Definition: **Run-Time Distribution (2)**

Given OLVA A' for optimisation problem Π':

- The *success probability* $P_s(RT_{A',\pi'} \leq t, SQ_{A',\pi'} \leq q)$ is the probability that A' finds a solution for a soluble instance $\pi' \in \Pi'$ of quality $\leq q$ in time $\leq t$.

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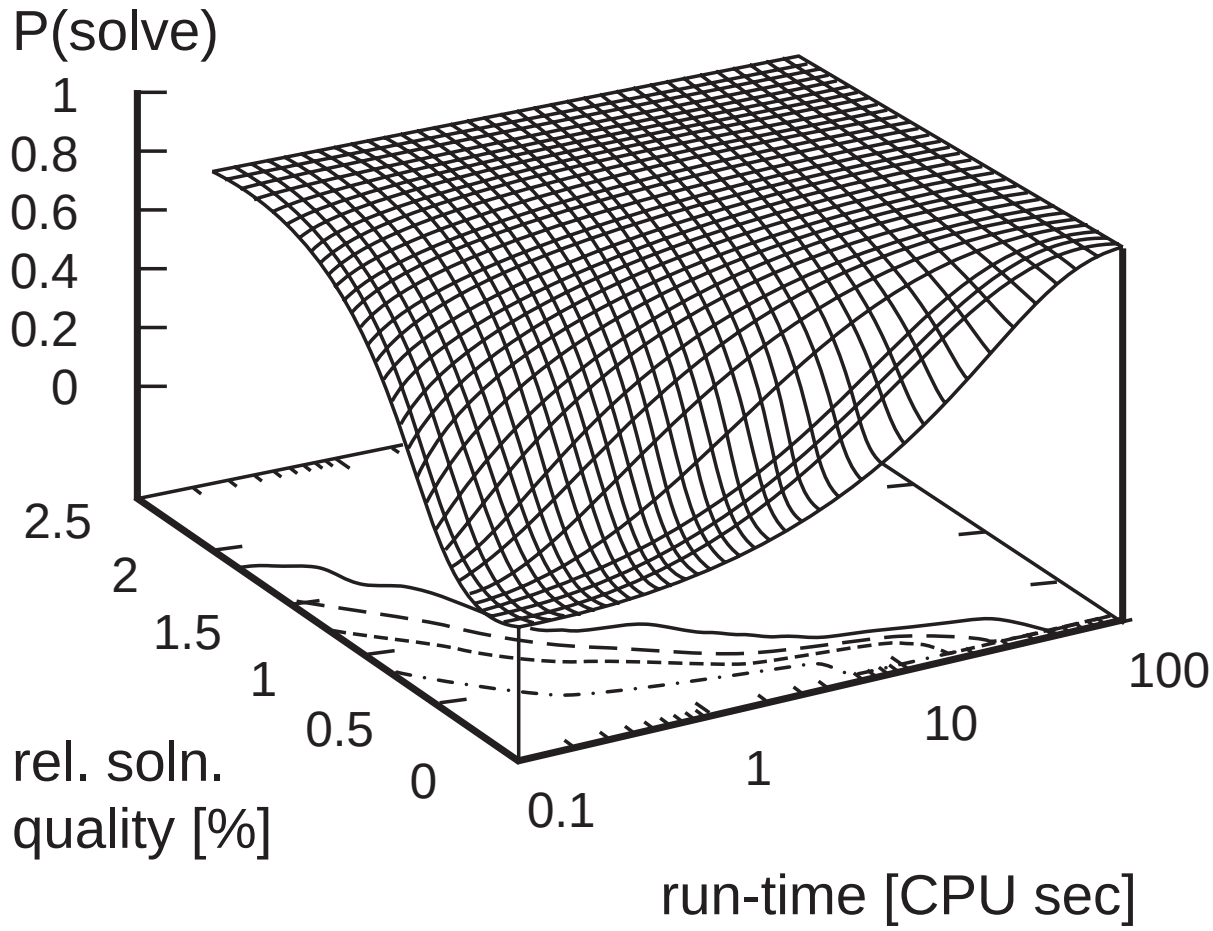
- The *run-time distribution function rtd* : $\mathbb{R}^+ \times \mathbb{R}^+ \mapsto [0,1]$, defined as $rtd(t,q) = P_s(RT_{A,\pi} \leq t, SQ_{A',\pi'} \leq q)$, completely characterises the RTD of A' on π'.

Typical run-time distribution for SLS algorithm applied to hard instance of combinatorial optimisation problem:



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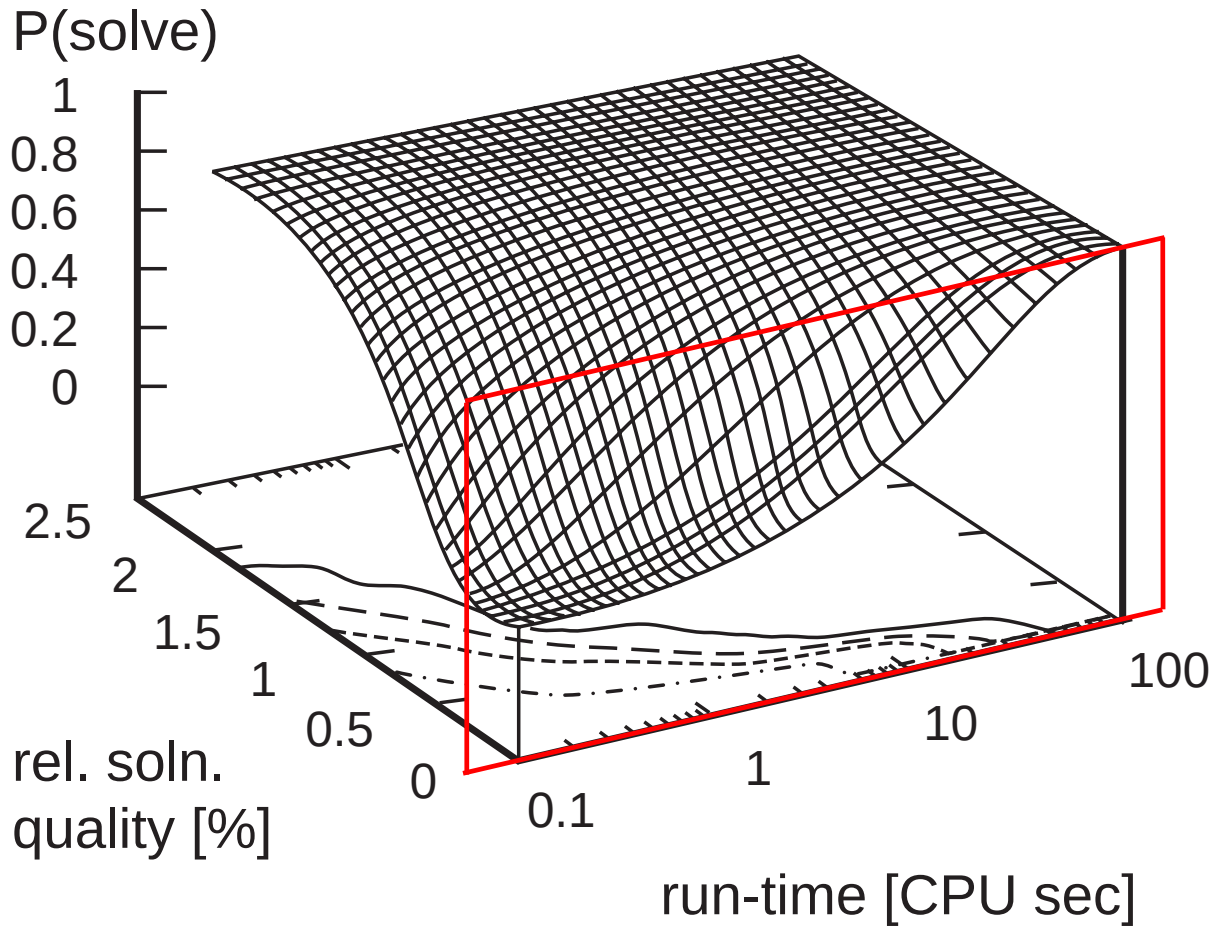

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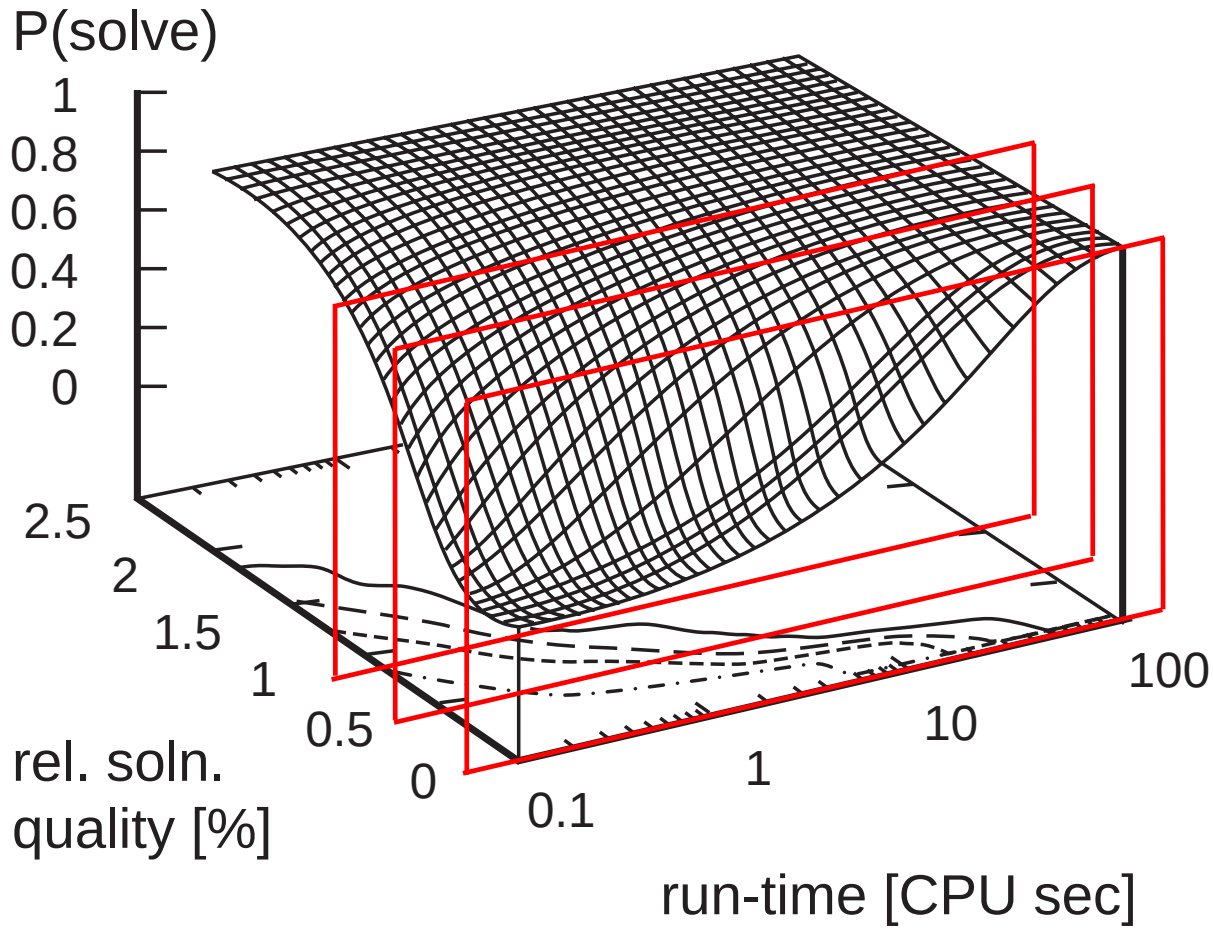
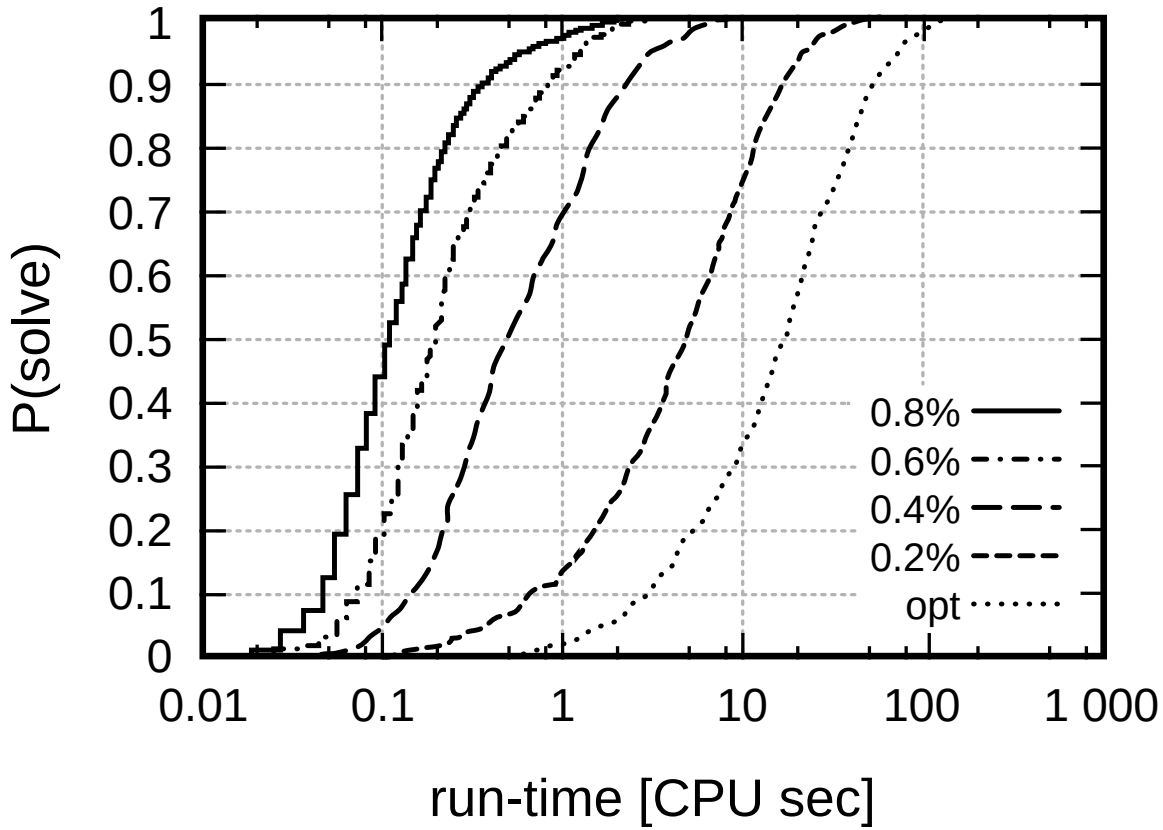


Qualified RTDs for various solution qualities:



Qualified run-time distributions (QRTDs)

- A *qualified run-time distribution (QRTD) of an OLVA A' applied to a given problem instance π' for solution quality q'* is a marginal distribution of the bivariate RTD $rtd(t, q)$ defined by:

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Note: Solution qualities q are often expressed as *relative solution qualities* $q/q^* - 1$, where $q^* =$ optimal solution quality for given problem instance.

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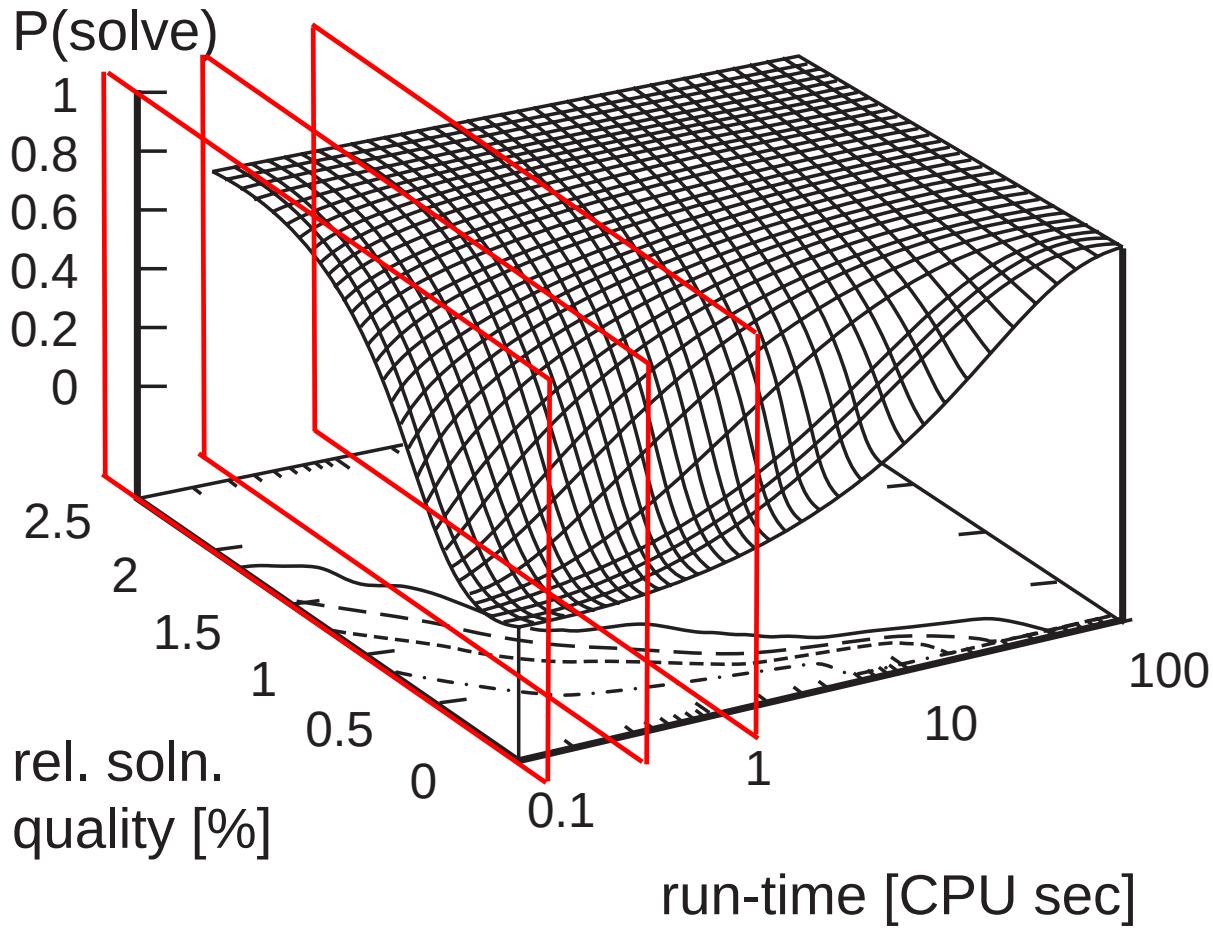
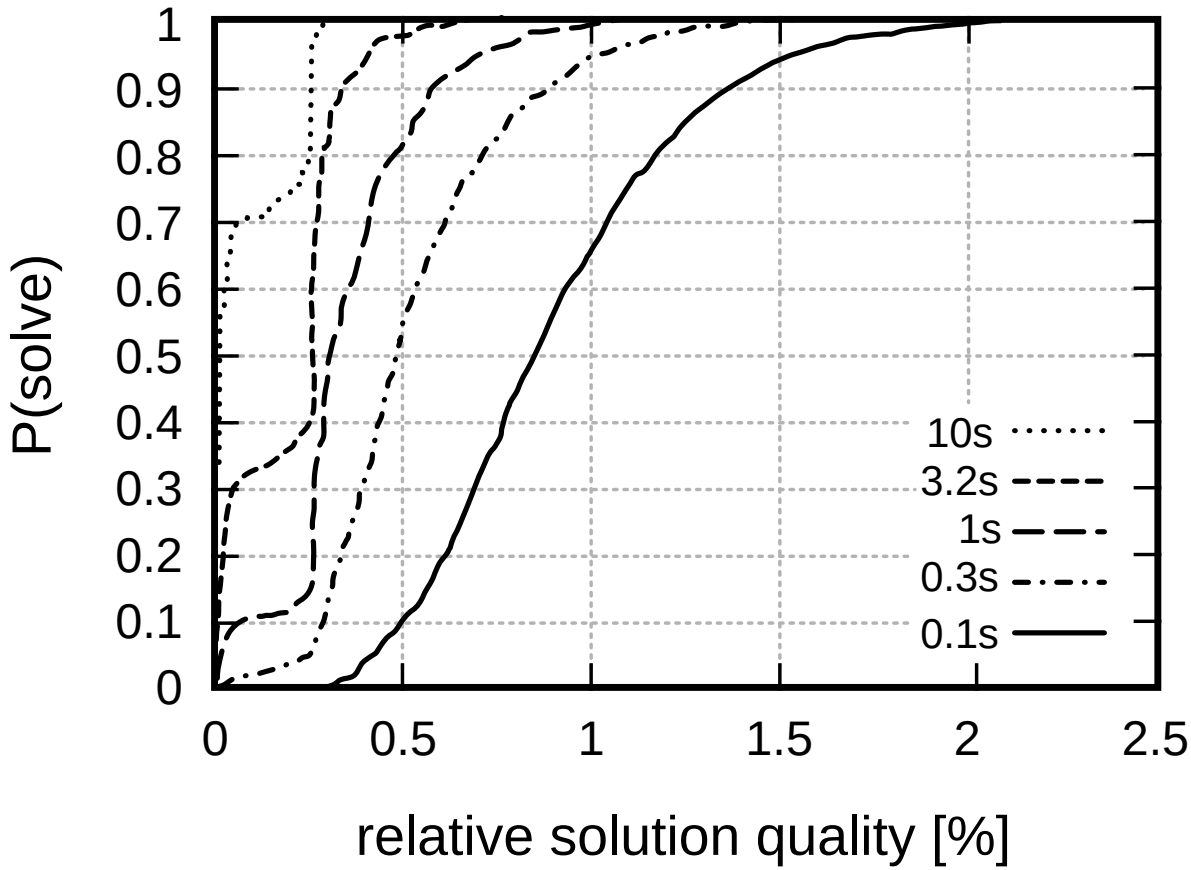


Solution quality distributions for various run-times:



Solution quality distributions (SQDs)

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- SQDs characterise the solution qualities achieved by a given SLS algorithm for a combinatorial optimisation problem within a given run-time bound (useful for type 2 application scenarios).

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- For sufficiently long run-times, increase in mean solution quality is often accompanied by decrease in solution quality variability.

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- For PAC algorithms, the SQDs for very large time-limits t' approach degenerate distributions that concentrate all probability on the optimal solution quality.
- For any essentially incomplete algorithm A' (such as Iterative Improvement) applied to a problem instance π', the SQDs for sufficiently large time-limits t' approach a non-degenerate distribution called the *asymptotic SQD of A' on π'*.

Solution quality statistic over time (SQTs)

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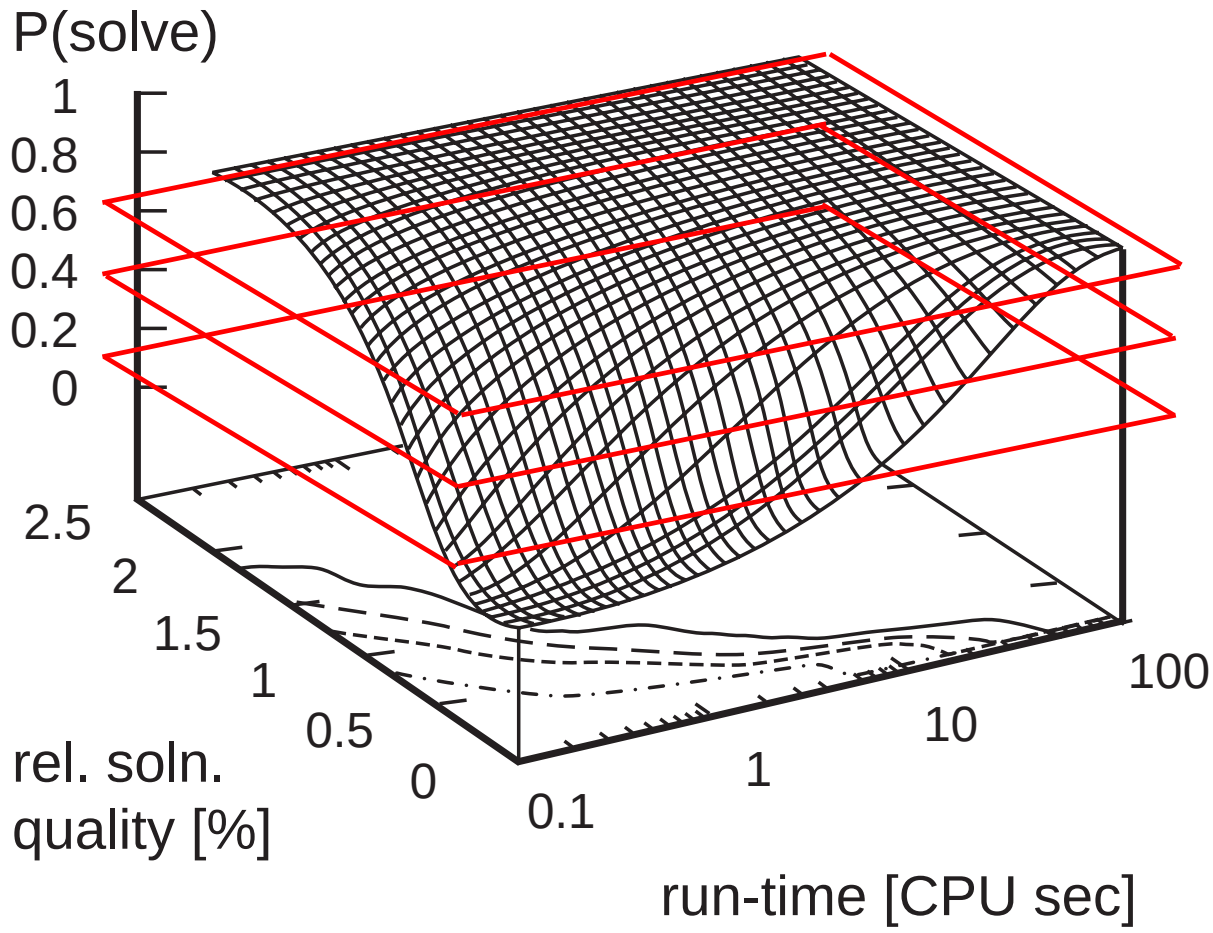
- SQT curves are widely used to illustrate the trade-off between run-time and solution quality for a given OLVA.
- **But:** Important aspects of an algorithm's run-time behaviour may be easily missed when basing an analysis solely on a single SQT curve.

Typical SQT curves for SLS optimisation algorithms applied to instance of hard combinatorial optimisation problem:



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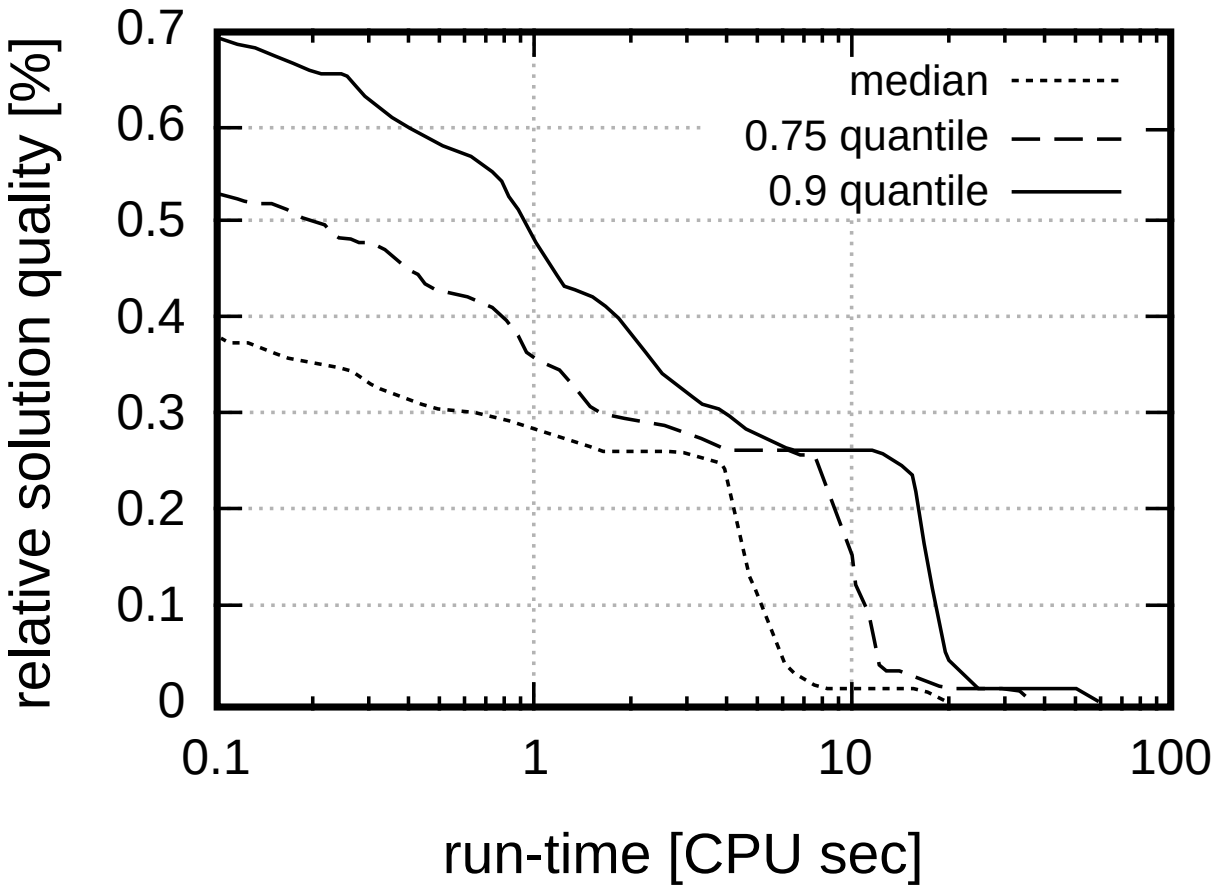

Note:

- The fraction of successful tuns, $sr := k'/k$, is called the *success ratio*; for large run-times t', it approximates the *asymptotic success probability* $p_s^* := lim_{t\to\infty} P_s(RT_{a,\pi} \leq t)$.

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  The mean run-time for a variant of the algorithm that restarts after time t' can be estimated as:

  $$\widehat{E}(RT_s) + (1/sr - 1) \cdot \widehat{E}(RT_f)$$

  where $\widehat{E}(RT_s)$ and $\widehat{E}(RT_f)$ are the average times of successful and failed runs, respectively.

  Note: $1/sr - 1$ is the expected number of failed runs required before a successful run is observed.

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- During each run, whenever the incumbent solution is improved, record the quality of the improved incumbent solution and the time at which the improvement was achieved in a *solution quality trace*.
- Let $sq(t', j)$ denote the best solution quality encountered in run j up to time t'. The cumulative empirical RTD of A' on π' is defined by $\widehat{P}_s(RT \leq t', SQ \leq q') := \#\{j \mid sq(t', j) \leq q'\}/k$.

Note: Qualified RTDs, SQDs and SQT curves can be easily derived from the same solution quality traces.