Convex Optimization for Big Data

Asian Conference on Machine Learning

Mark Schmidt

November 2014

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  - Seen across many fields of science and engineering.
  - Not gigabytes, but terabytes or petabytes (and beyond).

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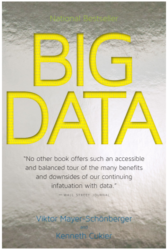

National Bestseller

BIG DATA

"No other book offers such an accessible and balanced tour of the many benefits and downsides of our continuing infatuation with data."
—WALL STREET JOURNAL

Viktor Mayer-Schönberger
and
Kenneth Cukier

- Many important aspects to the 'big data' puzzle:
    - Distributed data storage and management, parallel computation, software paradigms, data mining, machine learning, privacy and security issues, reacting to other agents, power management, summarization and visualization.

Context: Big Data and Big Models

- Machine learning uses big data to fit richer statistical models:
    - Vision, bioinformatics, speech, natural language, web, social.
    - Developping broadly applicable tools.
    - Output of models can be used for further analysis.

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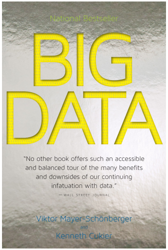

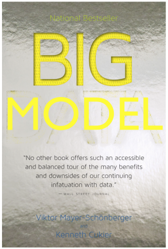

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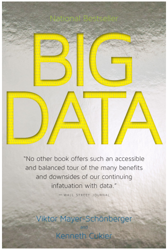

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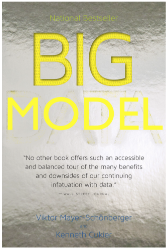

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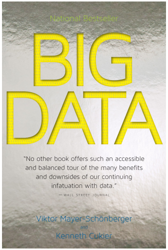

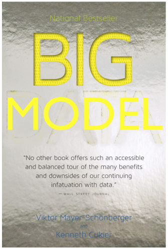

- Numerical optimization is at the core of many of these models.
- But, traditional 'black-box' methods have difficulty with:
  - the large data sizes.
  - the large model complexities.

Motivation: Why Learn about Convex Optimization?

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Why in particular learn about convex optimization?

- Among only *efficiently-solvable* continuous problems.
- You can do a lot with convex models.

  (least squares, lasso, generlized linear models, SVMs, CRFs)
- Empirically effective non-convex methods are often based methods with good properties for convex objectives.

  (functions are locally convex around minimizers)

Two Components of My Research

- The first component of my research focuses on computation:
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    - Can lead to enormous speedups for big data and complex models.

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- We can alternate between these two.

Outline

1. Convex Functions
2. Smooth Optimization
3. Non-Smooth Optimization
4. Randomized Algorithms
5. Parallel/Distributed Optimization

Convexity: Zero-order condition

A real-valued function is convex if

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y),$$

for all x, $y \in \mathbb{R}^n$ *and all* $0 \leq \theta \leq 1$.

- Function is *below a linear interpolation* from x to y.
- Implies that all local minima are global minima.

(contradiction otherwise)

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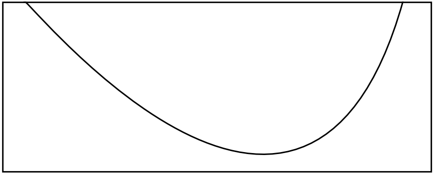

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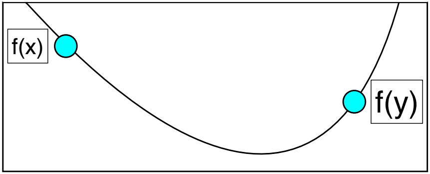

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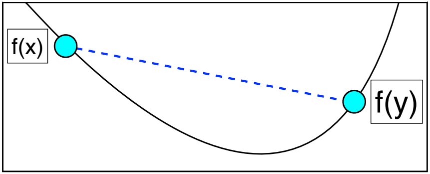

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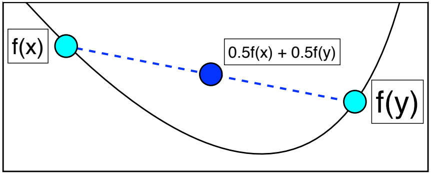

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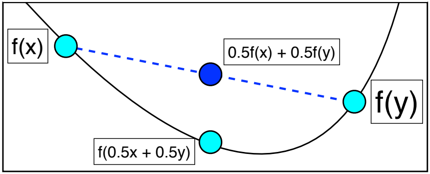

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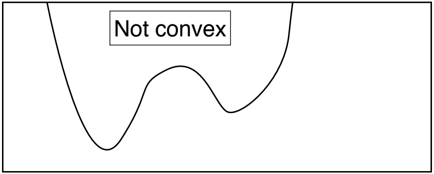

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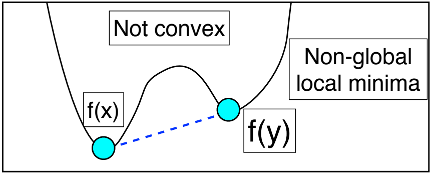

Convexity of Norms

We say that a function f is a **norm** *if:*

1. $f(0) = 0$.
2. $f(\theta x) = |\theta| f(x)$.
3. $f(x + y) \leq f(x) + f(y)$.

Examples:

$$\|x\|_2 = \sqrt{\sum_i x_i^2} = \sqrt{x^T x}$$

$$\|x\|_1 = \sum_i |x_i|$$

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Norms are convex:

$$\begin{aligned} f(\theta x + (1-\theta)y) &\leq f(\theta x) + f((1-\theta)y) && (3) \\ &= \theta f(x) + (1-\theta) f(y) && (2) \end{aligned}$$

Strict Convexity

A real-valued function is strictly convex if

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y),$$

for all $x \neq y \in \mathbb{R}^n$ *and all* $0 < \theta < 1$.

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- *Strictly below the linear interpolation* from x to y.
- Implies at most one global minimum.

(otherwise, could construct lower global minimum)

Convexity: First-order condition

A real-valued differentiable function is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x),$$

for all $x, y \in \mathbb{R}^n$.

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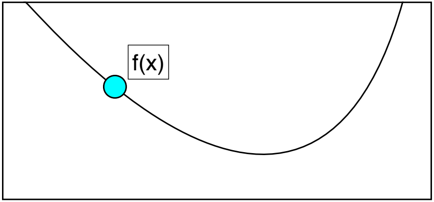

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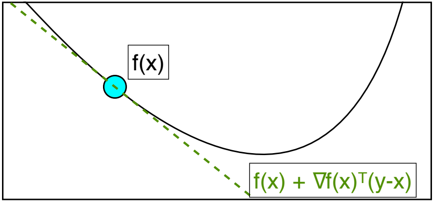

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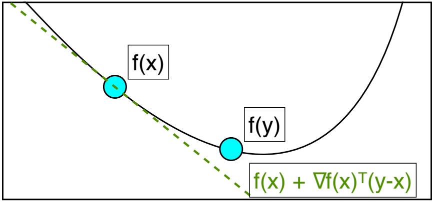

Convexity: Second-order condition

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A real-valued function f is a quadratic if it can be written in the form:

$$f(x) = \frac{1}{2}x^T A x + b^T x + c.$$

Since $\nabla^2 f(x) = A$, it is convex if $A \succeq 0$.
E.g., least squares has $\nabla^2 f(x) = A^T A \succeq 0$.

Examples of Convex Functions

Some simple convex functions:

- $f(x) = c$
- $f(x) = a^T x$
- $f(x) = ax^2 + b$ (for $a > 0$)
- $f(x) = \exp(ax)$
- $f(x) = x \log x$ (for $x > 0$)
- $f(x) = ||x||^2$
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Some other notable examples:

- $f(x, y) = \log(e^x + e^y)$
- $f(X) = \log \det X$ (for X positive-definite).
- $f(x, Y) = x^T Y^{-1} x$ (for Y positive-definite)

Operations that Preserve Convexity

1. Non-negative weighted sum:

$$f(x) = \theta_1 f_1(x) + \theta_2 f_2(x).$$

2. Composition with affine mapping:

$$g(x) = f(Ax + b).$$

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The first term has Hessian $I \succ 0$, for the second term use (3) on the two (convex) arguments, then use (1) to put it all together.

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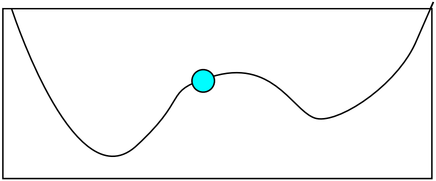

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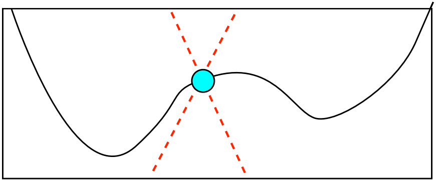

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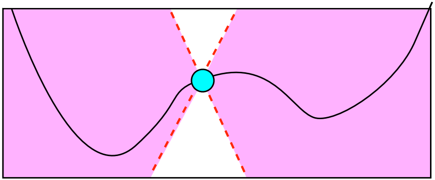

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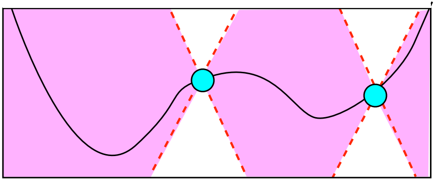

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- Optimization is hard, but assumptions make a big difference.

(we went from impossible to very slow)

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- However, these solvers require $O(n^2)$ or worse cost per iteration.
  - Infeasible for applications where n may be in the billions.
- Solving big problems has led to re-newed interest in simple first-order methods (gradient methods):

$$x^+ = x - \alpha \nabla f(x).$$

  - These only have $O(n)$ iteration costs.
  - But we must analyze how many iterations are needed.

ℓ_2-Regularized Logistic Regression

- Consider ℓ_2-regularized logistic regression:

$$f(x) = \sum_{i=1}^{n} \log(1 + exp(-b_i(x^T a_i))) + \frac{\lambda}{2}\|x\|^2.$$

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- Objective f is convex.
- First term is Lipschitz continuous.
- Second term is not Lipschitz continuous.
- But we have

$$\mu I \preceq \nabla^2 f(x) \preceq LI.$$

$(L = \frac{1}{4}\|A\|_2^2 + \lambda,\ \mu = \lambda)$

- Gradient is Lipschitz-continuous.
- Function is strongly-convex.

(implies strict convexity, and existence of unique solution)

Properties of Lipschitz-Continuous Gradient

- From Taylor's theorem, for some z we have:

$$f(y) = f(x) + \nabla f(x)^T(y-x) + \frac{1}{2}(y-x)^T\nabla^2 f(z)(y-x)$$

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- *Global quadratic upper bound on function value.*

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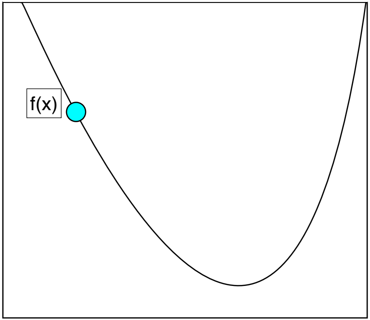

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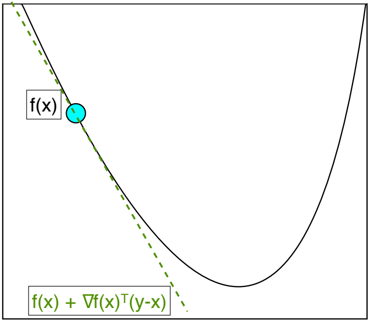

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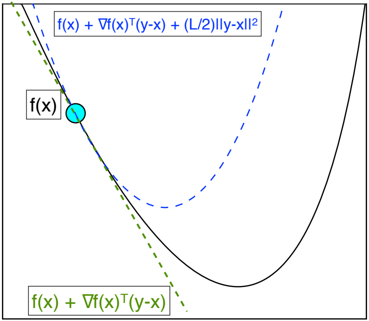

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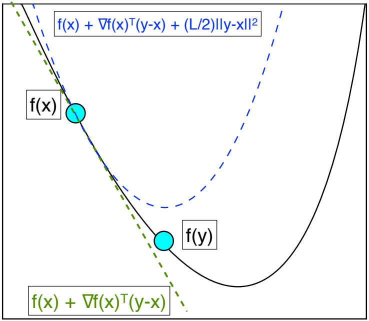

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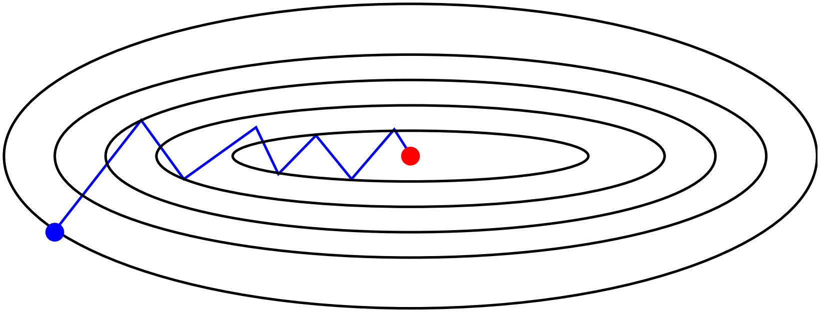

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- Set x^+ to minimize upper bound in terms of y:

$$x^+ = x - \frac{1}{L}\nabla f(x).$$

(gradient descent with step-size of $1/L$)

- Plugging this value in:

$$f(x^+) \leq f(x) - \frac{1}{2L}\|\nabla f(x)\|^2.$$

(decrease of *at least* $\frac{1}{2L}\|\nabla f(x)\|^2$)

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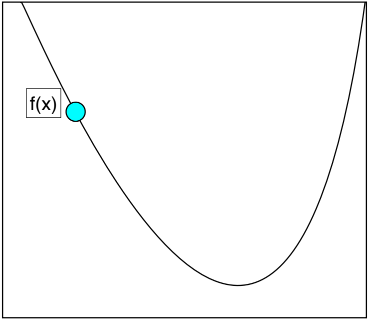

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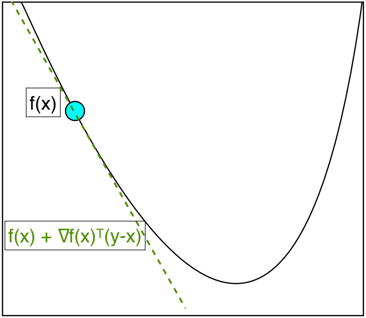

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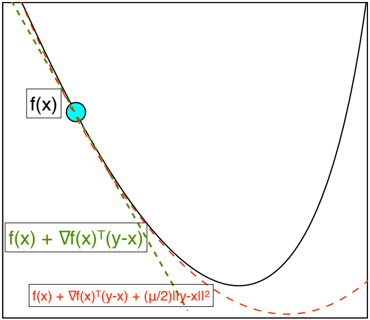

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- Minimize both sides in terms of y:

$$f(x^*) \geq f(x) - \frac{1}{2\mu}\|\nabla f(x)\|^2.$$

- Upper bound on how far we are from the solution.

Linear Convergence of Gradient Descent

- We have bounds on x^+ and x^*:

$$f(x^+) \leq f(x) - \frac{1}{2L}\|\nabla f(x)\|^2, \quad f(x^*) \geq f(x) - \frac{1}{2\mu}\|\nabla f(x)\|^2.$$

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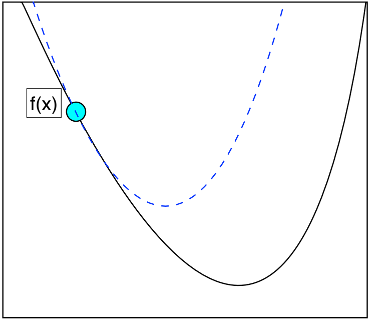

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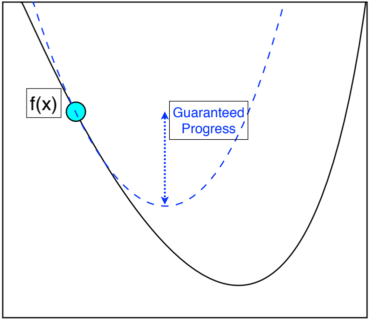

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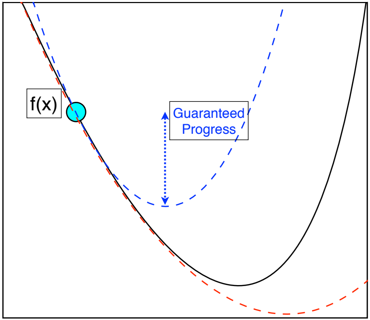

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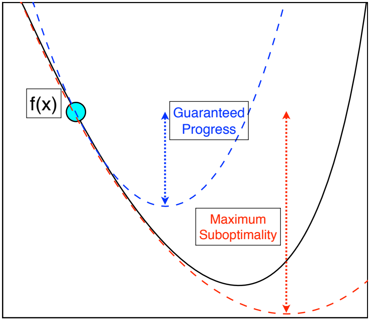

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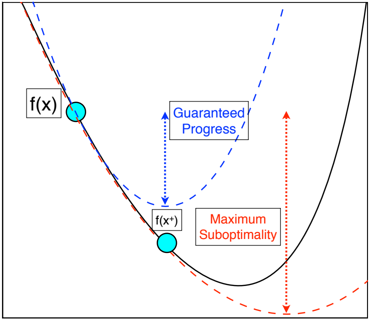

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- This gives a linear convergence rate:

$$f(x^t) - f(x^*) \leq \left(1 - \frac{\mu}{L}\right)^t [f(x^0) - f(x^*)]$$

- Each iteration multiplies the error by a fixed amount.

(very fast if μ/L is not too close to one)

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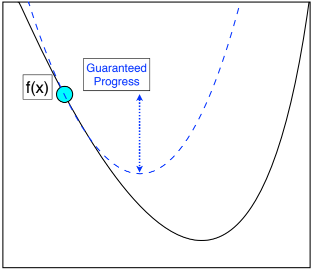

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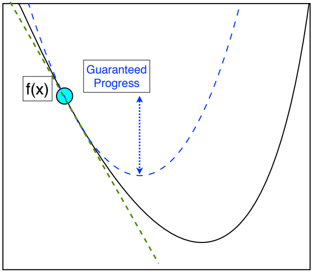

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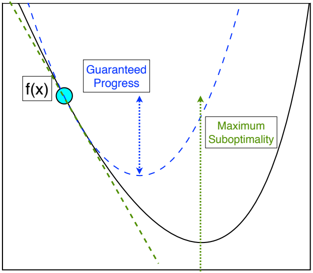

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- Also, check your derivative code!

$$\nabla_i f(x) \approx \frac{f(x + \delta e_i) - f(x)}{\delta}$$

- For large-scale problems you can check a random direction d:

$$\nabla f(x)^T d \approx \frac{f(x + \delta d) - f(x)}{\delta}$$

Convergence Rate of Gradient Method

We are going to explore the ‘convex optimization zoo’:

- Gradient method for smooth/convex: $O(1/t)$.
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- Is this the best algorithm under these assumptions?

Accelerated Gradient Method

- Nesterov's accelerated gradient method:

$$x_{t+1} = y_t - \alpha_t \nabla f(y_t),$$
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for appropriate α_t, β_t.

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- Motivation: "to make the math work"

  (but similar to heavy-ball/momentum and conjugate gradient method)

Convex Optimization Zoo

Algorithm	Assumptions	Rate
Gradient	Convex	$O(1/t)$
Nesterov	Convex	$O(1/t^2)$
Gradient	Strongly-Convex	$O((1-\mu/L)^t)$
Nesterov	Strongly-Convex	$O((1-\sqrt{\mu/L})^t)$

- $O(1/t^2)$ is optimal given only these assumptions.

  (sometimes called the *optimal* gradient method)
- The faster linear convergence rate is close to optimal.
- Also faster in practice, but implementation details matter.

Newton's Method

- The oldest differentiable optimization method is Newton's.

  (also called IRLS for functions of the form $f(Ax)$)

- Modern form uses the update

  $$x^+ = x - \alpha d,$$

  where d is a solution to the system

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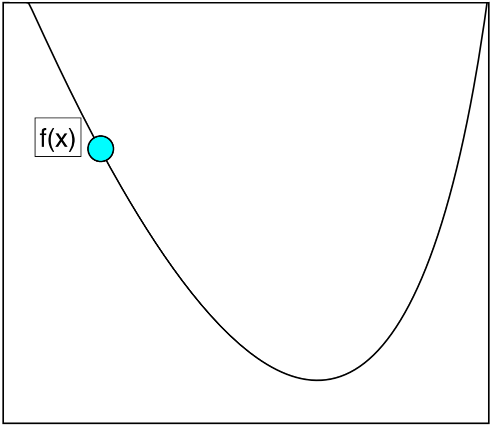
- We can generalize the Armijo condition to

  $$f(x^+) \leq f(x) + \gamma\alpha\nabla f(x)^T d.$$

- Has a natural step length of $\alpha = 1$.

  (always accepted when close to a minimizer)

Newton's Method



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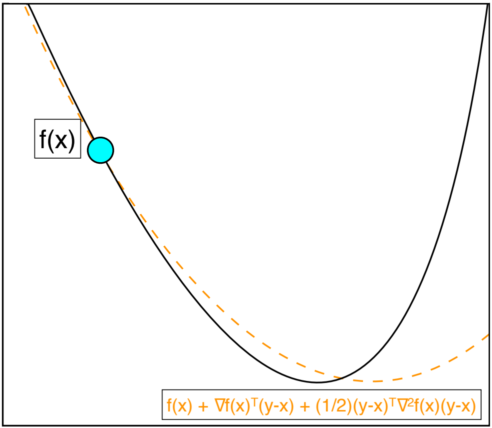

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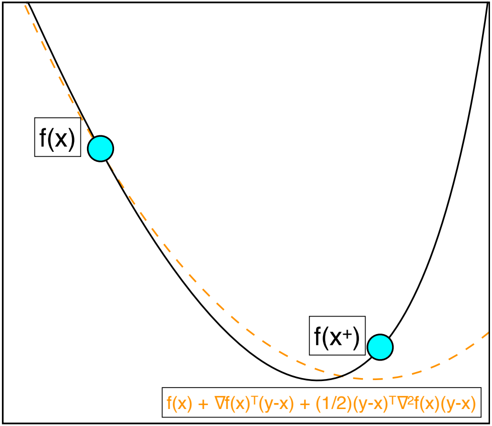

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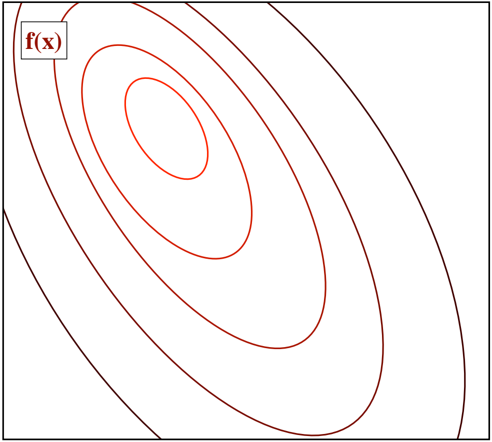

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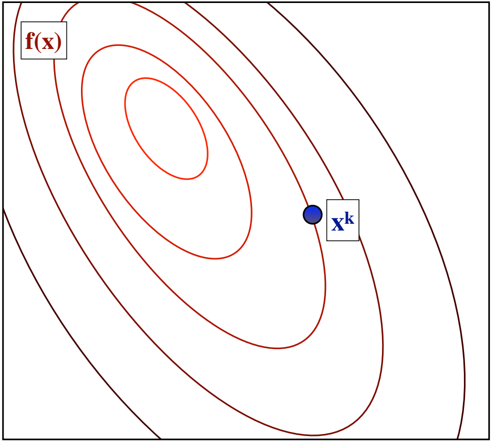

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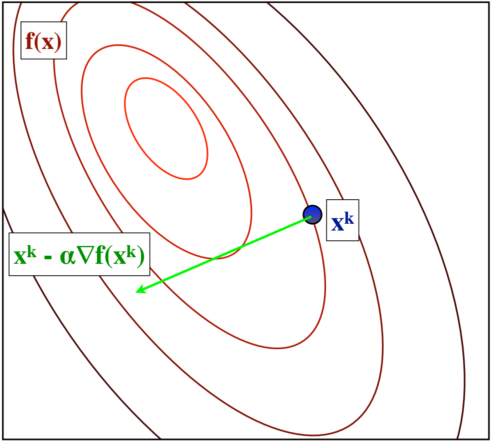

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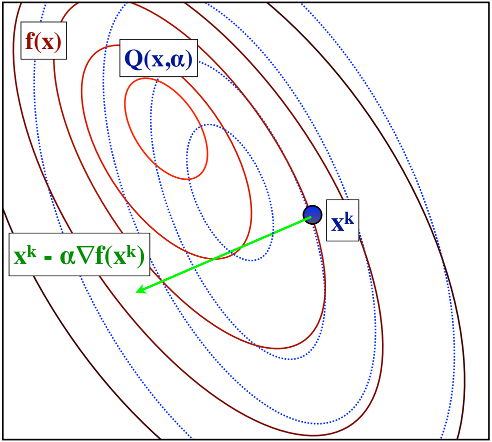

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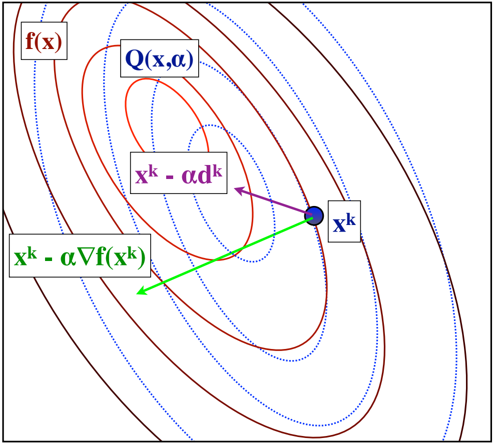

Convergence Rate of Newton's Method

- If $\nabla^2 f(x)$ is Lipschitz-continuous and $\nabla^2 f(x) \succeq \mu$, then close to x^* Newton's method has local superlinear convergence:

  $$f(x^{t+1}) - f(x^*) \leq \rho_t[f(x^t) - f(x^*)],$$

  with $\lim_{t\to\infty} \rho_t = 0$.
- Converges very fast, use it if you can!
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- But requires solving $\nabla^2 f(x) d = \nabla f(x)$.
- Get global rates under various assumptions (cubic-regularization/accelerated/self-concordant).

Newton's Method: Practical Issues

There are many practical variants of Newton's method:

- Modify the Hessian to be positive-definite.
- Only compute the Hessian every m iterations.
- Only use the diagonals of the Hessian.
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- Hessian-free: Compute d inexactly using Hessian-vector products:

$$\nabla^2 f(x)d = \lim_{\delta \to 0} \frac{\nabla f(x + \delta d) - \nabla f(x)}{\delta}$$

- Barzilai-Borwein: Choose a step-size that acts like the Hessian over the last iteration:

$$\alpha = \frac{(x^+ - x)^T(\nabla f(x^+) - \nabla f(x))}{\|\nabla f(x^+) - f(x)\|^2}$$

Another related method is nonlinear conjugate gradient.

Outline

1. Convex Functions
2. Smooth Optimization
3. Non-Smooth Optimization
4. Randomized Algorithms
5. Parallel/Distributed Optimization

Motivation: Sparse Regularization

- Consider ℓ_1-regularized optimization problems,

$$\min_x f(x) + \lambda\|x\|_1,$$

where f is differentiable.

- For example, ℓ_1-regularized least squares,

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- Regularizes and encourages sparsity in x
- The objective is non-differentiable when any $x_i = 0$.
- How can we solve non-smooth convex optimization problems?

Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y.$$

A vector d is a subgradient of a convex function f at x if

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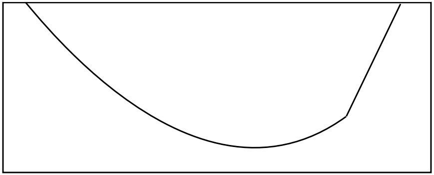

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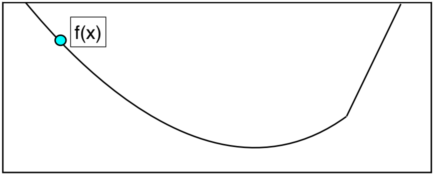

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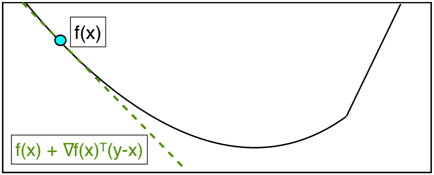

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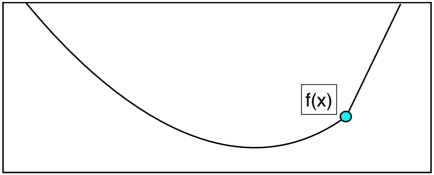

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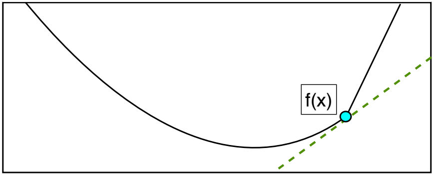

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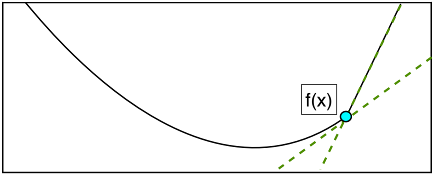

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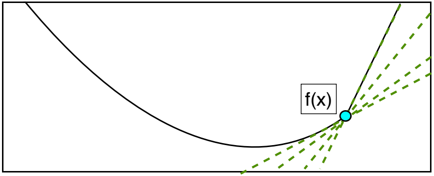

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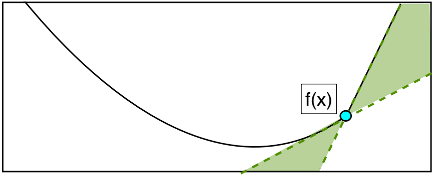

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- f is differentiable at x iff $\nabla f(x)$ is the only subgradient.
- At non-differentiable x, we have a set of subgradients.
- Set of subgradients is the sub-differential $\partial f(x)$.
- Note that $0 \in \partial f(x)$ iff x is a global minimum.

Sub-Differential of Absolute Value and Max Functions

- The sub-differential of the absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

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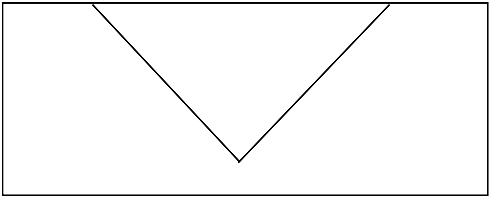

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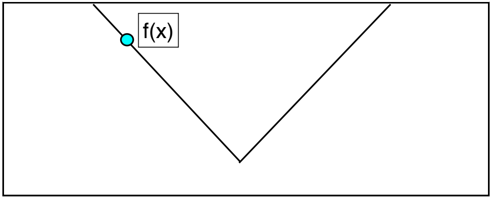

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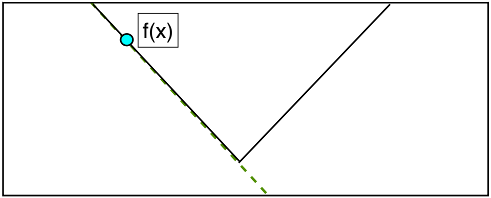

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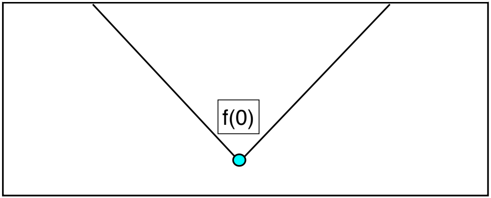

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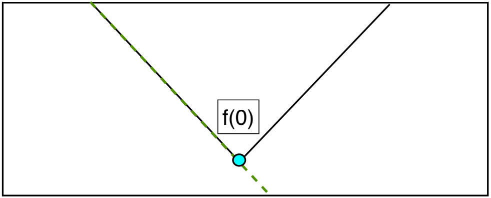

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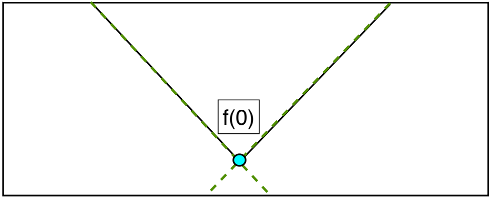

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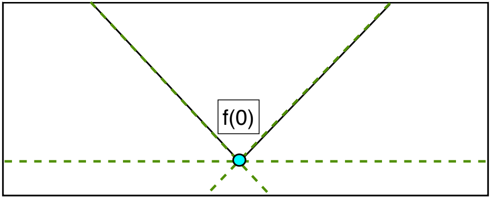

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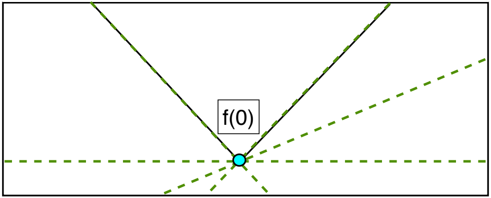

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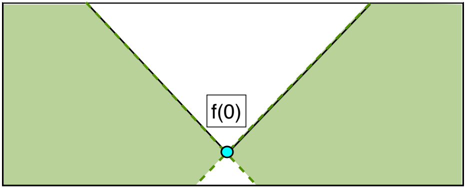

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- The sub-differential of the maximum of differentiable f_i:

$$\partial \max\{f_1(x), f_2(x)\} = \begin{cases} \nabla f_1(x) & f_1(x) > f_2(x) \\ \nabla f_2(x) & f_2(x) > f_1(x) \\ \theta \nabla f_1(x) + (1-\theta)\nabla f_2(x) & f_1(x) = f_2(x) \end{cases}$$

(any convex combination of the gradients of the argmax)

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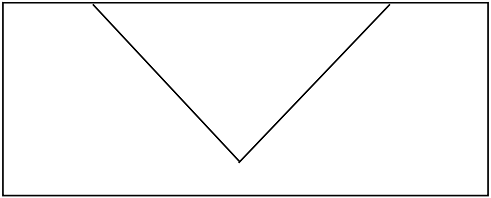

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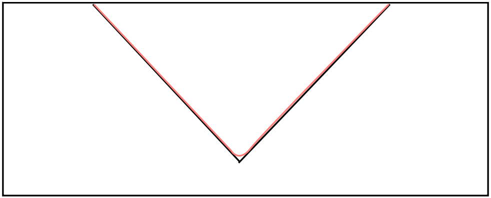

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- Generic smoothing strategy: strongly-convex regularization of convex conjugate.[Nesterov, 2005]

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- Smoothing is only faster if you use Nesterov's method.
- In practice, faster to slowly decrease smoothing level.
- You can get the $O(1/t)$ rate for $\min_x \max\{f_i(x)\}$ for f_i convex and smooth using *mirror-prox* method.[Nemirovski, 2004]

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- These are smooth objective with 'simple' constraints.

$$\min_{x \in \mathcal{C}} f(x).$$

Optimization with Simple Constraints

- Recall: gradient descent minimizes quadratic approximation:

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- Equivalent to projection of gradient descent:

$$x^{GD} = x - \alpha \nabla f(x),$$

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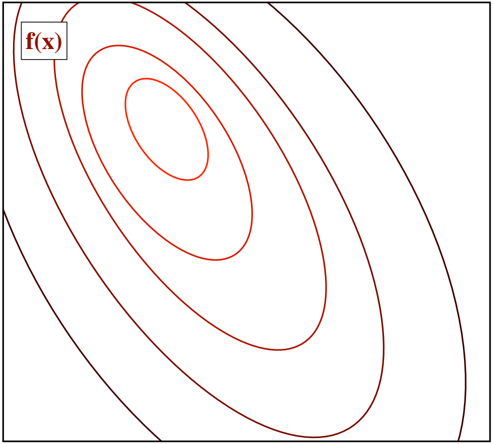
Gradient Projection



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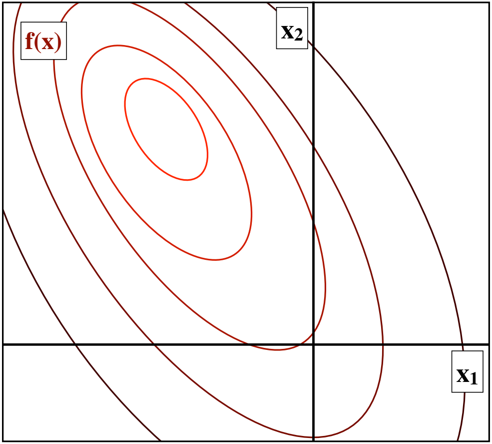

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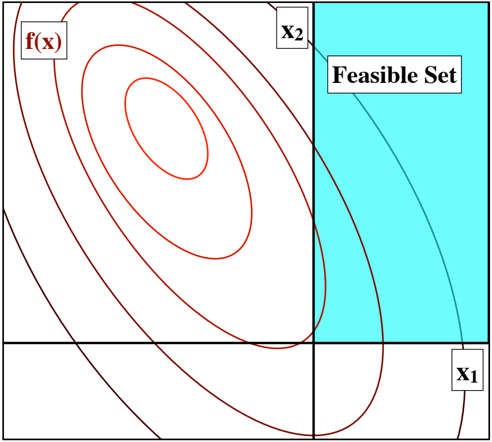

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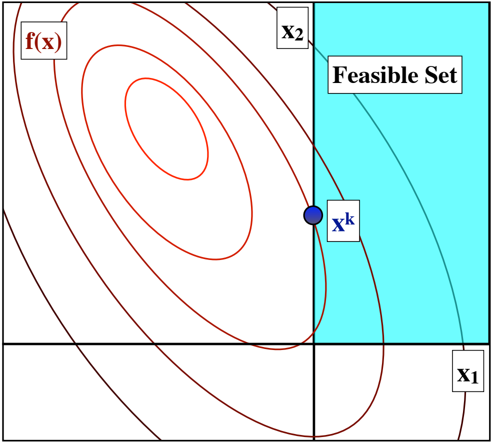

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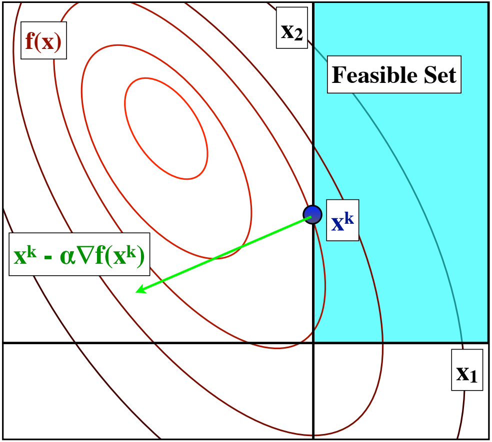

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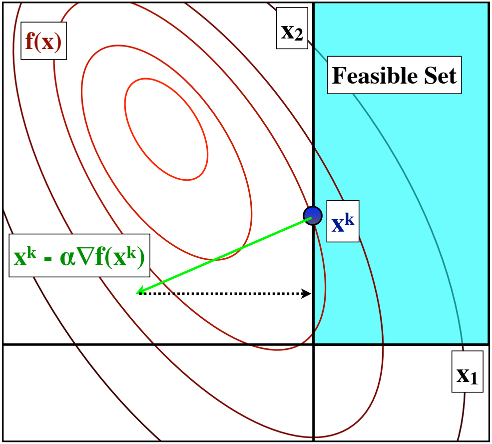

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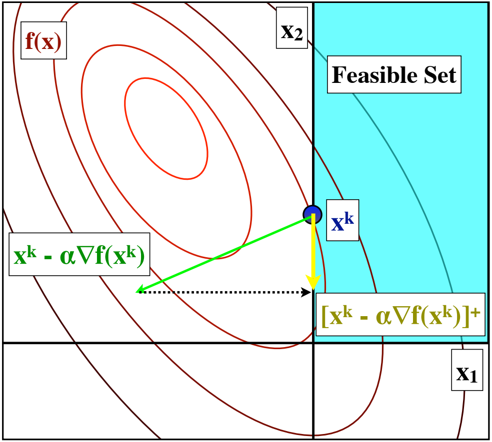

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- Intersection of simple sets: Dykstra's algorithm.

Convex Optimization Zoo

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P(Subgradient)	Convex	$O(1/\sqrt{t})$
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- For Newton, you need to project under $\|\cdot\|_{\nabla^2 f(x)}$

  (expensive, but special tricks for the case of simplex or lower/upper bounds)
- You don't need to compute the projection exactly.

Proximal-Gradient Method

- A generalization of projected-gradient is Proximal-gradient.
- The proximal-gradient method addresses problem of the form

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- Convergence rates are still the same as for minimizing f.

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$$\text{prox}_r[y] = \mathop{\arg\min}_{x \in \mathbb{R}^P} \; r(x) + \frac{1}{2}\|x - y\|^2.$$

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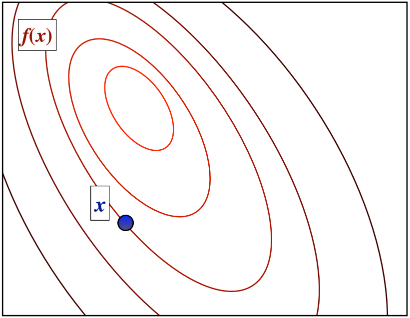

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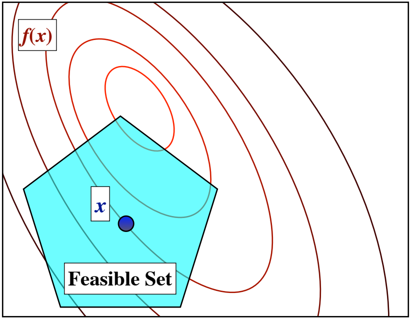

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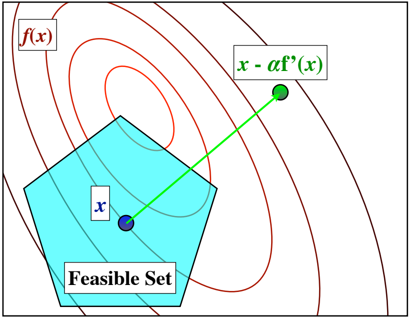

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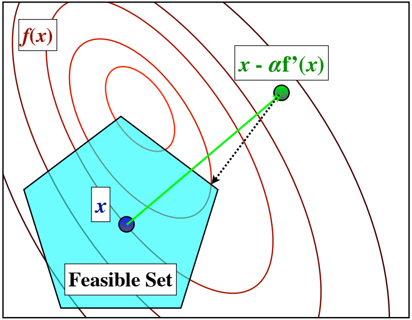

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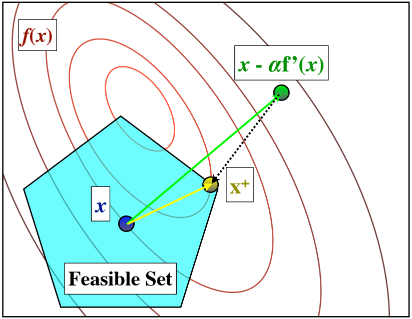

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- But for many problems we can not efficiently compute this operator.

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- Do inexact methods have the same rates?
  - *Yes, if the errors are appropriately controlled*. [Schmidt et al., 2011]

Convergence Rate of Inexact Proximal-Gradient

Proposition *[Schmidt et al., 2011] If the sequences of gradient errors* $\{||e_t||\}$ *and proximal errors* $\{\sqrt{\varepsilon_t}\}$ *are in* $\{O((1-\mu/L)^t)\}$*, then the inexact proximal-gradient method has an error of* $O((1-\mu/L)^t)$.

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- Classic result as a special case (constants are good).
- The rates degrades gracefully if the errors are larger.
- Similar analyses in convex case.
- Huge improvement in practice over black-box methods.
- Also exist accelerated and spectral proximal-gradient methods.

Discussion of Proximal-Gradient

- Theoretical justification for what works in practice.
- Significantly extends class of tractable problems.
- Many applications with inexact proximal operators:
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- But, it assumes computing $\nabla f(x)$ and $\text{prox}_h[x]$ have similar cost.
- Often $\nabla f(x)$ is much more expensive:
  - We may have a large dataset.
  - Data-fitting term might be complex.
- Particularly true for structured output prediction:
  - Text, biological sequences, speech, images, matchings, graphs.

Costly Data-Fitting Term, Simple Regularizer

- Consider fitting a conditional random field with ℓ_1-regularization:

$$\min_{x\in\mathbb{R}^P} \quad \frac{1}{N}\sum_{i=1}^{N} f_i(x) \; + \; r(x)$$

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- Inspiration from the smooth case:
  - For smooth high-dimensional problems, L-BFGS quasi-Newton algorithm outperforms accelerated/spectral gradient methods.

Quasi-Newton Methods

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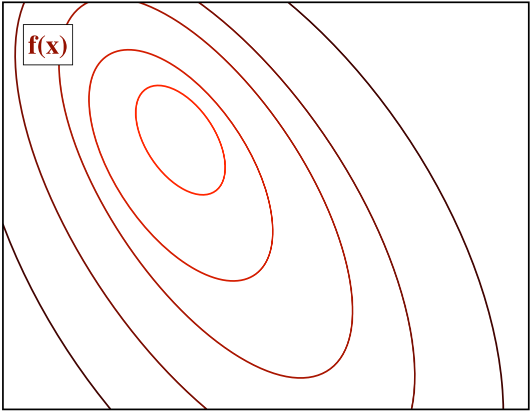
- H approximates the second-derivative matrix.
- L-BFGS is a particular strategy to choose the H values:
    - Based on gradient differences.
    - Linear storage and linear time.

`http://www.di.ens.fr/~mschmidt/Software/minFunc.html`

Gradient Method and Newton's Method



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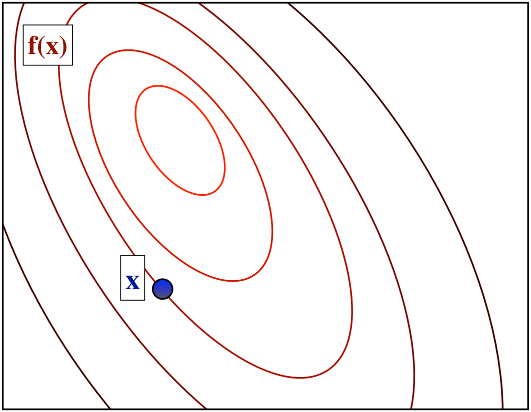

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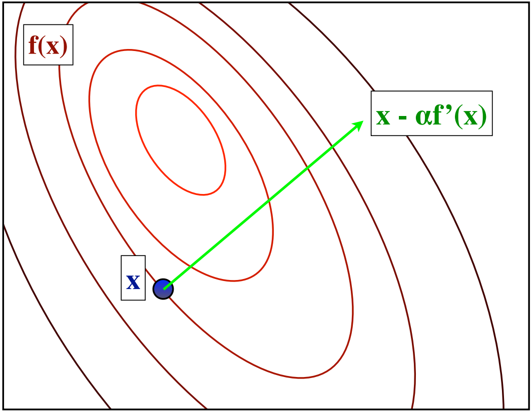

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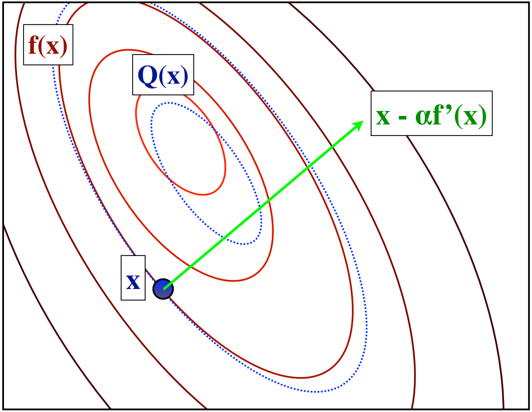

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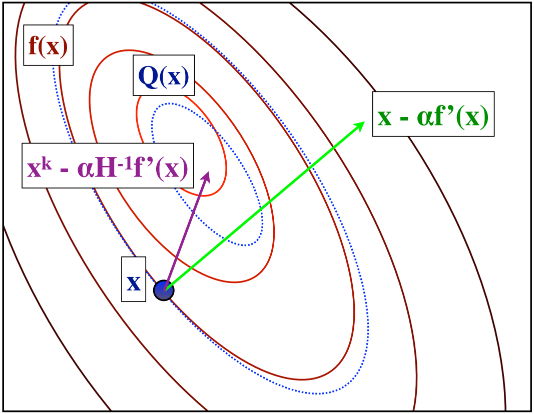

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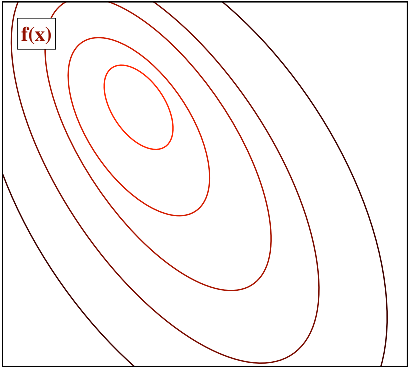

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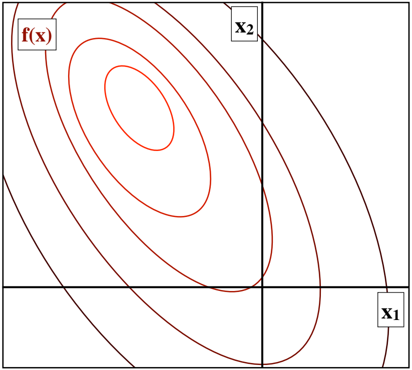

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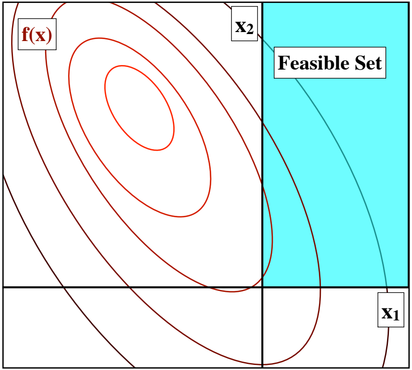

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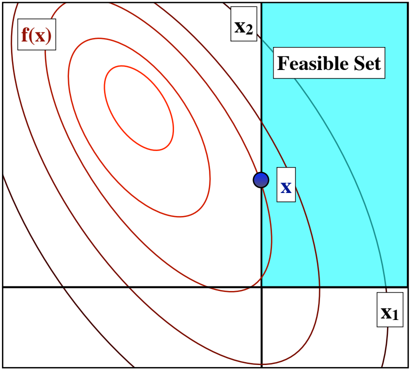

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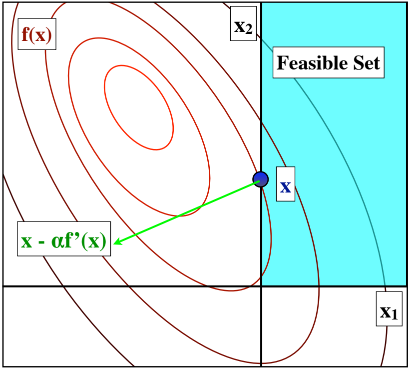

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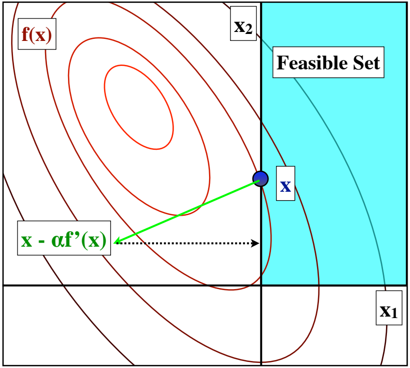

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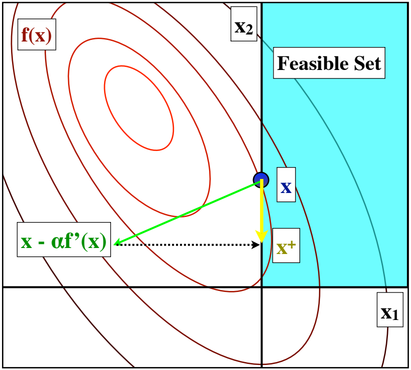

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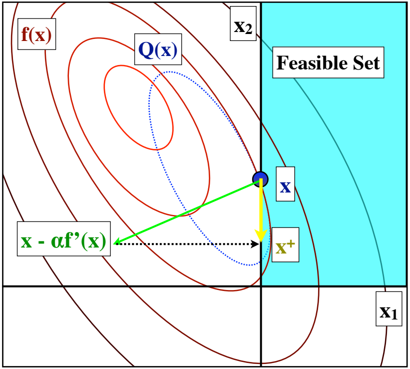

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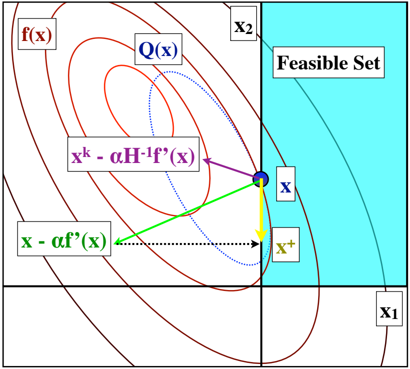

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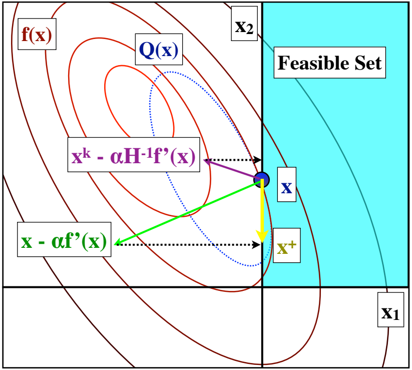

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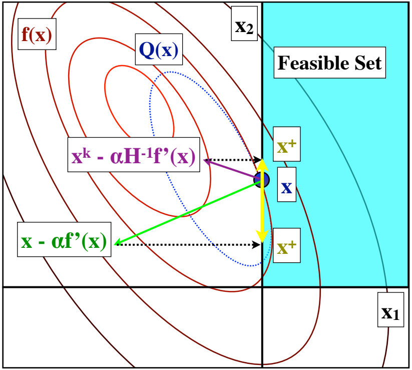


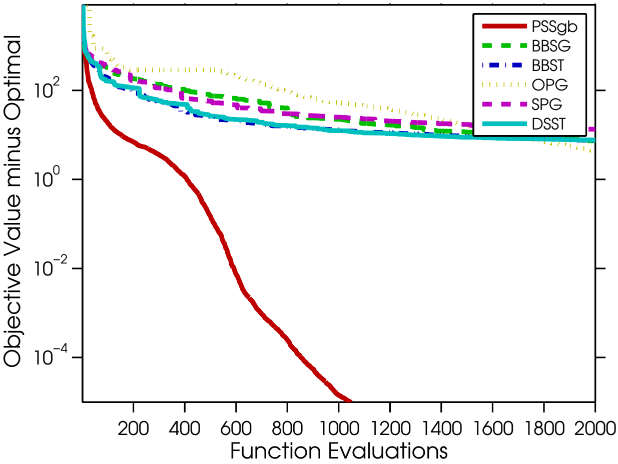
Two-Metric (Sub)Gradient Projection

- In some cases, we can modify H to make this work:
  - Bound constraints.
  - Probability constraints.
  - L1-regularization.
- Two-metric (sub)gradient projection.

  [Gafni & Bertskeas, 1984, Schmidt, 2010].
- Key idea: make H diagonal with respect to coordinates near non-differentiability.

Comparing to accelerated/spectral/diagonal gradient

Comparing to methods that do not use L-BFGS (sido data):



`http://www.di.ens.fr/~mschmidt/Software/L1General.html`

Inexact Proximal-Newton

- The broken proximal-Newton method:

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with the Euclidean proximal operator:

$$\text{prox}_{r}[y] = \mathop{\arg\min}_{x \in \mathbb{R}^P} \; r(x) + \frac{1}{2}\|x - y\|^2,$$

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- Solution: use a cheap approximate solution.

(e.g., spectral proximal-gradient)

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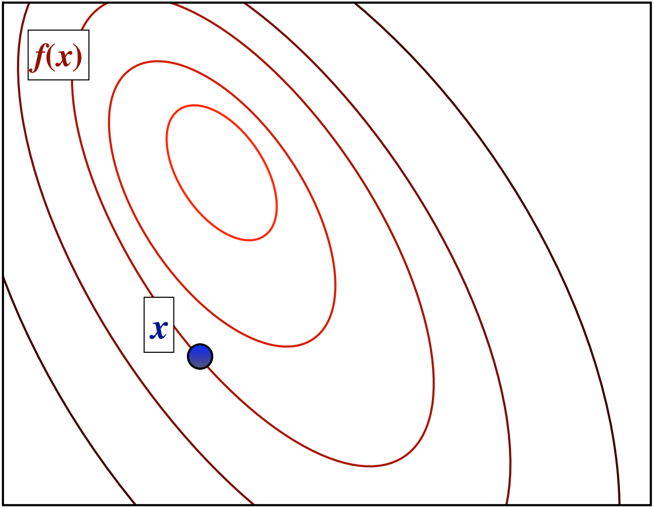

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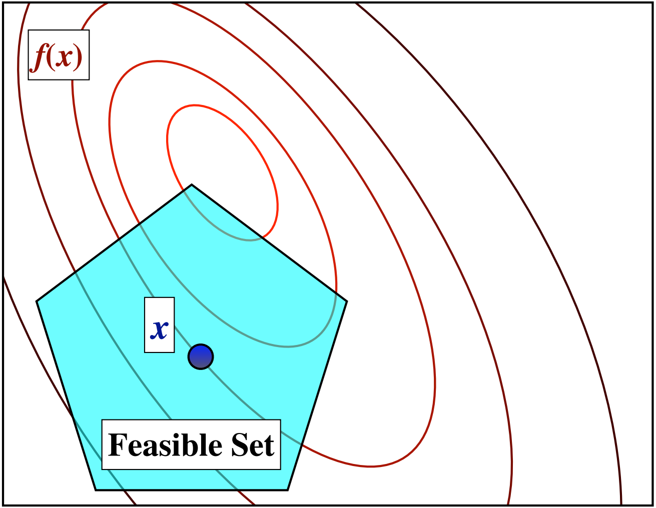

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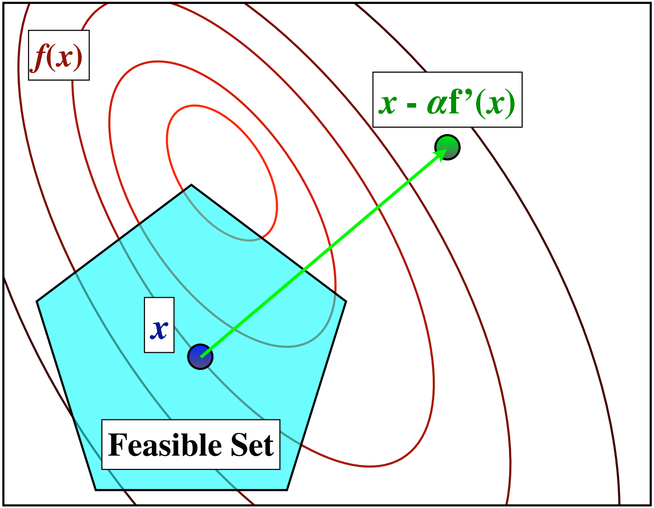

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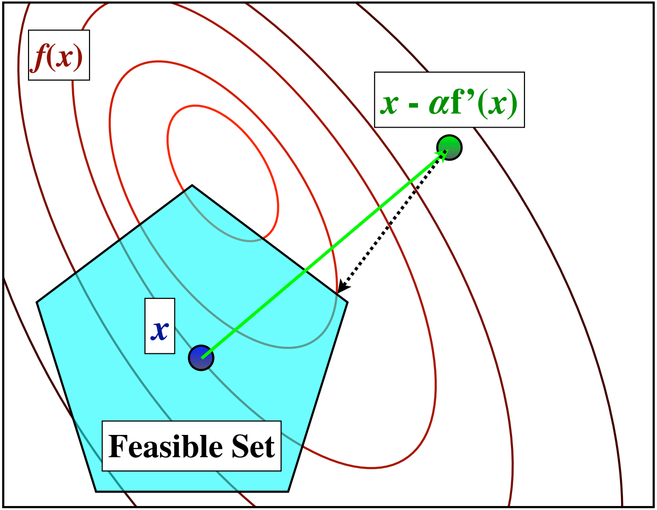

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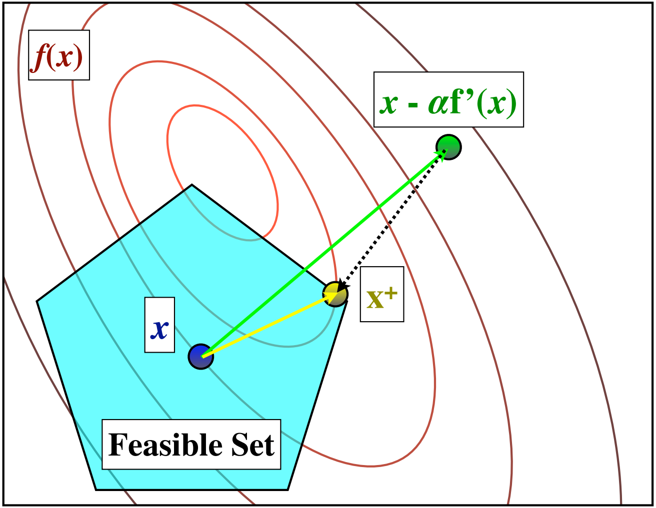

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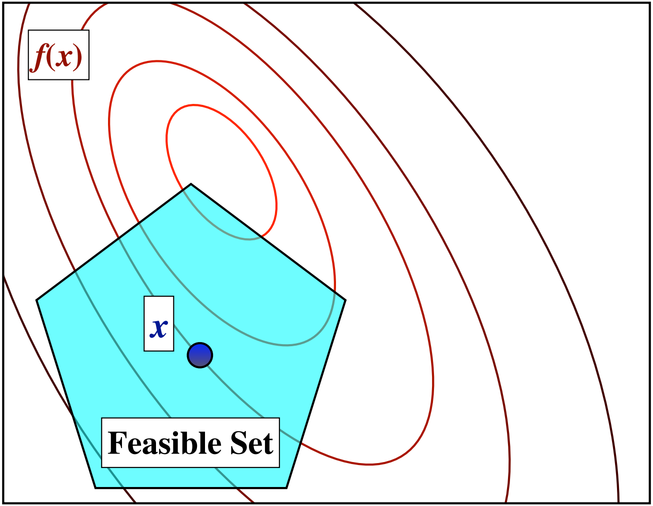

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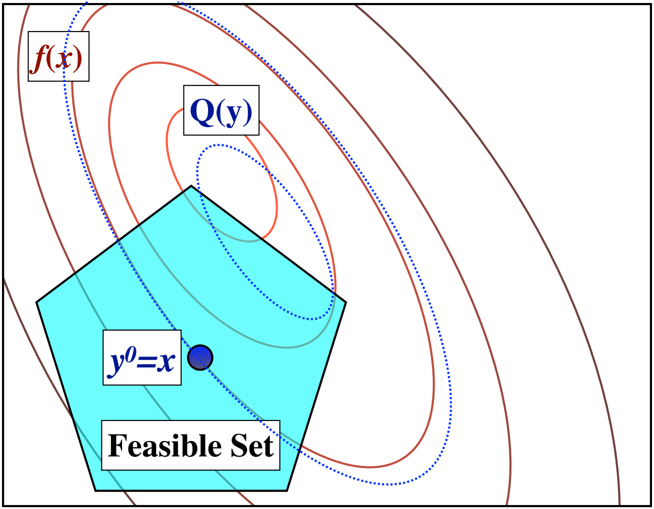

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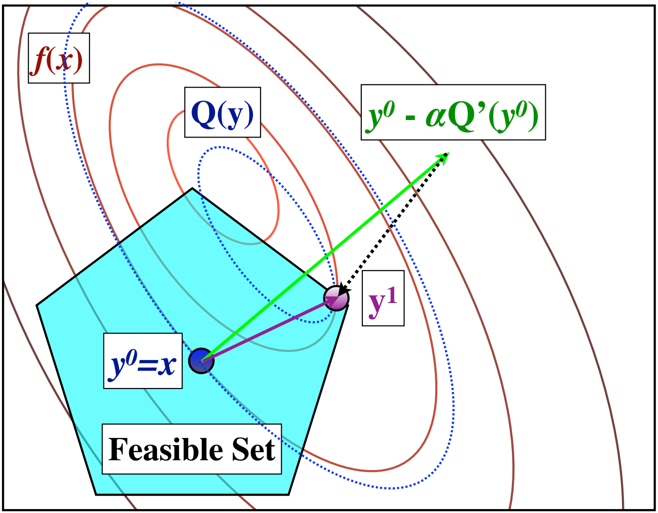

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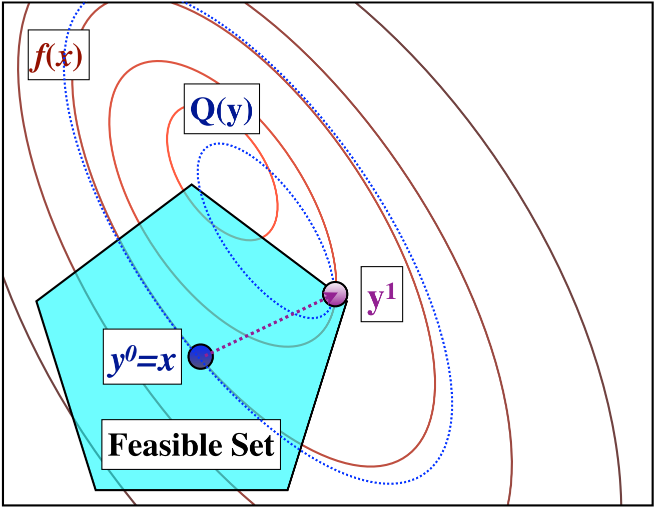

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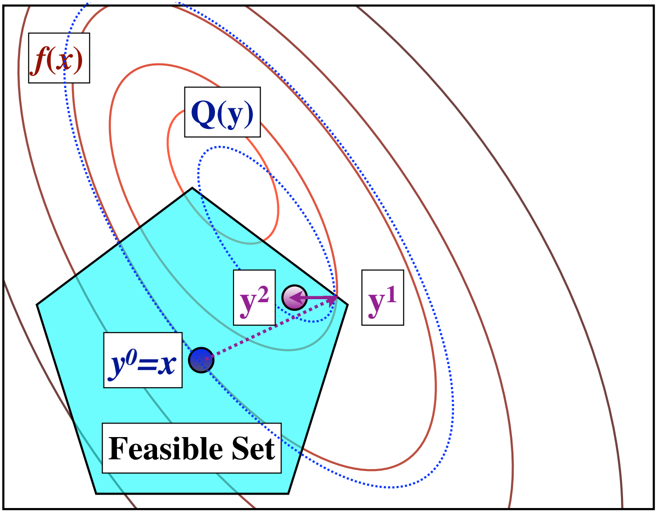

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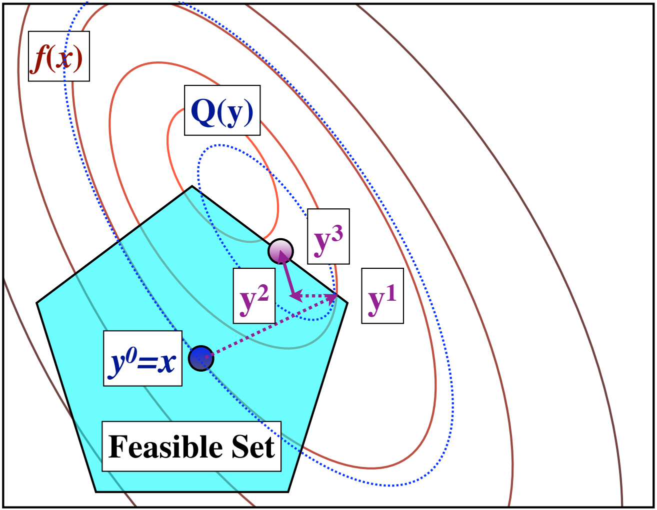

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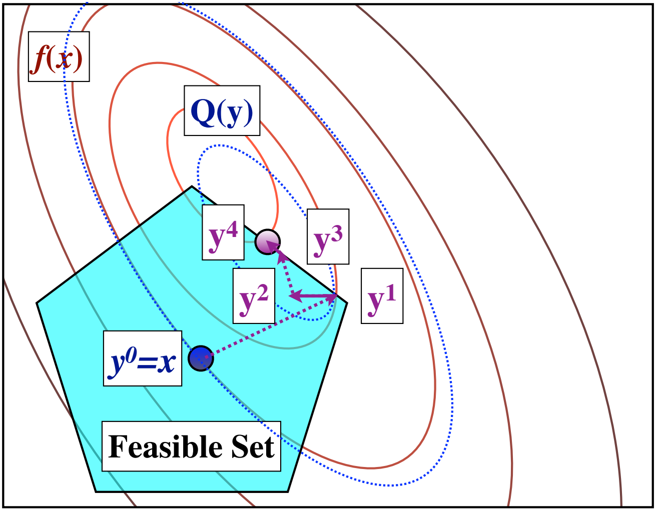

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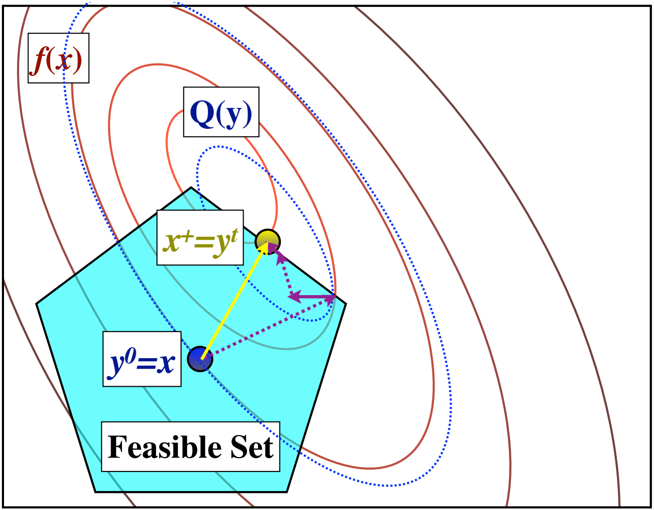

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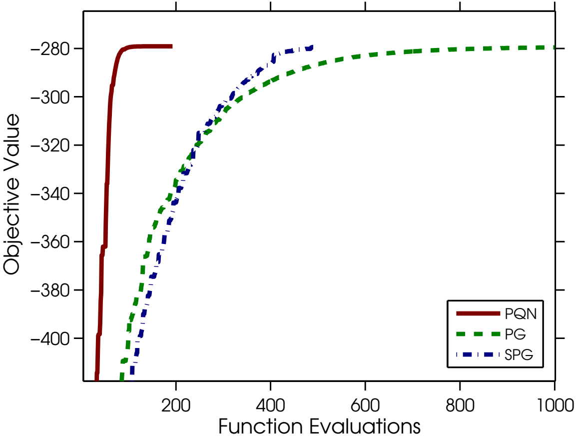
Graphical Model Structure Learning with Groups

Comparing PQN to first-order methods on a graphical model structure learning problem. [Gasch et al., 2000, Duchi et al., 2008].



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- If prox can not be computed exactly: Linearized ADMM.

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$$\min_{x} C \sum_{i=1}^{N} \max\{0, 1 - b_i a_i^T x\} + \frac{1}{2}\|x\|^2.$$

- SVM smooth dual:

$$\min_{0 \leq \alpha \leq C} \frac{1}{2}\alpha^T A A^T \alpha - \sum_{i=1}^{N} \alpha_i$$

- Smooth bound constrained problem:
  - Two-metric projection (efficient Newton-liked method).
  - Randomized coordinate descent (next section).

Discussion

- State of the art methods consider several other issues:
    - Shrinking: Identify variables likely to stay zero. [El Ghaoui et al., 2010].
    - Continuation: Start with a large λ and slowly decrease it. [Xiao and Zhang, 2012]
    - Frank-Wolfe: Using linear approximations to obtain efficient/sparse updates.

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- In some cases the projected gradient step

  $$x^+ = \mathop{\arg\min}_{y \in \mathcal{C}} \left\{ f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\alpha}\|y - x\|^2 \right\},$$

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- Iterate can be written as convex combination of vertices of $\mathcal{C}$.
- $O(1/t)$ rate for smooth convex objectives, some linear convergence results for smooth and strongly-convex.[Jaggi, 2013]

Outline

1 Convex Functions

2 Smooth Optimization

3 Non-Smooth Optimization

4 Randomized Algorithms

5 Parallel/Distributed Optimization

Big-N Problems

- We want to minimize the sum of a finite set of smooth functions:

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- Simple example is least-squares,

$$f_i(x) := (a_i^T x - b_i)^2.$$

- Other examples:
  - logistic regression, Huber regression, smooth SVMs, CRFs, etc.

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    - As in subgradient method, we require $\alpha_t \to 0$.
    - Classical choice is $\alpha_t = O(1/t)$.

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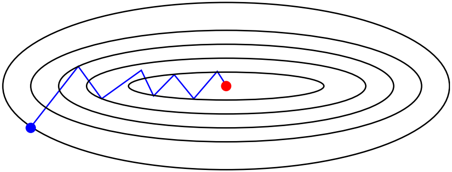

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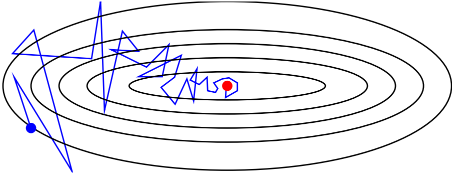

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- Bad news for smooth problems:
  - smoothness does not help stochastic methods.

Algorithm	Assumptions	Exact	Stochastic
Gradient	Convex	$O(1/t)$	$O(1/\sqrt{t})$
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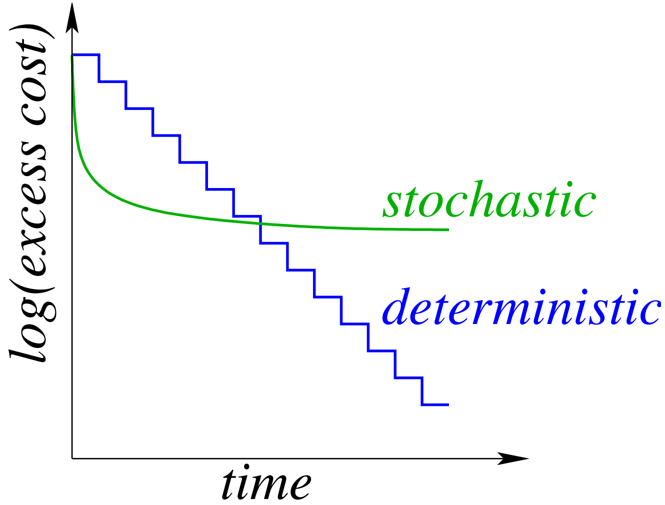
Deterministic vs. Stochastic Convergence Rates

Plot of convergence rates in smooth/strongly-convex case:



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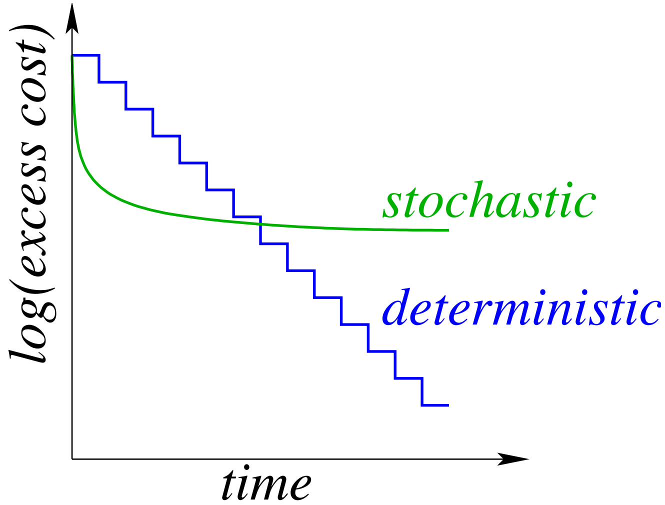
Motivation for Hybrid Methods for Smooth Problems



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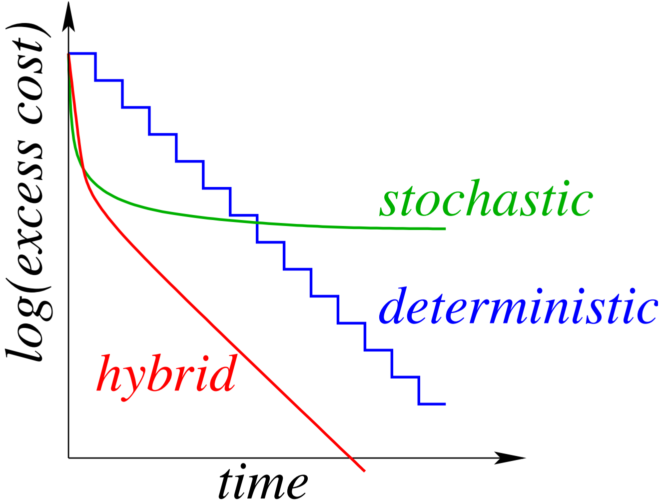

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  - Stochastic variant of increment average gradient (IAG). [Blatt et al., 2007]
  - Assumes gradients of non-selected examples don't change.
  - Assumption becomes accurate as $||x^{t+1} - x^t|| \to 0$.
  - Memory requirements reduced to $O(N)$ for many problems.

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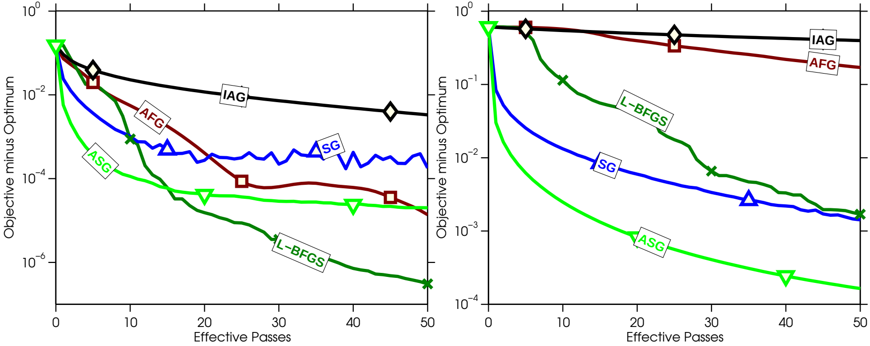
- L_i is the Lipschitz constant over all ∇f_i ($L_i \geq L$).
- SAG has a similar speed to the gradient method, but only looks at one training example per iteration.
- Recent work extends this result in various ways:
    - Similar rates for stochastic dual coordinate ascent. [Shalev-Schwartz & Zhang, 2013]
    - Memory-free variants. [Johnson & Zhang, 2013, Madavi et al., 2013]
    - Proximal-gradient variants. [Mairal, 2013]
    - ADMM variants. [Wong et al., 2013]
    - Improved constants. [Defazio et al., 2014]
    - Non-uniform sampling. [Schmidt et al., 2013]

Comparing FG and SG Methods

- quantum ($n = 50000$, $p = 78$) and rcv1 ($n = 697641$, $p = 47236$)



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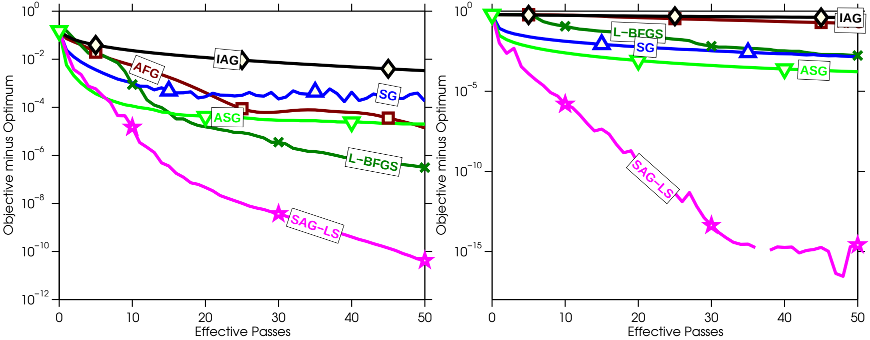

Coordinate Descent Methods

- Consider problems of the form

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- Appealing strategy for these problems is coordinate descent:

$$x_j{}^{+} = x_j - \alpha \nabla_j f(x).$$

  (i.e., update one variable at a time)

- We can typically perform a cheap and precise line-search for α.

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- May work quite badly if singular values decay slowly.

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Motivation for Parallel and Distributed

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Motivation for Parallel and Distributed

- Two recent trends:
  - We aren't making large gains in serial computation speed.
  - Datasets no longer fit on a single machine.
- Result: we must use parallel and distributed computation.
- Two major issues:
  - Synchronization: we can't wait for the slowest machine.
  - Communication: we can't transfer all information.

Embarassing Parallelism in Machine Learning

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- These allow optimal linear speedups.
  - You should always consider this first!

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- You need to decrease step-size in proportion to asynchrony.
- Convergence rate decays elegantly with delay m.[Niu et al., 2011]

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- Only needs to communicate single coordinates.
- Again need to decrease step-size for convergence.
- Speedup is based on density of graph.[Richtarik & Takac, 2013]

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- One solution: decentralized gradient method:
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    - Each processor has its own parameter vector x_c.
    - Each processor only communicates with a limited number of neighbours $\text{nei}(c)$.

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$$x_c = \frac{1}{|\text{nei}(c)|} \sum_{c' \in \text{nei}(c)} x_c - \frac{\alpha_c}{M} \sum_{i=1}^{M} \nabla f_i(x_c).$$

- Gradient descent is special case where all neighbours communicate.
- With modified update, rate decays gracefully as graph becomes sparse.[Shi et al., 2014]
- Can also consider communication failures.[Agarwal & Duchi, 2011]

Summary

Summary:

- Part 1: Convex functions have special properties that allow us to efficiently minimize them.
- Part 2: Gradient-based methods allow elegant scaling with problems sizes for smooth problems.
- Part 3: Tricks like proximal-gradient methods allow the same scaling for many non-smooth problems.
- Part 4: Randomized algorithms allow even further scaling for problem structures that commonly arise in machine learning.
- Part 5: The future will require parallel and distributed that are asynchronous and are careful about communication costs.

Thank you for coming and staying until the end!