The Renormalized Curvature Scale Space and the Evolution Properties of Planar Curves

Alan K. Mackworth† and Farzin Mokhtarian

Department of Computer Science
University of British Columbia
Vancouver, B.C., Canada V6T 1W5

Abstract

The Curvature Scale Space Image of a planar curve is computed by convolving a path-based parametric representation of the curve with a Gaussian function of variance σ^2, extracting the zeroes of curvature of the convolved curves and combining them in a scale space representation of the curve. For any given curve Γ, the process of generating the ordered sequence of curves $\{\Gamma_\sigma | \sigma \geq 0\}$ is the *evolution* of Γ. It is shown that the normalized arc length parameter of a curve is, in general, not the normalized arc length parameter of a convolved version of that curve. A new method of computing the curvature scale space image reparametrizes each convolved curve by its normalized arc length parameter. Zeroes of curvature are then expressed in that new parametrization. The result is the Renormalized Curvature Scale Space Image and is more suitable for matching curves similar in shape. Scaling properties of planar curves and the curvature scale space image are also investigated. It is shown that no new curvature zero-crossings are created at the higher scales of the curvature scale space image of a planar curve in C_1 if the curve remains in C_1 during evolution. Several positive and negative results are presented on the preservation of various properties of planar curves under the evolution process. Among these results is the fact that every polynomially represented planar curve in C_1 intersects itself just before forming a cusp point during evolution.

1. Introduction

In two previous papers (Mackworth and Mokhtarian, 1984) (Mokhtarian and Mackworth, 1986) we introduced a new shape representation for planar curves, the curvature scale space image, based on smoothing a path-based parametric representation of the curve. The representation has the property of being invariant to position and size changes of the curve and undergoes simple translations in response to changes in orientation and the level of detail provided (under certain conditions). It can also be used to recognize partially occluded and distorted versions of the curve using a coarse-to-fine optimal matching algorithm.

In this paper we explore, theoretically and experimentally, properties of the representation and present some new results. In particular, the main theoretical result is that the curvature scale space image is well-structured in the sense that, under certain assumptions, no new zeroes of curvature are introduced when the curve is convolved with a Gaussian function of arbitrary width. This result may appear to be a simple generalization of the one-dimensional result of (Babaud et al, 1986) and (Yuille and Poggio, 1986) but for reasons to be explained the obvious generalizations do not carry through.

† Fellow, Canadian Institute for Advanced Research

2. Path Representations of Planar Curves

A planar curve Γ is defined by a continuous, locally injective, vector mapping of an interval of $\mathbf{R}$ to $\mathbf{R}^2$ (Goetz, 1970). A curve is the set of points whose position vectors are the values of a continuous vector-valued function which is locally one-to-one. It is represented by the parametric vector equation

$$\mathbf{r}(u) = (x(u),\ y(u))$$

The vector function $\mathbf{r}(u)$ is a parametric representation of the curve, that is, a *path*. Any curve has an infinite number of distinct path representations. A *natural* path representation is one for which the parameter is the arc length s. If $\dot{\mathbf{r}}(u)$ exists and $|\dot{\mathbf{r}}(u)| > 0$ everywhere then there are no singular points on Γ and the curve is *regular*. For any parameterization $\mathbf{r}(u)$ for $u \in [a,b]$ of any regular curve, there is a conceptually, if not practically, straightforward reparametrization to one of a natural representation defined by the equation

$$s = \int_a^u |\dot{\mathbf{r}}(v)|\ dv$$

and the reparametrization is $\mathbf{r}(u(s))$.

The equivalent of the Frenet frame for a path representation for a regular planar curve is formed by the unit tangent vector $\mathbf{t}(u)$ and the orthogonal unit normal vector $\mathbf{n}(u)$ arranged in a right handed system. For any parametrization:

$$\dot{\mathbf{r}}(u) = (\dot{x}(u),\ \dot{y}(u))$$

$$\mathbf{t}(u) = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \left\{ \frac{\dot{x}}{(\dot{x}^2+\dot{y}^2)^{1/2}}\ ,\ \frac{\dot{y}}{(\dot{x}^2+\dot{y}^2)^{1/2}} \right\} \tag{1}$$

$$\mathbf{n}(u) = \left\{ \frac{-\dot{y}}{(\dot{x}^2+\dot{y}^2)^{1/2}}\ ,\ \frac{\dot{x}}{(\dot{x}^2+\dot{y}^2)^{1/2}} \right\} \tag{2}$$

For any planar curve the vectors $\mathbf{t}(u)$ and $\mathbf{n}(u)$ must satisfy the simplified Serret-Frenet vector equations:

$$\dot{\mathbf{t}}(u) = k(u)\mathbf{n}(u) \tag{3}$$

$$\dot{\mathbf{n}}(u) = -k(u)\mathbf{t}(u) \tag{4}$$

where $k(u)$, the curvature, uniquely characterizes the curve up to translation and rotation. From (3) we have

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$$k(u) = \dot{\mathbf{t}}(u)\cdot\mathbf{n}(u) \tag{5}$$

Differentiating (1):

$$\dot{\mathbf{t}}(u) = \left\{ \frac{-\dot{y}(\dot{x}\ddot{y} - \ddot{x}\dot{y})}{(\dot{x}^2+\dot{y}^2)^{3/2}}, \frac{\dot{x}(\dot{x}\ddot{y} - \ddot{x}\dot{y})}{(\dot{x}^2+\dot{y}^2)^{3/2}} \right\} \tag{6}$$

From (2), (5) and (6) we compute an expression for $k(u)$:

$$k(u) = \frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{3/2}}$$

Two special cases of the parametrization, of interest here, yield simplifications of these formulas. If we have a natural path representation with s, the arc length parameter, ranging over $[0,L]$ then:

$$|\,\dot{\mathbf{r}}(s)\,| = |\,(\dot{x}(s),\, \dot{y}(s))\,| = (\dot{x}^2(s)+\dot{y}^2(s))^{1/2} = 1$$

$$\mathbf{t}(s) = (\dot{x}(s),\, \dot{y}(s)) \qquad \dot{\mathbf{t}}(s) = (\ddot{x}(s),\, \ddot{y}(s))$$

$$\mathbf{n}(s) = (-\dot{y}(s),\, \dot{x}(s))$$

$$k(s) = \dot{\mathbf{t}}(s)\cdot\mathbf{n}(s) = \dot{x}(s)\ddot{y}(s) - \ddot{x}(s)\dot{y}(s)$$

Note also

$$k^2(s) = |\dot{\mathbf{t}}(s)|^2 = \ddot{x}^2(s) + \ddot{y}^2(s)$$

If the parameter is a linear rescaling of the arc length ranging over $[0,1]$, the normalized path length parameter w, then

$$w = \frac{s}{L} \qquad |\dot{\mathbf{r}}(w)| = L$$

$$\mathbf{t}(w) = \frac{1}{L}(\dot{x}(w),\, \dot{y}(w)) \qquad \mathbf{n}(w) = \frac{1}{L}(-\dot{y}(w),\, \dot{x}(w))$$

$$k(w) = \frac{1}{L^3}(\dot{x}(w)\ddot{y}(w)-\ddot{x}(w)\dot{y}(w))$$

and

$$k^2(w) = \frac{1}{L^4}(\ddot{x}^2(w)+\ddot{y}^2(w))$$

3. The Renormalized Curvature Scale Space

Following (Mokhtarian and Mackworth, 1986) a curve Γ is represented using the normalized arc length parameter w:

$$\Gamma = \{(x(w),\, y(w)) \mid w \in [0,1]\}$$

An *evolved* curve Γ_σ is defined by

$$\Gamma_\sigma = \{(X(u,\sigma),\, Y(u,\sigma)) \mid u \in [0,1]\}$$

where

$$X(u,\sigma) = x(u) \circledast g(u,\sigma) \qquad Y(u,\sigma) = y(u) \circledast g(u,\sigma)$$

and

$$g(u,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-u^2/2\sigma^2}$$

The curvature of Γ_σ is:

$$k(u,\sigma) = \frac{[X_u(u,\sigma)\,Y_{uu}(u,\sigma) - X_{uu}(u,\sigma)\,Y_u(u,\sigma)]}{(X_u(u,\sigma)^2 + Y_u(u,\sigma)^2)^{3/2}}. \tag{7}$$

The implicit function defined by

$$k(u,\sigma) = 0 \tag{8}$$

is the *curvature scale space image* of Γ (Mokhtarian and Mackworth, 1986). It is important to notice that, although w is the normalized arc length parameter for the original curve Γ, the parameter u is **not**, in general, the normalized arc length parameter for the evolved curve Γ_σ; however, u is a strictly increasing monotonic function of w, the normalized arc length parameter for Γ. The most general expression for $k(u,\sigma)$ must be used in (7).

For both theoretical and practical reasons, it is useful to reparametrize Γ_σ by its normalized arc length parameter w. Define

$$\mathbf{R}(u,\sigma) = (X(u,\sigma),\, Y(u,\sigma))$$

$$w = \Phi_\sigma(u)$$

where

$$\Phi_\sigma(u) = \frac{\int_0^u |\,\mathbf{R}_v(v,\sigma)\,|dv}{\int_0^1 |\mathbf{R}_v(v,\sigma)\,|dv}$$

then define

$$\hat{X}(w,\sigma) = X(\Phi_\sigma^{-1}(w),\sigma) \qquad \hat{Y}(w,\sigma) = Y(\Phi_\sigma^{-1}(w),\sigma) \tag{9}$$

Notice that

$$\Phi_\sigma(0) = 0 \qquad \Phi_\sigma(1) = 1$$

and

$$\frac{d\,\Phi_\sigma(u)}{du} = \frac{|\,\mathbf{R}_u(u,\sigma)\,|}{\int_0^1 |\,\mathbf{R}_v(v,\sigma)\,|\,dv} > 0 \text{ at non-singular points}$$

Also

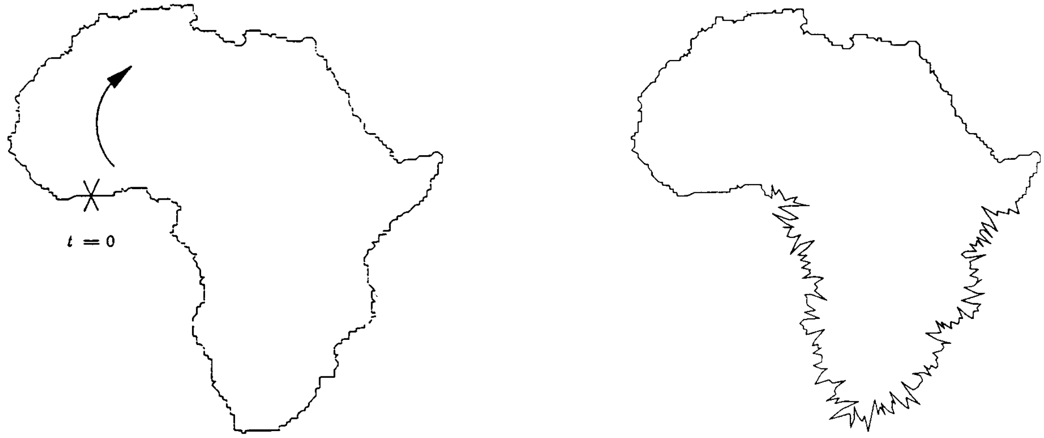
$$\Phi_0(u) = u$$

$\Phi_\sigma(u)$ deviates from the identity function $\Phi_\sigma(u) = u$ only to the extent to which the scale-related statistics deviate from stationarity along the original curve. Once we have changed parameters according to equations (9) then the curvature for the normalized path length parameters is:

$$k(w,\sigma) = \frac{1}{L^3}[\hat{X}_w(w,\sigma)\,\hat{Y}_{ww}(w,\sigma) - \hat{X}_{ww}(w,\sigma)\,\hat{Y}_w(w,\sigma)]$$

We now define the *renormalized curvature scale space image* of Γ to be the implicit function defined by $k(w,\sigma) = 0$.

As an example of these concepts we show the coastline of Africa in figure 1(a) and the curvature scale space and the renormalized curvature scale space of Africa in Figure 2(a) and 2(b) respectively. The difference between them is almost negligible. However, if part of the curve has radically different scale related phenomena than the remainder the difference is more important. In Figure 1(b) we have added considerable noise to half the contour in the normal direction and shown the corresponding curvature scale space images in figures 2(c) and 2(d) for comparison. Notice that the position of the major contours change substantially from Fig. 2(a) to Fig. 2(c) but are essentially unchanged from Fig. 2(b) to Fig. 2(d). This property of renormalized curvature scale space enhances its utility for shape matching of similar curves. The renormalization process corresponds to a continuous non-linear horizontal shearing of curvature scale space. Figure 3 shows a plot of renormalized arc length parameter on evolved Africa (σ=8) versus normalized arc length



(a) Coastline of Africa. (b) Coastline of Africa with added noise.

Figure 1. Two planar curves used as test data.

parameter of original Africa. As mentioned earlier, w on the evolved curve is an increasing monotonic function of u on the original curve.

4. Scaling Properties of the Curvature Scale Space

For the scale space image of a one dimensional function $f(s)$ an important monotonic property first observed and then proven is that zero crossings are never created as scale increases (Babaud *et al*, 1986; Yuille and Poggio, 1986). The generalization of the monotonic property to smoothing of two dimensional images is complex. Although a variant of the monotonic zero crossing property holds in that zero crossings are never created as scale increases, they do split and merge and in a cross section of the scale space the 1-D property may appear to be violated (Yuille and Poggio, 1986; Babaud *et al*, 1986). Accordingly it is by no means apparent that the extension of the monotonic property holds for curvature scale space. Moreover, if it does hold it may offer an advantage for path based smoothing of 2-D contours over direct 2-D smoothing of images. We now arrive at the main theoretical result of this paper.

Theorem 1: Let Γ be a planar curve in class C_1 and let Γ_σ be the evolved version of Γ. If all curves Γ_σ are in C_1, then all extrema occurring at regular points on contours in the curvature scale space image of Γ are maxima.

Proof: The proof will be carried out in the original curvature scale space but, since curvature is coordinate-frame invariant, the theorem also holds in renormalized curvature scale space.

By (8) on any contour in curvature scale space

$$k(u,\sigma) = 0$$

By (7) and the fact that all Γ_σ are in C_2 this is equivalent to:

$$\dot{X}(u,\sigma)\,\ddot{Y}(u,\sigma) - \ddot{X}(u,\sigma)\,\dot{Y}(u,\sigma) = 0$$

To exploit the properties of the heat equation (Hummel *et al*, 1987), it is convenient to change variables and let

$$t = \frac{1}{2}\sigma^2$$

so we define

$$x(u,t) = X(u,\sigma) \qquad y(u,t) = Y(u,\sigma)$$

$$\alpha(u,t) = x_u(u,t)\,y_{uu}(u,t) - x_{uu}(u,t)\,y_u(u,t) \tag{10}$$

The functions $x(u,t)$ and $y(u,t)$ are obtained by convolving $\frac{1}{\sqrt{4\pi t}} e^{-(1/4t)u^2}$ with the original curve coordinates $x(u)$ and $y(u)$ respectively, and so they satisfy the heat equation:

$$x_{uu}(u,t) = x_t(u,t) \tag{11}$$

$$y_{uu}(u,t) = y_t(u,t) \tag{12}$$

The implicit function theorem is satisfied on contours $\alpha(u,t) = 0$ because of the assumption that all Γ_σ are smooth. Therefore we can write:

$$t = t(u) \quad \text{and} \quad \dot{t}(u) = \frac{dt}{du} = \frac{-\alpha_u}{\alpha_t}$$

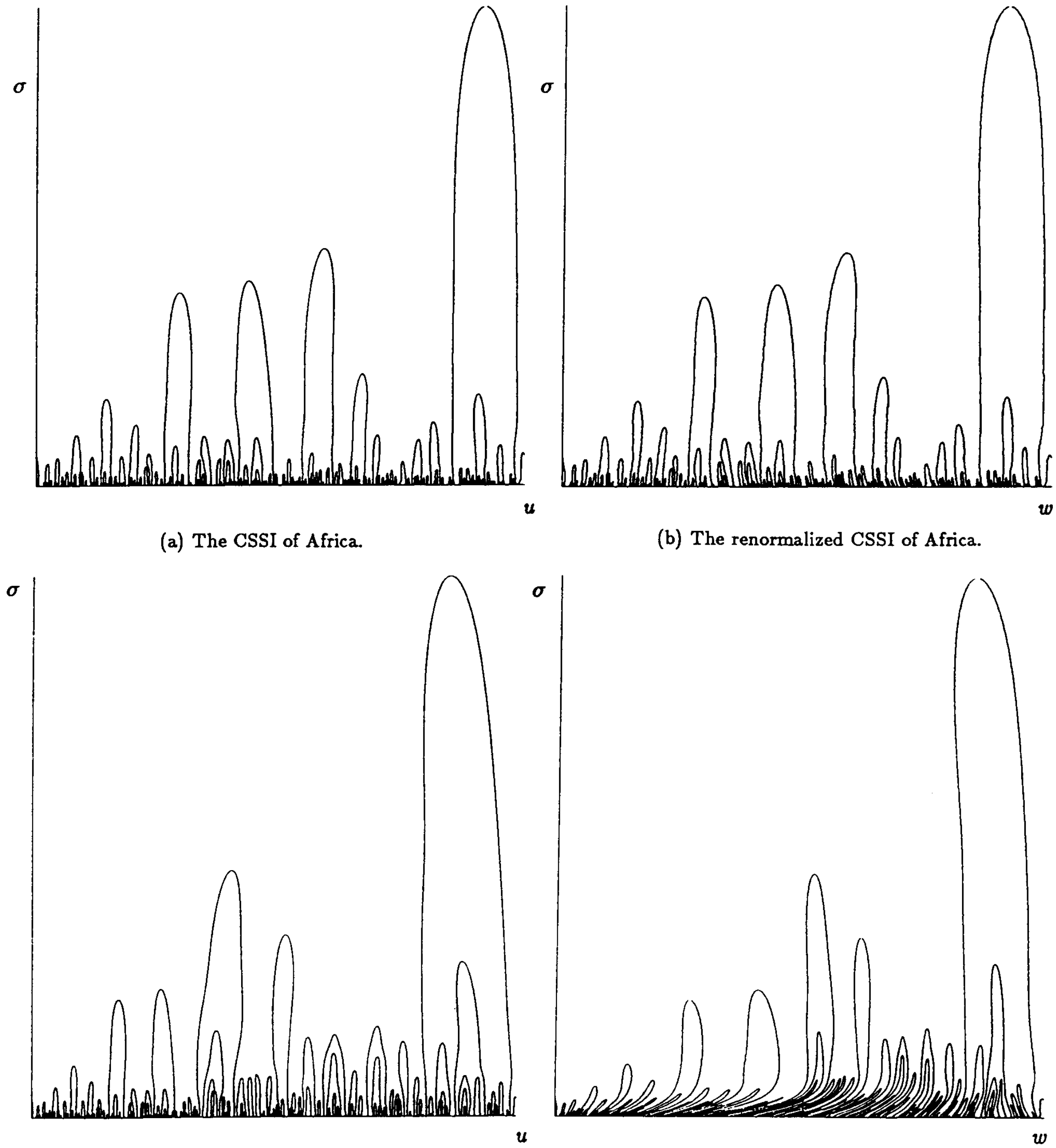
The theorem will be proven if we can show that for all points such that $\dot{t}(u) = 0$ we have $\ddot{t}(u) < 0$. Observe that

$$\dot{t}(u) = 0 \quad \text{iff} \quad \alpha_u(u,t) = 0 \tag{13}$$

At an extremum where (13) holds, we have

$$\ddot{t}(u) = \frac{d}{du}\left(\frac{-\alpha_u}{\alpha_t}\right) = \frac{\partial}{\partial u}\left(\frac{-\alpha_u}{\alpha_t}\right) + \frac{\partial}{\partial t}\left(\frac{-\alpha_u}{\alpha_t}\right)\frac{dt}{du} = \frac{-\alpha_{uu}}{\alpha_t}$$

So we must show that if $\alpha(u,t) = \alpha_u(u,t) = 0$ then



(a) The CSSI of Africa.

(b) The renormalized CSSI of Africa.

(c) The CSSI of noisy Africa.

(d) The renormalized CSSI of noisy Africa.

Figure 2. A comparison of regular and renormalized curvature scale space images.

$\dfrac{\alpha_{uu}}{\alpha_t} > 0$. We shall show that these conditions require $\dfrac{\alpha_{uu}}{\alpha_t} = 1$ which proves the theorem. From (10), (11) and (12) we have

$$\alpha = x_u y_t - x_t y_u$$

$$\alpha_u = x_{uu} y_t + x_u y_{ut} - x_{ut} y_u - x_t y_{uu}$$

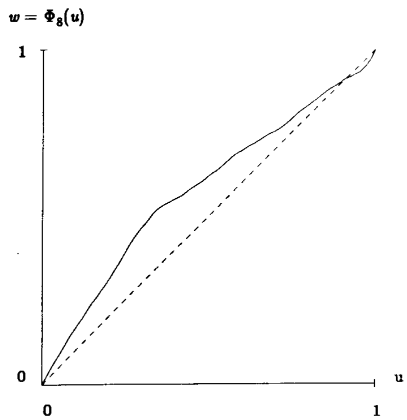


Figure 3. Plot of renormalized arc length on evolved Africa (σ=8) vs. arc length of original Africa.

But using (11)

$$\alpha_u = x_u y_{ut} - x_{ut} y_u \tag{14}$$

Similarly

$$\alpha_{uu} = (x_u y_{tt} - x_{tt} y_u) + (x_t y_{ut} - x_{ut} y_t)$$

If $\alpha = \alpha_u = 0$ then using (10) and (14)

$$x_t y_{ut} - x_{ut} y_t = x_t\left(y_{ut} - x_{ut}\frac{y_t}{x_t}\right) = x_t\left(y_{ut} - x_{ut}\frac{y_u}{x_u}\right) = \frac{x_t}{x_u}(x_u y_{ut} - x_{ut} y_u) = 0$$

so

$$\alpha_{uu} = x_u y_{tt} - x_{tt} y_u$$

We also have

$$\alpha_t = (x_u y_{tt} - x_{tt} y_u) - (x_t y_{ut} - x_{ut} y_t)$$

so

$$\alpha_t = x_u y_{tt} - x_{tt} y_u$$

and hence $\alpha_{uu} = \alpha_t$ as claimed. □

Notice, incidentally, that $\alpha(u,t)$ satisfies the diffusion equation at the maxima of the contours and that all such contours have a curvature of –1 at their maxima in (u,t) curvature scale space.

5. Some planar curves and their scaling properties

In this section we will investigate some of the scaling properties of planar curves. Sub-section **5.1** contains a number of theoretical results on scaling properties of planar curves and sub-section **5.2** contains some observations and examples of behavior of a few planar curves during evolution.

5.1. Theoretical results

We first present three lemmas concerning fundamental properties of evolution of planar curves.

Lemma 1: Evolution is invariant under rotation, uniform scaling and translation of the curve.

Proof: We will show that evolution is invariant under a general affine transform which includes transformations consisting of rotation, uniform scaling and translation.

Let $\Gamma = (x(u),\ y(u))$ be a planar curve and let $\Gamma_\sigma = (X(u,\sigma),\ Y(u,\sigma))$ be its evolved version. If Γ_σ is transformed according to an affine transform, then the following relationships hold between its old coordinates, $X(u,\sigma)$ and $Y(u,\sigma)$, and its new coordinates, $x_1(u,\sigma)$ and $y_1(u,\sigma)$:

$$x_1(u,\sigma) = a X(u,\sigma) + b\, Y(u,\sigma) + c$$
$$y_1(u,\sigma) = d X(u,\sigma) + e\, Y(u,\sigma) + f$$

Now suppose Γ is transformed according to an affine transform and then evolved. The coordinates, $x_2(u)$, $y_2(u)$ of the new curve are:

$$x_2(u,\sigma) = (a x(u) + b y(u) + c) \circledast g(u,\sigma)$$

$$y_2(u,\sigma) = (d x(u) + e y(u) + f) \circledast g(u,\sigma)$$

Because the convolution operator is distributive [Kecs 1982], it follows that

$$x_2(u,\sigma) = a\,(x(u) \circledast g(u,\sigma)) + b\,(y(u) \circledast g(u,\sigma)) + c = x_1(u,\sigma)$$

$$y_2(u,\sigma) = d\,(x(u) \circledast g(u,\sigma)) + e\,(y(u) \circledast g(u,\sigma)) + f = y_1(u,\sigma)$$

Note that this result holds for any convolution operator not just the Gaussian. □

Lemma 2: A connected planar curve remains connected during evolution.

Proof: Let $\Gamma = (x(u),\ y(u))$ be a connected planar curve and let $\Gamma_\sigma = (X(u,\sigma),\ Y(u,\sigma))$ be its evolved version. We will show that Γ_σ is also a connected curve.

Since Γ is connected, $x(u)$, $y(u)$ and therefore $X(u,\sigma)$ and $Y(u,\sigma)$ are continuous functions. Let u_0 be any value of parameter u and let x_0 and y_0 be the values of $X(u,\sigma)$ and $Y(u,\sigma)$ at u_0 respectively. It follows that if u goes through an infinitesimal change, then $X(u,\sigma)$ and $Y(u,\sigma)$ will also go through infinitesimal changes

$$X(u_0,\sigma) \rightarrow x_0 + \delta \qquad Y(u_0,\sigma) \rightarrow y_0 + \xi$$

As a result, point $P(x_0,\ y_0)$ on Γ_σ goes to point $P'(x_0 + \delta,\ y_0 + \xi)$. Let the distance between P and P' be D. Then

$$D = \sqrt{(x_0 + \delta - x_0)^2 + (y_0 + \xi - y_0)^2} = \sqrt{\delta^2 + \xi^2} \le \delta\sqrt{2}.$$

assuming $|\delta|$ is the larger of $|\delta|$ and $|\xi|$. It follows that an infinitesimal change in parameter u also results in an infinitesimal change in the value of the vector-valued function Γ_σ. Therefore Γ_σ is a connected curve. □

Lemma 3: A closed planar curve remains closed during evolution.

Proof: A closed curve has $(x(0), y(0)) = (x(1), y(1))$. It follows that $(X(0,\sigma),\ Y(0,\sigma)) = (X(1,\sigma),\ Y(1,\sigma))$. □

If one smoothes the curvature function (Asada and Brady, 1986) then closed curves apparently may not remain closed (Horn and Weldon, 1986).

The next theorem concerns a property of all planar curves during evolution.

Theorem 2: Let $\Gamma = (x(u),\ y(u))$ be a planar curve in C_1 and let $x(u)$ and $y(u)$ be polynomial functions of u. Let $\Gamma_\sigma = (X(u,\sigma),\ Y(u,\sigma))$ be the first evolved version of Γ with a cusp point at u_0, then there is a $\delta > 0$ such that $\Gamma_{\sigma-\delta}$ intersects itself in a neighborhood of point u_0.

Proof: Since the class of polynomial functions is closed under convolution with a Gaussian [Hummel *et al.*, 1987], it follows that $X(u,\sigma)$ and $Y(u,\sigma)$ are also polynomial functions:

$$X(u,\sigma) = a_0 + a_1 u + a_2 u^2 + a_3 u^3 + \cdots$$

$$Y(u,\sigma) = b_0 + b_1 u + b_2 u^2 + b_3 u^3 + \cdots$$

Suppose that Γ_σ goes through the origin of the coordinate system at $u_0 = 0$. It follows that $a_0 = b_0 = 0$. Assume further that there is a singularity on Γ_σ at u_0. We have:

$$X_u(u,\sigma) = a_1 + 2a_2 u + 3a_3 u^2 + 4a_4 u^3 + \cdots$$

$$Y_u(u,\sigma) = b_1 + 2b_2 u + 3b_3 u^2 + 4b_4 u^3 + \cdots$$

Since $X_u(u,\sigma)$ and $Y_u(u,\sigma)$ are zero at a singular point, it also follows that $a_1 = b_1 = 0$. We will now perform a case analysis of the singular point at u_0 to determine when it corresponds to a cusp point. Since we will examine a small neighborhood of point u_0, we will approximate the curve using the lowest degree terms in $X(u,\sigma)$ and $Y(u,\sigma)$:

$$\Gamma_\sigma = (u^m,\ u^n)$$

Assume w.l.o.g. that $n > m$. From above we know that $m > 1$. Using

$$\mathbf{r}_u(u,\sigma) = (X_u(u,\sigma),\ Y_u(u,\sigma)) = (m u^{m-1},\ n u^{n-1})$$

it follows that

$$\mathbf{r}_u(\epsilon,\sigma) = (m\epsilon^{m-1},\ n\epsilon^{n-1}) = \epsilon^{m-1}(m,\ n\epsilon^{n-m})$$

and

$$\mathbf{r}_u(-\epsilon,\sigma) = (m(-\epsilon)^{m-1},\ n(-\epsilon)^{n-1})$$

We can now analyze the singular point in each of the four possible cases:

1. m and n are both even: $m-1$ and $n-1$ are both odd. So

$$\mathbf{r}_u(-\epsilon,\sigma) = (-m\epsilon^{m-1},\ -n\epsilon^{n-1}) = -\epsilon^{m-1}(m,\ n\epsilon^{n-m})$$

A comparison of $\mathbf{r}_u(\epsilon,\sigma)$ and $\mathbf{r}_u(-\epsilon,\sigma)$ shows that an infinitesimal change in the parameter u results in a large change in the direction of the tangent vector. Therefore the singular point *is* also a cusp point in this case.

2. m and n are both odd: $m-1$ and $n-1$ are both even. So

$$\mathbf{r}_u(-\epsilon,\sigma) = (m\epsilon^{m-1},\ n\epsilon^{n-1}) = \epsilon^{m-1}(m,\ n\epsilon^{n-m})$$

Comparing $\mathbf{r}_u(\epsilon,\sigma)$ to $\mathbf{r}_u(-\epsilon,\sigma)$ now shows that the tangent direction does not change with u in a small neighborhood of the singular point. Therefore this singular point is *not* a cusp point.

3. m is odd and n is even: $m-1$ is even and $n-1$ is odd. So

$$\mathbf{r}_u(-\epsilon,\sigma) = (m\epsilon^{m-1},\ -n\epsilon^{n-1}) = \epsilon^{m-1}(m,\ -n\epsilon^{n-m})$$

An infinitesimal change in u also results in an infinitesimal change in the tangent direction. Again, this singular point is *not* a cusp point.

4. m is even and n is odd: $m-1$ is odd and $n-1$ is even. So

$$\mathbf{r}_u(-\epsilon,\sigma) = (-m\epsilon^{m-1},\ n\epsilon^{n-1}) = \epsilon^{m-1}(-m,\ n\epsilon^{n-m})$$

An infinitesimal change in u now results in a large change in the tangent direction. Therefore this singular point *is* a cusp point.

It follows from the case analysis above that only the singular points in cases **1** and **4** are cusp points. We will now derive analytical expressions for the curve $\Gamma_{\sigma-\delta}$ so that it can be analyzed in a small neighborhood of the cusp point.

To deblur function $f(u) = u^k$, we convolve a rescaled version of that function with the function $\frac{2}{\sqrt{\pi}} e^{-v^2}(1-v^2)$, an approximation to the deblurring operator derived in (Hummel *et al.*, 1987), as follows:

$$(D_t f)(u) = \int_{-\infty}^{\infty} \frac{2}{\sqrt{\pi}} e^{-v^2}(1-v^2) f(u + 2v\sqrt{t})\, dv$$

or

$$(D_t f)(u) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2}(1-v^2)(u + 2v\sqrt{t})^k dv$$

where t is the scale factor and controls the amount of deblurring. Note that our approximation is good for small t. Solving the integral above yields

$$(D_t f)(u) = \sum_{\substack{p=0 \\ (p\ even)}}^{k} 1.3 \ldots (p-1) \frac{(2t)^{p/2} k(k-1)\ldots(k-p+1)}{p!} (1-p) u^{k-p} \qquad (15)$$

The following are four functions of the form $f(u) = u^k$ and their deblurred versions:

a. $f(u) = u^2$ $\quad (D_t f)(u) = u^2 - 2t$
b. $f(u) = u^3$ $\quad (D_t f)(u) = u^3 - 6tu$
c. $f(u) = u^4$ $\quad (D_t f)(u) = u^4 - 12tu^2 - 36t^2$
d. $f(u) = u^5$ $\quad (D_t f)(u) = u^5 - 20tu^3 - 180t^2 u$

We can now analyze the cusp points identified in cases **1** and **4** above. In case **1**, the curve Γ_σ is approximated by $(u^m,\ u^n)$ where m and n are both even numbers. We now deblur the curve to obtain:

$$(D_t x)(u) = u^m - c_1 t u^{m-2} - \cdots - c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^2 - c_{\frac{m}{2}} t^{\frac{m}{2}}$$

$$(D_t y)(u) = u^n - c'_1 t u^{n-2} - \cdots - c'_{\frac{n-2}{2}} t^{\frac{n-2}{2}} u^2 - c'_{\frac{n}{2}} t^{\frac{n}{2}}$$

Note that all powers of u are even and the constants c_j and c'_j are all positive as follows from an examination of (15). It follows that

$$(D_t\dot{x})(u) = mu^{m-1} - (m-2)c_1tu^{m-3} - \cdots - 2c_{\frac{m-2}{2}}t^{\frac{m-2}{2}}u$$

$$(D_t\dot{y})(u) = nu^{n-1} - (n-2)c'_1tu^{n-3} - \cdots - 2c'_{\frac{n-2}{2}}t^{\frac{n-2}{2}}u$$

contain only odd powers of u and $(D_t\dot{\mathbf{r}})(\epsilon) = -(D_t\dot{\mathbf{r}})(-\epsilon)$. Hence there is also a cusp point on the curve $\Gamma_{\sigma-\delta}$ at u_0. This is a contradiction of the assumption that Γ_σ is the first evolved version of Γ with a cusp at u_0. It follows that Γ_σ can not have a cusp point of this kind at u_0.

We shall now turn to the cusp points encountered in case **4**. Recall that, in that case, the curve Γ_σ is approximated, in a small neighborhood of the cusp point, by (u^m,u^n) where m is even and n is odd. Again we deblur the curve to obtain:

$$(D_tx)(u) = u^m - c_1tu^{m-2} - \cdots - c_{\frac{m-2}{2}}t^{\frac{m-2}{2}}u^2 - c_{\frac{m}{2}}t^{\frac{m}{2}}$$

$$(D_ty)(u) = u^n - c'_1tu^{n-2} - c'_2t^2u^{n-4} - \cdots - c'_{\frac{n-1}{2}}t^{\frac{n-1}{2}}u$$

Again note that constants c_j and c'_j are all positive. The deblurred curve intersects itself if there are two values of u, u_1 and u_2, such that

$$x(u_1) = x(u_2) \quad (16)$$

$$y(u_1) = y(u_2) \quad (17)$$

Since $(D_tx)(u)$ contains even powers of u only, it follows from (16) that $u_1 = -u_2$. Since $(D_ty)(u)$ contains odd powers of u only, substituting $u_1 = -u_2$ in (17), letting $t = \delta$, and simplifying yields:

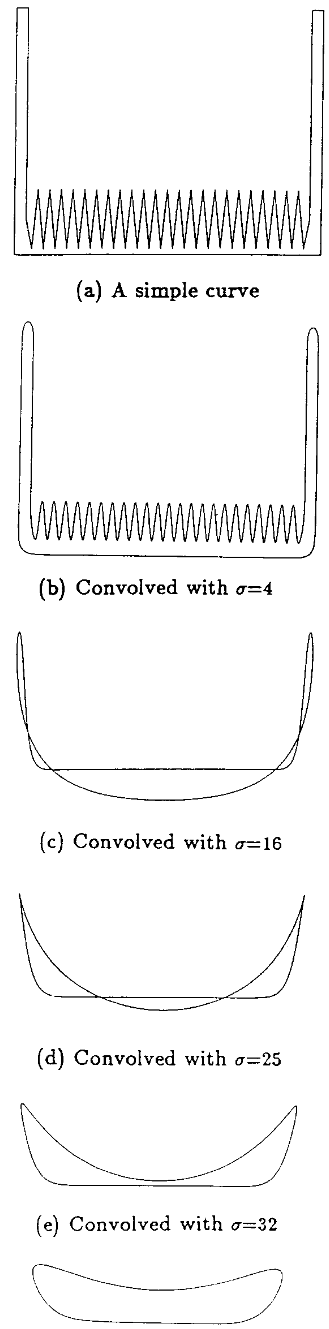
$$u_1^n - c'_1\delta u_1^{n-2} - c'_2\delta^2u_1^{n-4} - \cdots - c'_{\frac{n-1}{2}}\delta^{\frac{n-1}{2}}u_1 = 0 \quad (18)$$

$u_1=0$ is one of the roots of this equation. For very small values of u_1, the LHS of (18) is negative since the first term will be smaller than each of the other terms (which are negative). As u_1 grows larger, the first term becomes larger than the sum of all other terms and therefore the LHS of (18) becomes positive. It follows that there exists a positive value of u_1 at which (18) is satisfied. Therefore $\Gamma_{\sigma-\delta}$ is self-intersecting in a small neighborhood of the cusp point of Γ_σ. □

Theorem **2** is of practical importance. If one can demonstrate that a curve remains simple during evolution then it follows from this theorem that it can not develop a cusp.

5.2. Observations and examples

While most simple curves remain in C_1 during evolution, some with very irregular shapes may not show that property. Figure 4 shows a simple curve which forms a pair of cusp points during evolution. This is an important curve because it is a counter-example to the hypothesis that the class of simple curves is closed under evolution.



(a) A simple curve

(b) Convolved with σ=4

(c) Convolved with σ=16

(d) Convolved with σ=25

(e) Convolved with σ=32

(f) Convolved with σ=48

Figure 4. A simple curve which crosses itself and then develops cusps during evolution.

It serves as an illustration of the need for the careful wording of the statement of Theorem **1** and as an example of the situation covered by Theorem **2**.

The class of self-crossing curves also has some members which remain in C_1 during evolution and other members which develop cusps during evolution. Figure 5 shows a self-crossing curve which forms a cusp point during evolution. In this example, the formation of the cusp is followed by the creation of two new zero-crossings which eventually disappear as the curve evolves further. Figure 6 shows a convex but self-intersecting curve which also forms a cusp during evolution.

Convex curves are those planar curves which have positive curvature at every point. Gage and Hamilton [1986] have shown that simple convex curves remain simple and convex during evolution and tend to a circle. This fact demonstrates that we are correct in ending the curvature scale space filtering of a planar curve as soon as it becomes a simple convex curve since it is guaranteed that it will remain convex from that point on and therefore no new curvature zero-crossings will be found at larger scales.

6. Conclusions

We introduced the renormalized curvature scale space image which corrects for the non-linear shrinking of arc length when a planar curve evolves. This new representation is more suitable than the curvature scale space image for matching a planar curve to an evolved version of itself, or for matching two similar curves.

We also showed that no new curvature zero-crossings are created at the higher scales of the curvature scale space image of a planar curve provided that the curve remains in C_1 during evolution. The scaling properties of a few other categories of planar curves were also investigated. Among these is a result concerning the behavior of curves that develop cusps during evolution.

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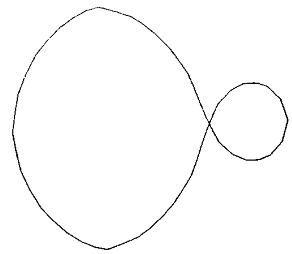
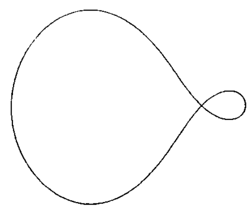
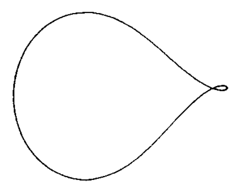


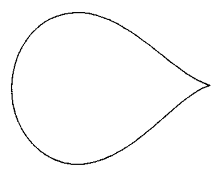
(a) A self-crossing curve



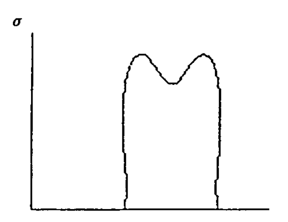
(b) Convolved with σ=8



(c) Convolved with σ=12

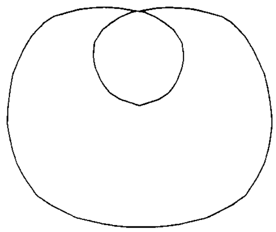


(d) Convolved with σ=14

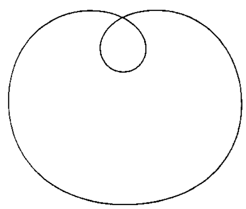


(e) The curvature scale-space image of the curve

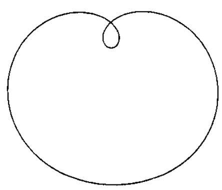
Figure 5. A self-crossing curve that develops a cusp point during evolution.



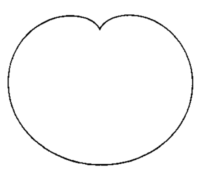
(a) A convex curve



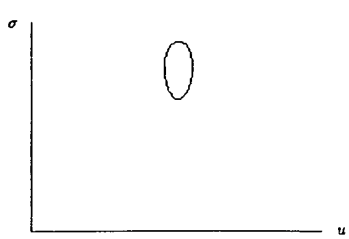
(b) Convolved with σ=16



(c) Convolved with σ=24



(d) Convolved with σ=32



(e) The curvature scale-space image of the curve

Figure 6. A convex curve
that develops a cusp point during evolution.

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