

# Bar $k$ -Visibility Graphs

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## Abstract

Let  $S$  be a set of horizontal line segments, or bars, in the plane. We say that  $G$  is a bar visibility graph, and  $S$  its bar visibility representation, if there exists a one-to-one correspondence between vertices of  $G$  and bars in  $S$ , such that there is an edge between two vertices in  $G$  if and only if there exists an unobstructed vertical line of sight between their corresponding bars. If bars are allowed to see through each other, the graphs representable in this way are precisely the interval graphs. We consider representations in which bars are allowed to see through at most  $k$  other bars. Since all bar visibility graphs are planar, we seek measurements of closeness to planarity for bar  $k$ -visibility graphs. We obtain an upper bound on the number of edges in a bar  $k$ -visibility graph. As a consequence, we obtain an upper bound of 12 on the chromatic number of bar 1-visibility graphs, and a tight upper bound of 8 on the size of the largest complete bar 1-visibility graph. We conjecture that bar 1-visibility graphs have thickness at most 2.

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## 1 Introduction

Let  $S$  be a set of disjoint horizontal line segments, or *bars*, in the plane. We say that a graph  $G$  is a *bar visibility graph*, and  $S$  a *bar visibility representation* of  $G$ , if there exists a one-to-one correspondence between vertices of  $G$  and bars in  $S$ , such that there is an edge between two vertices in  $G$  if and only if there exists an unobstructed vertical line of sight between their corresponding bars. Bar visibility graphs were introduced in the 1980s [9, 13] as a modeling tool for VLSI layout problems. These graphs have been fully characterized as those planar graphs having a plane embedding with all cut points on the outer face [12, 16, 19].

Recent attention has been drawn to a variety of generalizations and restrictions of bar visibility graphs, including unit bar visibility graphs, arc visibility graphs, rectangle visibility graphs, and others [2, 3, 5, 4, 6, 7, 10, 11, 14, 15]. In this paper, we define a new generalization of bar visibility graphs, called *bar  $k$ -visibility graphs*, and discuss their properties. In what follows, we use the standard graph theory terminology found in [8, 18].

Let  $G$  be a bar visibility graph, and let  $S$  be a bar visibility layout of  $G$ . If each line of sight is required to be a rectangle of positive width, then  $S$  is an  *$\varepsilon$ -visibility representation* of  $G$ , and when each line of sight is a line segment (with width 0), then  $S$  is a *strong visibility representation* of  $G$  [16]. In general, these definitions are not equivalent;  $K_{2,3}$  admits an  $\varepsilon$ -visibility representation but not a strong visibility representation, as shown in Figure 1.

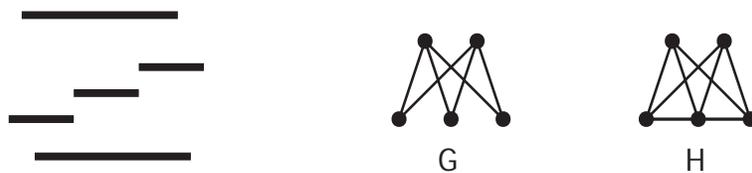


Figure 1: The bar visibility representation shown is an  $\varepsilon$ -visibility representation of  $G$  and a strong visibility representation of  $H$ .

Given a set of bars  $S$  in the plane, suppose that an endpoint of a bar  $B$  and an endpoint of a bar  $C$  in  $S$  have the same  $x$ -coordinate. We elongate one of these two bars so that their endpoints have distinct  $x$ -coordinates. If  $S$  is a strong visibility representation of a graph  $G$ , then we may perform this elongation so that  $S$  is still a strong visibility representation of  $G$ . If  $S$  is an  $\varepsilon$ -visibility representation of  $G$ , then we may perform this elongation so that  $S$  is an  $\varepsilon$ -visibility representation of a new graph  $H$  with  $G \subseteq H$ . Since we are interested in the maximum number of edges obtainable in a representation, we may consider the graph  $H$  instead of the graph  $G$ . Repeating this process yields a set of bars with pairwise distinct endpoint  $x$ -coordinates. For the remainder of this paper, we assume that all bar visibility representations are of this form.

If a set of bars  $S$  has all endpoint  $x$ -coordinates distinct, the graphs  $G$  and  $H$  which have  $S$  as a strong bar visibility representation and an  $\varepsilon$ -visibility representation, respectively, are isomorphic. Hence without loss of generality, for the remainder of the paper, all bar visibility representations are strong bar visibility representations. Formally, two vertices  $x$  and  $y$  in  $G$  are adjacent if and only if, for their corresponding bars  $X$  and  $Y$  in  $S$ , there exists a vertical line segment  $L$ , called a *line of sight*, whose endpoints are contained in  $X$  and  $Y$ , respectively, and which does not intersect any other bar in  $S$ .

By contrast, suppose that  $S$  is a set of closed intervals on the real line. The graph  $G$  is called an *interval graph* and  $S$  an *interval representation* of  $G$  if there exists a one-to-one correspondence between vertices of  $G$  and intervals in  $S$ , such that  $x$  and  $y$  are adjacent in  $G$  if and only if their corresponding intervals intersect. Suppose we call a set  $S$  of horizontal bars in the plane an  *$x$ -ray-visibility representation* if we allow sight lines to intersect arbitrarily many bars in  $S$ . Then we can easily transform an  $x$ -ray-visibility representation into an interval representation by vertically translating the bars in  $S$ , and vice-versa. Therefore  $G$  is an  $x$ -ray-visibility graph if and only if  $G$  is an interval graph.

Motivated by this correspondence, we define a *bar  $k$ -visibility graph* to

be a graph with a bar visibility representation in which a sight line between bars  $X$  and  $Y$  intersects at most  $k$  additional bars. We are interested in characterizing the graphs that are bar  $k$ -visibility graphs. In particular, since all bar visibility graphs are planar, we seek measurements of closeness to planarity for bar  $k$ -visibility graphs.

## 2 An Edge Bound for Bar 1-Visibility Graphs

Suppose  $G$  is a graph with  $n$  vertices, and  $S$  is a bar 1-visibility representation of  $G$ . Since we consider  $S$  to be a strong visibility representation of  $G$ , without loss of generality, we may assume that all endpoints of all bars in  $S$  have distinct  $x$ -coordinates, and all bars in  $S$  have distinct  $y$ -coordinates.

It will be convenient to use four different labeling systems for the bars in  $S$ . Label the bars  $1_l, 2_l, \dots, n_l$  in increasing order of the  $x$ -coordinate of their left endpoint. Label them  $1_r, 2_r, \dots, n_r$  in decreasing order of the  $x$ -coordinate of their right endpoint. Label them  $1_b, 2_b, \dots, n_b$  in increasing order of their  $y$ -coordinate. Finally, label them  $1_t, 2_t, \dots, n_t$  in decreasing order of their  $y$ -coordinate. So the bar  $1_l$  has leftmost left endpoint, the bar  $1_r$  has rightmost right endpoint, the bar  $1_b = n_t$  is bottommost in the representation, and the bar  $1_t = n_b$  is topmost in the representation.

**Remark 1** *Suppose  $S$  is a bar  $k$ -visibility representation of a graph  $G$ . For the remainder of the paper we assume that  $1_t = 1_r = 1_l$  and  $1_b = 2_r = 2_l$ , since we may always elongate the top and bottom bars in  $S$  without reducing the number of edges in  $G$ .*

**Theorem 2** *If  $G$  is a bar 1-visibility graph with  $n \geq 5$  vertices, then  $G$  has at most  $6n - 20$  edges.*

**Proof:** Suppose  $G$  is a graph with  $n \geq 5$  vertices, and let  $S$  be a bar 1-visibility representation of  $G$ . We define the following correspondence between bars in  $S$  and edges of  $G$ . Let  $\{u, v\}$  be an edge in  $G$ , and let  $U$  and  $V$

be the bars in  $S$  associated with  $u$  and  $v$ , respectively. Denote by  $\ell(\{u, v\})$  the vertical line segment from the point in  $U$  to the point in  $V$  whose  $x$ -coordinate is the infimum of  $x$ -coordinates of lines of sight between  $U$  and  $V$ . The edge  $\{u, v\}$  is called a *left edge* of  $U$  (respectively  $V$ ) if  $\ell(\{u, v\})$  contains the left endpoint of  $U$  (respectively  $V$ ). If  $\ell(\{u, v\})$  contains neither  $U$  nor  $V$ 's left endpoint then it must be a 1-visibility edge, and it must contain the right endpoint of some bar  $B$  that blocks the 1-visibility of  $U$  from  $V$  to the left of that point. In this case, we call  $\{u, v\}$  a *right edge* of  $B$ . Note that a right edge of  $B$  is *not* incident to the vertex  $b$  of  $G$  corresponding to the bar  $B$ . Each bar  $B$  can have at most at most 2 right edges, as shown in Figure 2, and at most 4 left edges (two to adjacent bars above  $B$  in  $S$  and two to adjacent bars below  $B$  in  $S$ ).

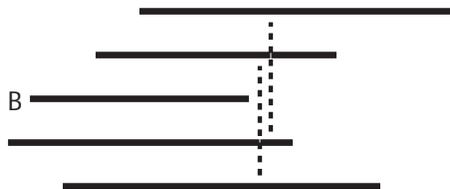


Figure 2: The two right edges associated to bar  $B$ .

Counting both left and right edges, each bar in  $S$  is associated with at most 6 edges, giving an upper bound of  $6n$  edges in  $G$ . However, the bars  $1_l$ ,  $2_l$ ,  $3_l$ , and  $4_l$  have at most 0, 1, 2, and 3 left edges, respectively. Similarly, the bars  $1_r$ ,  $2_r$ ,  $3_r$ , and  $4_r$  have at most 0, 0, 0, and 1 right edges, respectively. Therefore there are at most  $4n - 10$  left edges and at most  $2n - 7$  right edges, for a total of at most  $6n - 17$  edges in  $G$ .

By our assumption that  $1_t$  and  $1_b$  are the leftmost edges, the edge  $\{1_t, 1_b\}$  will always be a left edge. Since the edge associated with the right endpoint of the bar  $4_r$  can only be this edge, the bar  $4_r$  must have 0 right edges. So there are at most  $2n - 8$  right edges in  $G$ , and at most  $6n - 18$  edges total.

We call the set of left endpoints of the bars  $\{1_l, \dots, 4_l\}$  the *outer* left endpoints of  $S$ , and the remaining left endpoints the *inner* left endpoints

of  $S$ . Similarly, the right endpoints of the bars  $\{1_r, \dots, 4_r\}$  are the *outer* right endpoints of  $S$ , and the remaining right endpoints are the *inner* right endpoints of  $S$ . Note that  $G$  has  $6n - 18$  edges if and only if all bars in  $S$  with an inner left endpoint have exactly four left edges, and all bars in  $S$  with an inner right endpoint have exactly two right edges.

Consider the bars  $2_t$  and  $2_b$ , which are distinct bars since  $n \geq 5$ . Each of these two bars has at most three left edges and at most one right edge. Hence if any two of their endpoints are inner endpoints,  $G$  has at most  $6n - 20$  edges. There are two more cases to consider.

*Case 1: All four endpoints of  $2_t$  and  $2_b$  are outer endpoints.* Then the edges  $\{1_t, 2_b\}$  and  $\{1_b, 2_t\}$  are left edges. Since the rightmost bars in the representation are  $1_t$ ,  $1_b$ ,  $2_t$ , and  $2_b$ , the bar  $5_r$  has no right edges, and  $G$  has at most  $6n - 20$  edges.

*Case 2: One endpoint of  $2_t$  is an inner endpoint, and the remaining three endpoints of  $2_t$  and  $2_b$  are outer endpoints.* In this case, the bar  $2_b$  has at most three left edges. Also, since the edge  $\{1_t, 2_b\}$  is a left edge, the bar  $5_r$  has at most one right edge by the argument in Case 1. Hence  $G$  has at most  $6n - 20$  edges.  $\square$

**Corollary 3** *The graph  $K_9$  is not a bar 1-visibility graph.*

**Proof:** Any bar 1-visibility graph with 9 vertices has at most 34 edges, whereas  $K_9$  has 36 edges.  $\square$

**Theorem 4** *For each  $n$ ,  $1 \leq n \leq 8$ , the complete graph  $K_n$  is a bar 1-visibility graph. For each  $n \geq 8$ , there exists a bar 1-visibility graph with  $6n - 20$  edges.*

**Proof:** The graph with representation shown in Figure 3 is a bar 1-visibility graph with  $n = 11$  vertices and  $6n - 20 = 46$  edges. For ease of counting, the left and right endpoints of bars in this representation are labeled with the number of left and right edges associated to each bar. Note that this representation has  $4n - 11$  left edges and  $2n - 9$  right edges.

A layout of  $K_8$  is achieved by removing all but the top four and bottom four bars, and then any of these bars can be removed to obtain a smaller complete graph. The remaining three bars in Figure 3 can be increased or decreased in number to obtain, for  $n \geq 8$ , a bar 1-visibility graph with  $6n - 20$  edges.  $\square$

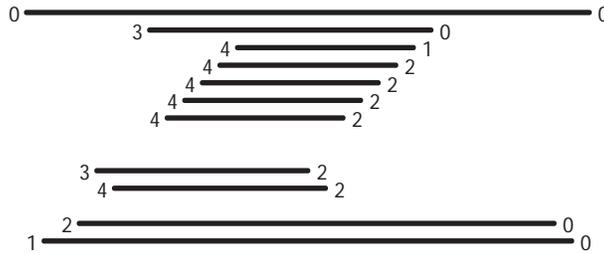


Figure 3: A bar 1-visibility representation with  $6n - 20$  edges.

By Theorem 4, if  $G$  is a bar 1-visibility graph, then  $\chi(G)$  may be 8. No bar 1-visibility graph is known with chromatic number 9. The standard example of a graph with chromatic number 9 but clique number smaller than 9 is the Sulanke graph  $K_6 \vee C_5$  [18], which is not a bar 1-visibility graph since it has 11 vertices and 50 edges.

### 3 Edge Bounds on Bar $k$ -Visibility Graphs

The following theorem generalizes the technique used in the proof of Theorem 2 for  $k > 1$ .

**Theorem 5** *Let  $G$  be a bar  $k$ -visibility graph with  $n$  vertices, where  $n \geq 5$ ,  $k \geq 1$ , and  $n \geq 2k + 2$ . Then  $G$  has at most  $(k + 1)(3n - 2k - 2) - 12$  edges.*

**Proof:** We proceed by induction on  $k$ . When  $k = 1$  and  $n \geq 5$ ,  $G$  has at most  $6n - 20 = (1 + 1)(3n - 2 - 2) - 12$  edges, by Theorem 2.

Now suppose that  $k \geq 1$ , and assume the statement is true for  $k$ . We consider a bar  $(k + 1)$ -visibility graph  $G$ , and show that  $G$  has at most

$((k+1)+1)(3n-2(k+1)-2)-12 = (k+2)(3n-2k-4)-12$  edges. Let  $S$  be a bar  $(k+1)$ -visibility representation of  $G$ .

Consider the graph  $H$  with bar  $k$ -visibility representation  $S$ . By induction,  $H$  has at most  $(k+1)(3n-2k-2)-12$  edges.  $G$  has all of the edges of  $H$ , plus the additional edges obtained by the extra visibility. We associate each of the edges of  $G$  with a left or right endpoint of a bar in  $S$ , as in the proof of Theorem 2.

Each bar in  $S$  has at most two additional left edges and at most one additional right edge in  $G$ . However, the topmost  $k+2$  bars in  $S$  and the bottommost  $k+2$  bars in  $S$  have at most one additional left edge in  $G$ , and the topmost  $k+1$  bars in  $S$  and the bottommost  $k+1$  bars in  $S$  have no additional right edges in  $G$ . Therefore, since  $n \geq 2k+2$ , we have  $|E(G)| \leq |E(H)| + 3n - 2(k+2) - 2(k+1) \leq (k+1)(3n-2k-2) - 12 + 3n - 4k - 6 = (k+2)(3n-2k-4) - 12$ .  $\square$

We now improve the bound given in Theorem 5 for  $k > 1$  using a different technique.

**Theorem 6** *Let  $G$  be a bar  $k$ -visibility graph with  $n$  vertices, where  $n \geq 5$ ,  $k \geq 1$ , and  $n \geq 2k+2$ . Then  $G$  has at most  $(k+1)(3n - \frac{7}{2}k - 5) - 1$  edges.*

**Proof:** Let  $S$  be a bar- $k$  visibility representation of  $G$ . Recall that we may assume that all  $x$ -coordinates of endpoints of bars in  $S$  are distinct. We also assume that  $S$  is of the form given in Remark 1.

We sweep a vertical line left-to-right over the representation, and consider how many new edges can be added each time a new endpoint is encountered. We first consider the left endpoints of the bars in  $S$  in the order  $l_1, \dots, l_n$ . When we encounter a new left endpoint, its bar can only increase distances between existing bars, so the only new visibilities are the ones involving this new bar. Hence at most  $2(k+1)$  new edges are added, comprising  $(k+1)$  neighbors above the new bar and  $(k+1)$  neighbors below it. However, the first  $2(k+1)$  left endpoints add fewer edges. In particular, the first left endpoint adds no new edges, the second left endpoint adds at most one, the

third left endpoint adds at most two, etc. In other words, the total number of edges added by encountering the left endpoint of new bars is at most the following, since  $n \geq 2k + 2$ :

$$\begin{aligned} & 0 + 1 + 2 + \dots + 2k + (2k + 1) + (2k + 2) + \dots (2k + 2) \\ &= \frac{(2k + 1)(2k + 2)}{2} + 2(k + 1)(n - 2k - 2) \\ &= (k + 1)(2n - 2k - 3) \end{aligned}$$

Now we count the number of edges added by reaching the right endpoint of a bar. The first bar that can have a positive number of right edges is  $(k + 3)_r$ , since the edge is between two additional bars that see each other through  $k$  other bars, not including this bar, for a total of  $k + 3$  bars. The number of edges added at each subsequent right endpoint increases by at most one, to a maximum of  $k + 1$ . Thus the total number of new edges produced at the right endpoints of bars is at most the following, in which the first  $k + 2$  terms are zero:

$$\begin{aligned} & 0 + \dots + 0 + 1 + \dots + k + (k + 1)(n - 2k - 2) \\ &= \frac{k(k + 1)}{2} + (k + 1)(n - 2k - 2) \\ &= (k + 1)(n - 3k/2 - 2) \end{aligned}$$

Thus the total number of possible edges in  $G$  is at most the following:

$$\begin{aligned} & (k + 1)(2n - 2k - 3) + (k + 1)(n - 3k/2 - 2) \\ &= (k + 1)(3n - 7k/2 - 5) \end{aligned}$$

Finally, by Remark 1, the leftmost left edge is between bars  $1_t$  and  $1_b$ . Since  $1_t = 1_r$  and  $1_b = 2_r$ , this edge is the only possible right edge of  $(k + 3)_r$ . Hence we counted this edge twice in Theorem 6, making the final total at most  $(k + 1)(3n - \frac{7}{2}k - 5) - 1$ .  $\square$

**Theorem 7** *For  $k \geq 0$  and  $n \geq 4k + 4$ , there exist bar  $k$ -visibility graphs with  $n$  vertices and  $(k + 1)(3n - 4k - 6)$  edges.*

**Proof:** Figure 4 shows a bar  $k$ -visibility representation of a graph with  $n$  vertices and  $(k + 1)(3n - 4k - 6)$  edges. As in Figure 3, the left and right endpoints of bars in this representation are labeled with the number of left and right edges associated to each bar. Although  $n = 4k + 4$  in this representation, the number of bars in the group labeled  $A$  can be increased arbitrarily to create a representation with  $n$  bars for any  $n \geq 4k + 4$ .  $\square$

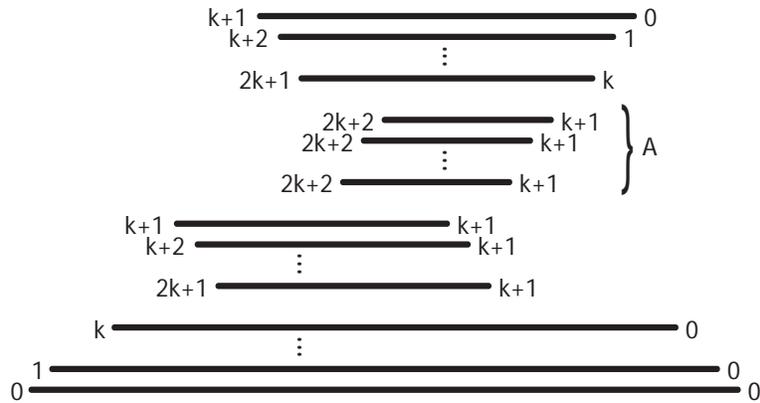


Figure 4: A bar  $k$ -visibility graph with  $n$  vertices and  $(k + 1)(3n - 4k - 6)$  edges.

Note that Theorem 7 gives the largest number of edges in a bar  $k$ -visibility graph for  $k = 0, 1$ . We believe that this is the case for larger  $k$  as well. We state this as a conjecture.

**Conjecture 8** *If  $G$  is a bar  $k$ -visibility graph with  $n \geq 2k + 2$  vertices, then  $G$  has at most  $(k + 1)(3n - 4k - 6)$  edges.*

**Corollary 9** *If  $G$  is a bar  $k$ -visibility graph, then  $\chi(G) \leq 6k + 6$ .*

**Proof:** We proceed, for fixed  $k$ , by induction on  $n$ . The result is obvious when  $n \leq 6k + 6$ . For  $n > 6k + 6$  assume that all bar  $k$ -visibility graphs with  $n - 1$  vertices have  $\chi \leq 6k + 6$ , and suppose that  $G$  is a bar  $k$ -visibility graph

with  $n$  vertices. By Theorem 6,  $\sum_{v \in V(G)} \deg(v) < (6k + 6)n$ , so the average degree of a vertex in  $G$  is strictly less than  $6k + 6$ . Then there must exist a vertex  $v$  in  $G$  of degree at most  $6k + 5$ . We consider the graph  $G - v$ . Although this graph may not be a bar  $k$ -visibility graph, it is a subgraph of the graph  $G'$  with bar  $k$ -visibility representation obtained from a representation of  $G$  by deleting the bar corresponding to  $v$ . Therefore the edge bound in Theorem 6 still applies to  $H$ . By the induction hypothesis, we may color the vertices of  $H$  with  $6k + 6$  colors, replace  $v$ , and color  $v$  with a color not used on its neighbors.  $\square$

The following theorem is a corollary of Theorem 6.

**Theorem 10**  $K_{5k+5}$  is not a bar  $k$ -visibility graph.

**Proof:** If  $G$  is a graph with  $n = 5k + 5$  vertices, then by Theorem 6,  $G$  has at most  $(k + 1)(3(5k + 5) - \frac{7}{2}k - 5) - 1 = \frac{1}{2}(23k^2 + 37k + 3)$  edges. However,  $K_{5k+5}$  has  $\binom{5k+5}{2} = \frac{1}{2}(25k^2 + 45k + 5)$  edges, which exceeds the bound by  $k^2 + k + 1$  edges.  $\square$

Note that if Conjecture 8 is true, we immediately obtain the following conjecture as a corollary.

**Conjecture 11**  $K_{4k+4}$  is the largest complete bar  $k$ -visibility graph.

**Proof:** Figure 4 shows a bar  $k$ -visibility representation of  $K_{4k+4}$ . Conversely, suppose that  $G$  is a graph with  $n = 4k + 5$  vertices. Then by Conjecture 8,  $G$  has at most  $(k + 1)(3(4k + 5) - 4k - 6) = 8k^2 + 17k + 9$  edges. However,  $K_{4k+5}$  has  $\binom{4k+5}{2} = 8k^2 + 18k + 10$  edges.  $\square$

Conjecture 8 is not required to prove Conjecture 11 when  $k = 0$  or  $1$ ; we have already proved these cases in the previous section. Note also that the graph  $K_{4k+4}$  exactly achieves the bound given by Conjecture 8. So if this conjecture is correct, the family of complete graphs  $K_{4k+4}$  is an example of a family of bar  $k$ -visibility graphs with the maximum number of edges.

Table 1 shows the two proven upper bounds on the number of edges in a bar  $k$ -visibility graph, together with the conjectured exact bound.

$k$	$(k+1)(3n-2k-2)-12$	$(k+1)(3n-\frac{7}{2}k-5)$	$(k+1)(3n-4k-6)$
0	N/A	$3n-6$	$3n-6$
1	$6n-20$	$6n-17$	$6n-20$
2	$9n-30$	$9n-33$	$9n-42$
3	$12n-44$	$12n-54$	$12n-72$
4	$15n-62$	$15n-80$	$15n-110$

Table 1: Two proven upper bounds and the conjectured exact bound

## 4 Thickness of Bar $k$ -Visibility Graphs

By Theorem 4,  $K_8$  is a bar 1-visibility graph, and thus there are non-planar bar 1-visibility graphs. Motivated by the fact that all bar 0-visibility graphs are planar, we are interested in measuring the closeness to planarity of bar 1-visibility graphs. The *thickness*  $\Theta(G)$  of a graph  $G$  is the minimum number of planar graphs whose union is  $G$ .  $K_8$  has thickness 2 [1, 17], so there exist bar 1-visibility graphs with thickness 2.

**Theorem 12** *There are thickness-2 graphs with  $n$  vertices that are not bar 1-visibility graphs for all  $n \geq 15$ .*

**Proof:** Consider the graph  $G_{15}$ , whose partition into two plane layers is shown in Figure 5. This graph has 15 vertices and  $6 \times 15 - 12 = 78$  edges. Note that no thickness-2 graph with  $n$  vertices has more than  $6n - 12$  edges, since if  $G$  has thickness 2 then  $G$  is the union of two planar graphs, each of which has at most  $3n - 6$  edges. Beginning with the graph  $G_n, n = 15$ , we repeatedly add vertex  $n + 1$  to produce a new thickness-2 graph  $G_{n+1}$  with  $6(n + 1) - 12$  edges.

Let  $L_1$  and  $L_2$  be the two plane layers of  $G_{15}$  that are shown in Figure 5. In general, we form  $G_n$  from  $G_{n-1}$  by choosing two vertex-disjoint triangles from  $G_{n-1}$ , one in  $L_1$  and the other in  $L_2$ , and then taking the join of each triangle with the new vertex  $n$ .  $G_{16}$  is obtained from  $G_{15}$  by adding to it the

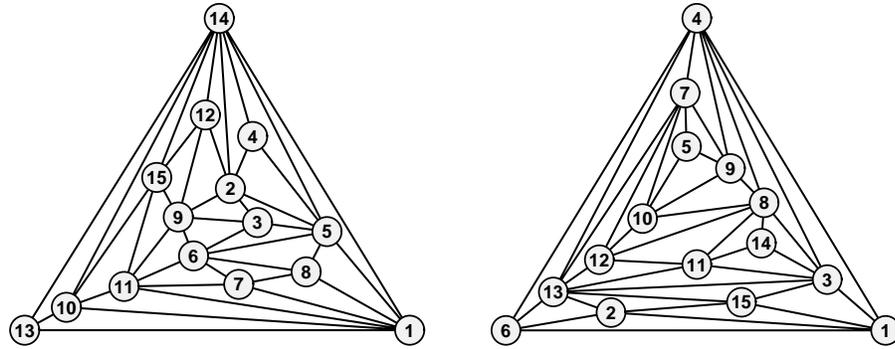


Figure 5: Two planar graphs whose union is not a bar 1-visibility graph.

	$G_{16}$	$G_{17}$	$G_{18}$	$G_{19}$
$L_1$	$\{1, 5, 8\}$	$\{9, 11, 15\}$	$\{1, 5, 16\}$	$\{9, 11, 17\}$
$L_2$	$\{2, 6, 13\}$	$\{7, 10, 12\}$	$\{7, 10, 17\}$	$\{2, 6, 16\}$
	$G_{20}$	$G_{21}$	$G_{22}$	$G_{23}$
$L_1$	$\{1, 5, 18\}$	$\{9, 11, 19\}$	$\{1, 5, 20\}$	$\{9, 11, 21\}$
$L_2$	$\{2, 6, 19\}$	$\{7, 10, 18\}$	$\{7, 10, 21\}$	$\{2, 6, 20\}$
	$G_{2n}$	$G_{2n+1}$	$G_{2n+2}$	$G_{2n+3}$
$L_1$	$\{1, 5, 2n - 2\}$	$\{9, 11, 2n - 1\}$	$\{1, 5, 2n\}$	$\{9, 11, 2n + 1\}$
$L_2$	$\{2, 6, 2n - 1\}$	$\{7, 10, 2n - 2\}$	$\{7, 10, 2n + 1\}$	$\{2, 6, 2n\}$

Table 2: Construction of edge-maximal thickness-2 graphs

edges of  $\{1, 5, 8\} \vee \{16\}$  and  $\{2, 6, 13\} \vee \{16\}$ , and  $G_{17}$  is obtained from  $G_{16}$  by adding to it the edges of  $\{9, 11, 15\} \vee \{17\}$  and  $\{7, 10, 12\} \vee \{17\}$ .

The entries of Table 2 show how to continue this process indefinitely, always choosing two vertex-disjoint triangles from  $G_{n-1}$ , one in  $L_1$  and one in  $L_2$ , and taking the join of each triangle with  $\{n\}$  to form  $G_n$ . A pattern for choosing the two rectangles is established in the last four entries of Table 2. For each  $n \geq 15$ ,  $L_1$  and  $L_2$  are plane triangulations, so  $G_n$  has  $6n - 12$  edges.

□

Suppose  $G$  is a bar  $k$ -visibility graph, and  $S$  is a bar  $k$ -visibility representation of  $G$ . We define the *underlying bar  $i$ -visibility graph*  $G_i$  of  $S$  to be the graph with bar  $i$ -visibility representation  $S$ .

It is not known whether every bar 1-visibility graph has thickness  $\leq 2$ . For example, the complement of the bar 0-visibility edges in a bar 1-visibility graph need not induce a planar graph. Figure 6 shows two bar 1-visibility graphs, each with a non-planar subgraph induced by “pure” 1-visibility edges, i.e., edges that are not in the underlying bar 0-visibility graph. In the first layout, the non-planar graph is induced by only *left* 1-visibility edges, and in second layout only *right* 1-visibility edges are needed to induce a non-planar subgraph. For each layout there is a subdivided  $K_5$  on the vertices corresponding to the bars  $\{A, B, C, D, E\}$ , using the vertices corresponding to the numbered bars for the subdivided edges. Theorem 13 bounds the thickness of bar 1-visibility graphs by 4, and Theorem 14 gives an upper bound on the thickness of bar  $k$ -visibility graphs.

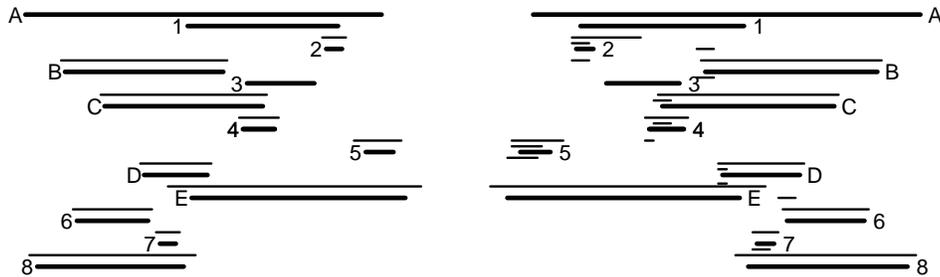


Figure 6: Bar 1-visibility graphs whose left 1-visibility edges and right 1-visibility edges, respectively, induce a non-planar subgraph

The following two theorems relate the thickness of a bar  $k$ -visibility graph  $G$  to the chromatic number of  $G_{k-1}$ . Note that since  $G_{k-1}$  depends on the choice of bar  $k$ -visibility representation  $S$  of  $G$ , different representations might yield different bounds on the thickness of a particular bar  $k$ -visibility graph  $G$ .

The proofs of Theorems 13 and 14 use a drawing of  $G$  in the plane, induced by the representation, with the property that each edge is a polyline with two bends, and such that the middle linear section of each edge is a vertical line segment corresponding to a line of sight. This drawing, which generalizes one given in [8] for bar visibility graphs, and which may have crossing edges, is obtained as follows. First we fatten each bar corresponding to a vertex  $v$  into a rectangle  $R_v$  and draw the vertex  $v$  in the center of  $R_v$ . Next, for each edge  $e = \{u, v\}$  of  $G$  with  $R_u$  above  $R_v$ , we choose a vertical line of sight represented by a vertical line segment  $\ell$  from the upper border of  $R_v$  to the lower border of  $R_u$ . We choose these lines of sight so that no two vertical segments overlap, and so that no vertex has the same  $x$ -coordinate as the line of sight for one of its incident edges. Finally, we let the drawing of the edge  $e = \{u, v\}$  comprise the chosen vertical segment  $\ell$  plus the two non-vertical line segments connecting  $u$  and  $v$  to the endpoints of  $\ell$ ; see Fig. 7. We refer to this (non-unique) plane drawing of  $G$  as a *2-bend drawing of  $G$  induced by the bar  $k$ -visibility representation*.

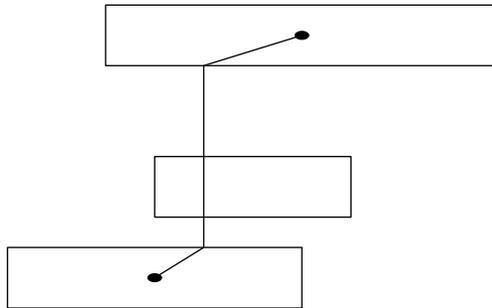


Figure 7: A two-bend edge induced by a bar 1-visibility representation

**Theorem 13** *If  $G$  is a bar 1-visibility graph then  $\theta(G) \leq 4$ .*

**Proof:** Suppose  $G$  is a bar 1-visibility graph. Let  $L$  be a bar 1-visibility representation of  $G$ , and let  $D$  be an induced 2-bend drawing of  $G$  in the plane. The underlying bar-visibility graph  $G_0$  is planar and thus has chromatic number at most 4. We choose an arbitrary 4-coloring  $C_0$  of  $G_0$ , and

we use this vertex-coloring to define a 4-coloring of the edges of  $G$  such that each color class of edges induces a planar graph.

Define an edge  $e$  of  $G$  to be a 1-visibility edge if its chosen line of sight passes through a bar in  $S$ ; otherwise it is a 0-visibility edge. Each 1-visibility edge  $e$  of  $G$  receives the color of the bar that it passes through. Since the two endpoints of  $e$  are both visible to this bar in  $G_0$ , neither endpoint has the same color as  $e$ . Each 0-visibility edge is assigned an arbitrary color different from those of its endpoints.

We claim that any two crossing edges have different colors. If two edges cross, then the vertical section of one edge  $e$  crosses a non-vertical section of the other edge  $f$ . This non-vertical section lies within the rectangle  $R_v$  corresponding to an endpoint  $v$  of  $f$ , hence  $v$  and  $f$  have different colors. Since  $e$  passes through  $R_v$ , it has the same color as  $v$ , and thus  $e$  and  $f$  have different colors. Therefore each color class induces a planar graph, and the thickness of  $G$  is at most 4.  $\square$

The proof of Theorem 13 can be generalized to bound the thickness of a bar  $k$ -visibility graph  $G$  by a quadratic function of  $k$ .

**Theorem 14** *If  $G$  is a bar  $k$ -visibility graph, then the thickness  $\theta(G)$  satisfies  $\theta(G) \leq 2k(9k - 1)$ .*

**Proof:** If  $G$  is a bar  $k$ -visibility graph and  $L$  a bar  $k$ -visibility representation of  $G$  in the plane, let  $D$  be an induced 2-bend drawing of  $G$  in the plane. Choose a vertex-coloring of  $G_{k-1}$  using  $\chi(G_{k-1})$  colors. The vertical section of any edge  $e$  in  $G$  passes through at most  $k$  bars; the vertices corresponding to these bars and the two endpoints of  $e$  are all adjacent, so they must all have different colors. If the two endpoints of  $e$  have colors  $c_1$  and  $c_2$  (which may be the same), we assign as a color to  $e$  the set  $\{c_1, c_2\}$ , which has one or two elements. If  $e$  crosses another edge  $f$  with color  $\{d_1, d_2\}$ , then, without loss of generality, the vertical section of  $e$  passes through the bar corresponding to one of the endpoints of  $e$ . If this endpoint has color  $d_i$ , then neither endpoint of  $e$  can have color  $d_i$ . Hence  $\{c_1, c_2\} \neq \{d_1, d_2\}$ ,

and each edge-color class induces a planar graph. It follows that  $\theta(G) \leq \chi(G_{k-1}) + \chi(G_{k-1})(\chi(G_{k-1}) - 1)/2$ . By Corollary 9,  $\chi(G_{k-1}) \leq 6k$ , so that  $\theta(G) \leq k + 3k(6k - 1) = 2k(9k - 1)$ .  $\square$

Note that we could use this proof when  $k = 1$ , but we get a better result (4 versus 16) by partitioning the edges according to the color on the vertical segment.

## 5 Future Work

We end with a list of open problems inspired by the results of this paper.

1. What is the largest number of edges in a bar 2-visibility graph with  $n$  vertices?
2. What is the largest number of edges in a bar  $k$ -visibility graph with  $n$  vertices?
3. Are there bar 1-visibility graphs with thickness 3?
4. More generally, what is the largest thickness of a bar  $k$ -visibility graph? Is it  $k + 1$ ?
5. Are there bar 1-visibility graphs with chromatic number 9?
6. More generally, what is the largest chromatic number of a bar  $k$ -visibility graph?
7. What is the largest crossing number of a bar  $k$ -visibility graph?
8. What is the largest genus of a bar  $k$ -visibility graph?
9. *Rectangle visibility graphs* are studied in [6, 7, 11, 15]. Generalize the results of this paper to rectangle visibility graphs.
10. *Arc-* and *circle-visibility graphs* are defined in [10]. Generalize the results of this paper to arc- and circle-visibility graphs.

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