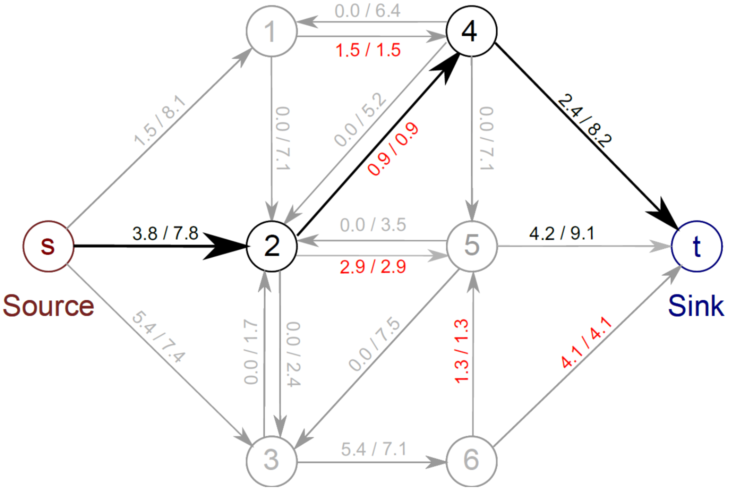

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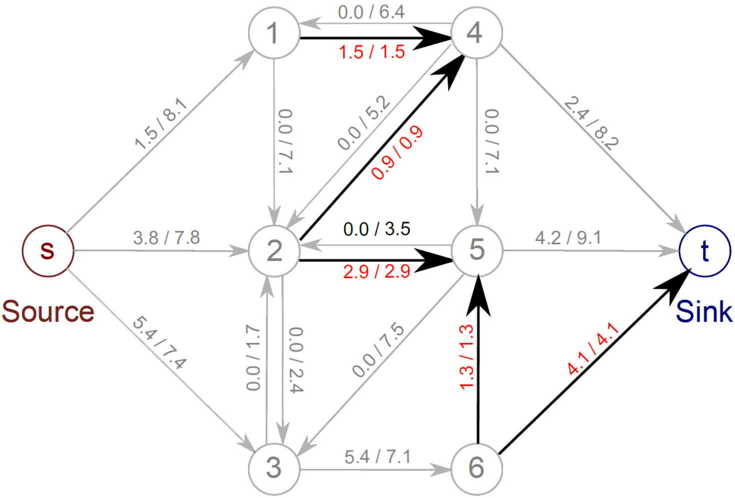

- No further ‘augmenting path’ exists.

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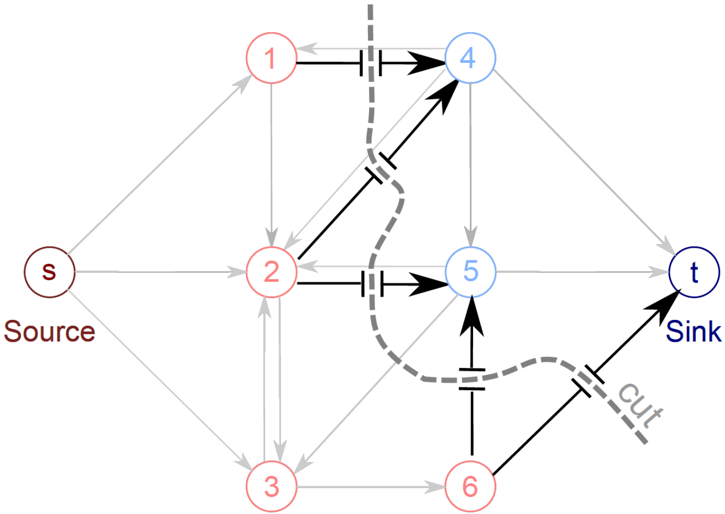

- The saturated edges partition the graph into two subgraphs.

The binary MRFs

- In the simplest form, let us constrain the pairwise potentials for adjacent nodes m, n to be:
  - $\phi_{m,n}(0,0) = \phi_{m,n}(1,1) = 0$.
  - $\phi_{m,n}(1,0) = \theta_{10}$.
  - $\phi_{m,n}(0,1) = \theta_{01}$.

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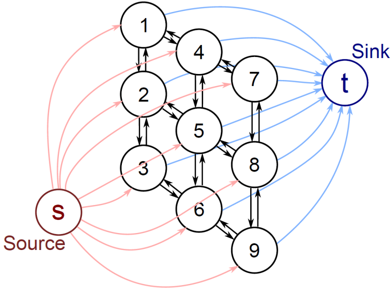
- Will make a graph such that each cut corresponds to a configuration.



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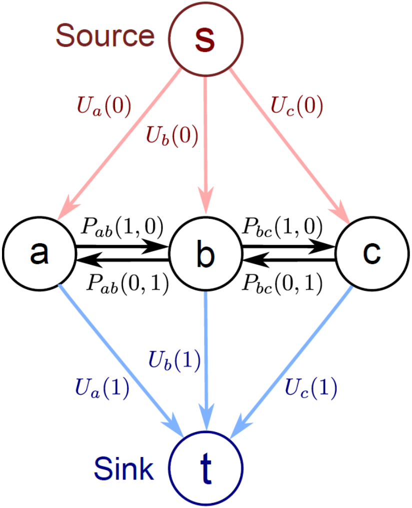

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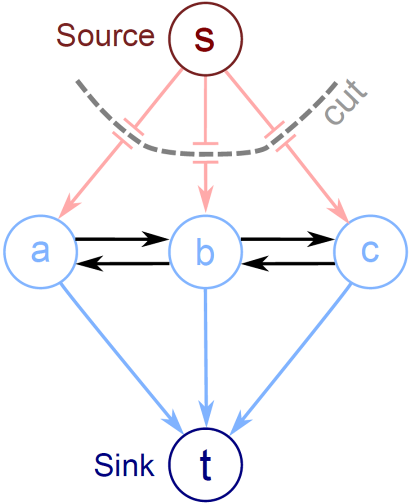


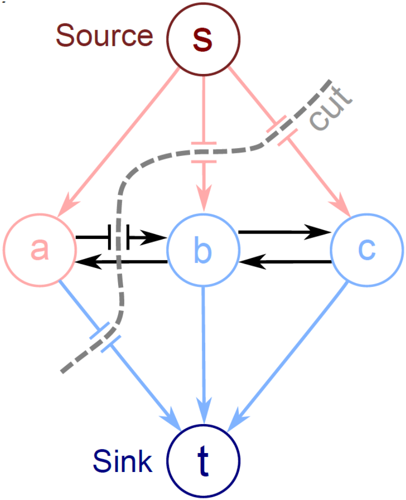
Solution

0	0	0

Cost

$$U_a(0) + U_b(0) + U_c(0)$$

The binary MRFs



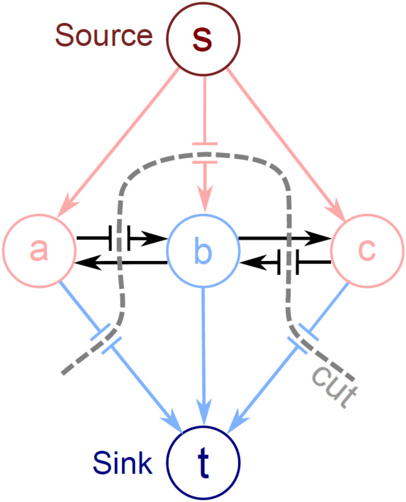
Solution

1	0	0

Cost

$$U_a(1) + U_b(0) + U_c(0)$$
$$+P_{bc}(1,0)$$

The binary MRFs



Solution

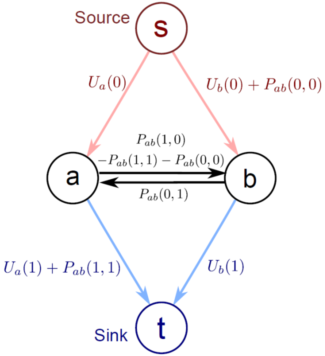
1	0	1

Cost

$$U_a(0) + U_b(0) + U_c(0)$$
$$+P_{ab}(1,0) + P_{bc}(0,1)$$

The binary MRFs

- In the general case:



The binary MRFs

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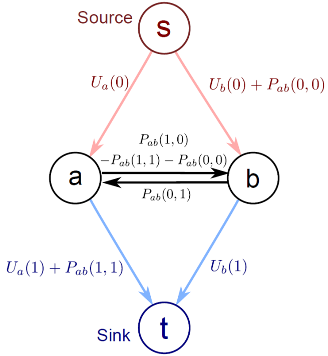

- Constraint $\theta_{10} + \theta_{01} > \theta_{11} + \theta_{00}$ (attraction).

The binary MRFs

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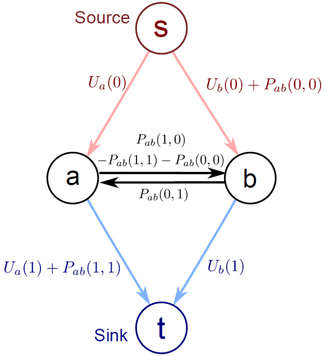
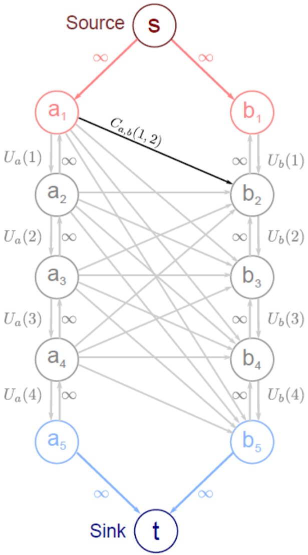


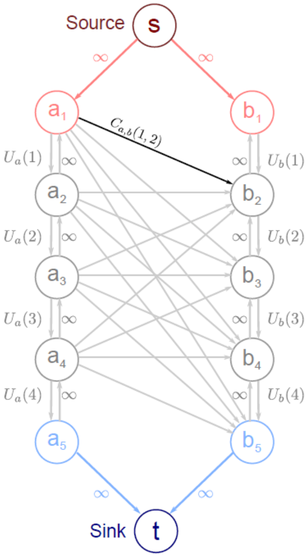
- Constraint $\theta_{10} + \theta_{01} > \theta_{11} + \theta_{00}$ (attraction).
- If met, the problem is called "submodular" and we can solve it in polynomial time.

Other cases



Source
s
∞
∞
a_1
b_1
$C_{a,b}(1,2)$
$U_a(1)$
∞
∞
$U_b(1)$
a_2
b_2
$U_a(2)$
∞
∞
$U_b(2)$
a_3
b_3
$U_a(3)$
∞
∞
$U_b(3)$
a_4
b_4
$U_a(4)$
∞
∞
$U_b(4)$
a_5
b_5
∞
∞
Sink
t

Other cases



$$P_{ab}(\beta,\gamma) + P_{ab}(\alpha,\delta) - P_{a,b}(\beta,\delta) - P_{ab}(\alpha,\gamma) \geq 0,$$

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  - if the pairwise potential is a metric

$$
\begin{aligned}
P(\alpha,\beta) = & \quad 0 \quad \Leftrightarrow \alpha = \beta \\
P(\alpha,\beta) & = P(\beta,\alpha) > 0 \\
P(\alpha,\beta) & \leq P(\alpha,\gamma) + P(\gamma,\beta)
\end{aligned}
$$

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- Alpha Expansion Algorithm (next week) uses the max-flow idea as a subroutine to do coordinate descent in the label space.

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  - Exact solution in multi-label case if submodular.
  - Approximate solution in multi-label case if a metric.

Thank you!

Questions?