The Left-Right Planarity Test

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Abstract

A graph is planar if and only if it can be drawn in the plane without crossings. I give a detailed exposition of a simple and efficient, yet poorly known algorithm for planarity testing and embedding based on the left-right characterization of planarity. This is complemented by a new Kuratowski subgraph extraction algorithm.

Key words: Graph algorithms, planarity testing, planar embedding, Kuratowski subgraphs

1 Introduction

Two things appear to constitute the folklore about graph planarity testing:

(1) There are two main strands of linear-time algorithms, the *vertex-addition approach* pioneered by Lempel, Even, and Cederbaum (1967), and the *path-addition approach* pioneered by Hopcroft and Tarjan (1974).
(2) Both are a real challenge to understand, implement, and teach.

This article is not a review of the exciting history of planarity testing at large, however, but of the lesser known *left-right approach*, which is generally considered to be different from the above and associated with de Fraysseix and Rosenstiehl (1982). And even though the development from its origins in Wu (1955) to its latest version in de Fraysseix, Ossona de Mendez, and Rosenstiehl (2006) and de Fraysseix (2008) appears to be another interesting story, my only goal here is to meet the apparent demand for an accessible

⋆ I would like to thank Sabine Cornelsen, Giuseppe Di Battista, Bernd Gaertner, Daniel Kaiser, Martin Mader, and Maurizio Patrignani for helpful comments, suggestions, and corrections. I am particularly grateful to an anonymous reviewer who pointed out a gap in the first version of the Kuratowski subgraph extraction method.

exposition. As it turns out, the left-right approach simplifies and improves upon the approach of Hopcroft and Tarjan (1974).

The left-right approach is remarkably elementary and does not require tricky data structures (e.g., Booth and Lueker 1976), a complicated embedding phase (e.g., Mehlhorn and Mutzel 1996), or even special treatment of biconnected components. Moreover, it was found to be extremely fast (Boyer, Cortese, Patrignani, and Di Battista, 2004) and can be augmented easily to return a Kuratowski subgraph if the input is not planar.

This work is motivated by the stark contrast between the elegance and simplicity of the left-right approach and its minimal adoption. It yields, I am convinced, the simplest linear-time planarity algorithms known to date, but to the best of my knowledge, there is not a single exposition or implementation independent from the original group of authors.

The absence of an easily readable, yet fully detailed description may be the main cause for its lack of popularity. In an attempt to remedy this situation, the original description of de Fraysseix, Ossona de Mendez, and Rosenstiehl (2006) is simplified with minor corrections, and it is extended by a new motivation, implementation-level pseudo-code, and more straightforward Kuratowski subgraph extraction. While the planarity test given here differs from the original paper, similar improvements have been introduced independently into the only previous implementation, available in PIGALE (de Fraysseix and Ossona de Mendez, 2002).

From the present description it should be possible to teach the algorithm in no more than two sessions of an advanced algorithms course. With a planar graph data structure at hand, transforming the pseudo-code into an implementation should be a matter of hours.

The remainder is organized such that readers solely interested in understanding the left-right approach can stop reading after Section 6. Therefore, only minimal background on graph planarity and the associated algorithmic problems is provided in Section 2. A new motivation for the left-right approach is given in Section 3, and the planarity characterization on which it is based in Section 4. We derive a simple polynomial-time planarity test in Section 5, before giving the linear-time left-right algorithm for planarity testing and planar embedding in Section 6, including detailed pseudo-code. Kuratowski subgraph extraction for non-planar graphs is treated separately in Section 7, and the relation to other planarity criteria and algorithms as well as some notes on the history of the left-right approach conclude the paper in Section 8.

2 Planarity

We consider simple undirected graphs $G = (V, E)$, since directions, loops and multiple edges have no effect on planarity, and denote $n = n(G) = |V|$ and $m = m(G) = |E| \leq \frac{n(n-1)}{2}$ throughout. Bollobás (1998) and Diestel (2005) are excellent textbooks on graph theory.

A *drawing* of a graph is a mapping of its vertices onto points in the plane (or, equivalently, the surface of a sphere), and its edges onto curves connecting their endpoints. Where possible without confusion, we neglect the distinction between vertices, edges, etc., and their drawings. A drawing of a graph is *planar*, if edges do not intersect except at common endpoints. A graph is planar, if it admits a planar drawing.

A planar drawing divides the plane into connected regions, called *faces*. Each bounded face is an *inner* face, and the single unbounded one is called the *outer* face. We use $f = f(G)$ to denote the number of faces in a planar drawing of G. The following classic result assures, in particular, that the number of faces $f = f(G)$ is the same for every planar drawing of G. For a proof see, e.g., Aigner and Ziegler (2009, Chapter 12).

Theorem 1 (Euler's Formula) *For connected planar graphs, $n-m+f = 2$.*

As an immediate consequence, we can rule out graphs that are too dense to be planar from further consideration.

Corollary 2 *For planar graphs with $n > 2$ vertices, $m \leq 3n - 6$.*

PROOF. In a planar drawing of a planar graph, an edge is in the boundary of at most two faces, while every face has at least three edges. It follows that $f \leq \frac{2m}{3}$ and hence $m \leq 3n - 6$ by Euler's Formula. □

A *(combinatorial) embedding* consists of cyclic orderings of the incident edges at each vertex. An embedding is *realized* by a drawing, if the clockwise ordering of the edges around each vertex in the drawing agrees with the embedding. Note that an embedding represents an equivalence class of drawings that realize it. An embedding is planar, if it can be realized by a planar drawing.

Given a graph G, there are four major algorithmic problems related to planarity:

(1) Decide whether G is planar.
(2) If G is planar, determine a planar embedding.

(3) If G is not planar, determine a Kuratowski subgraph.
(4) Given a planar embedding of G, determine a planar drawing realizing it.

A Kuratowski subgraph is an inclusion-minimal non-planar subgraph. Since the problem of determining a Kuratowski subgraph it is of less general interest, the topic is deferred to Section 7.

Moreover, we will not consider the fourth problem. Note, however, that realizations of a given planar embedding may be subject to various criteria such as integer coordinates, straight-line edges, small area, polygonal edges with few bends and/or slopes, etc., and there are many algorithms for drawing planar graphs according to such criteria (see, e.g., Nishizeki and Rahman 2004).

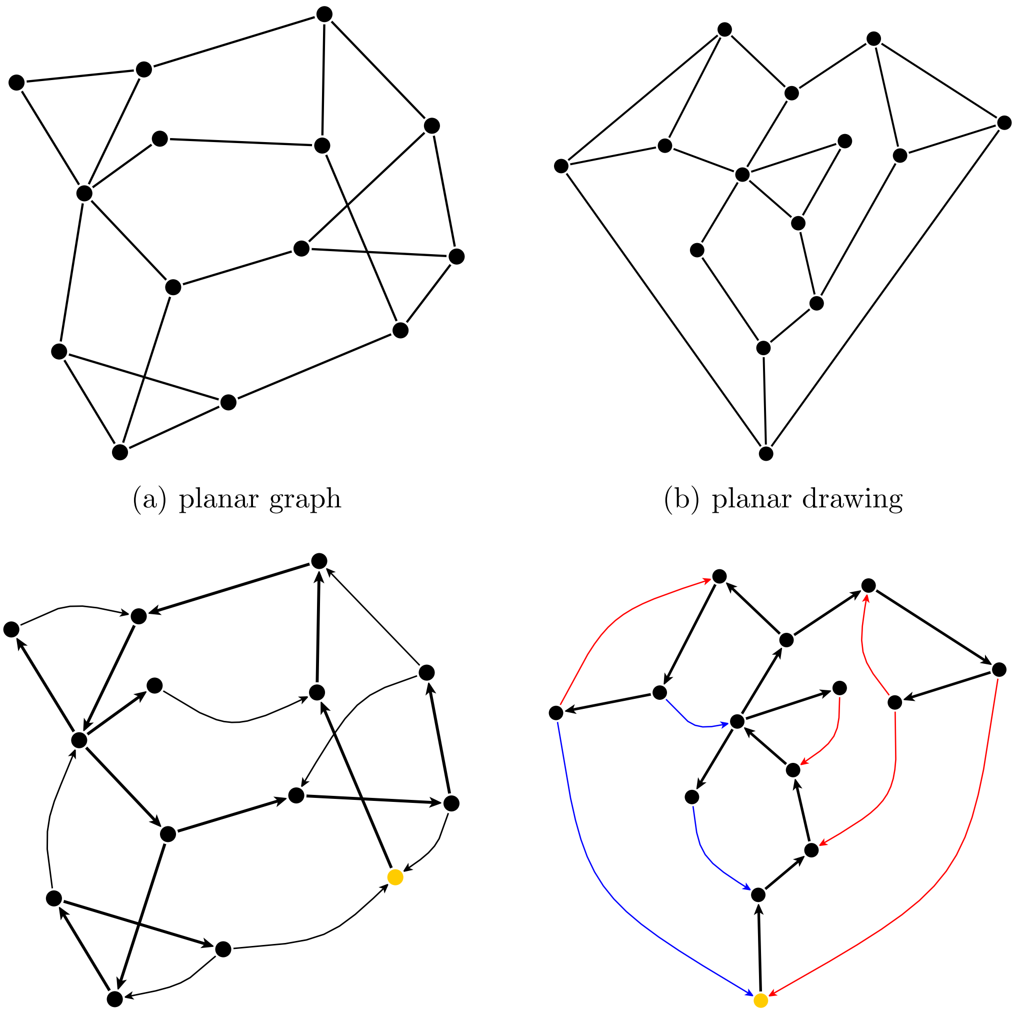
The linear-time testing and embedding algorithm described in Section 6 is based on a rather intuitive criterion that is motivated and established in the next two sections, respectively.

3 Motivation

Since planarity is about the absence of crossings, cycles are the root cause of difficulties: cycles yield closed curves that disconnect regions of the plane, so that care has to be taken which other parts of the graph are placed inside or outside such cycles.

There are only two significantly different ways to draw a simple cycle planarly, namely clockwise or counterclockwise. Fixing an orientation, however, may impose constraints on the orientation of other, overlapping cycles via the ordering of edges around vertices. In fact, testing planarity amounts to deciding whether there is a consistent simultaneous orientation of all cycles. Despite a potentially exponential number of cycles, this can be done efficiently, because we will see that constraints need to be resolved only for a small set of cycles representing the entire cycle structure.

This small set of representative cycles is determined from a depth-first search as described next. We then motivate how apparent orientation constraints can be used to relate cycle orientations to embeddings. In Section 4, this is made more precise in order to characterize planar graphs via cycle orientations. The proof is constructive and yields a planar combinatorial embedding, if one exists.



(a) planar graph (b) planar drawing

(c) DFS orientation (d) LR partition

Fig. 1. Example of a planar graph (from Cai, Han, and Tarjan 1993). In both the planar and non-planar drawing, the same depth-first search (DFS) orientation is shown with thick tree edges and curved back edges. In any planar drawing the back edges can be partitioned into left and right, depending on whether their fundamental cycle is counterclockwise or clockwise. Note that the non-planar drawing contains self-intersecting fundamental cycles for both back edges entering the DFS root.

3.1 Depth-first search

The left-right planarity criterion is inherently related to depth-first search (DFS). Important aspects of this relation are hinted at in this section, and DFS terminology is introduced along the way.

Recall that a depth-first search on a connected undirected graph $\bar{G} = (V, \bar{E})$ yields a *DFS orientation* of $\bar{G}$, i.e., a directed graph $\vec{G} = (V, \vec{E})$ in which each undirected edge is oriented according to its traversal direction. Once the graph is oriented, we will only work with its directed version and hence neglect the distinction between $\bar{E}$ and $\vec{E}$. In the oriented graph, we denote by $E^+(v) = \{(v,w) \in E \,:\, w \in V\}$ the set of all outgoing edges of $v \in V$, so that $E = \bigcup_{v \in V} E^+(v)$.

In addition to an orientation, a DFS traversal yields a bipartition $E = T \uplus B$ of the edges, where those in T are called *tree edges* and induce a rooted spanning tree (the *DFS tree*), and the non-tree edges in B are called *back edges*. See Figure 3. We write $u \to v$ and $v \hookrightarrow w$ for $(u,v) \in T$ and $(v,w) \in B$. Also, we use $\stackrel{+}{\to}$ for the transitive and $\stackrel{*}{\to}$ for the reflexive and transitive closure of $\to$. The unique sequence of edges inducing $u \stackrel{*}{\to} v$ is called a *tree path*. While the tree path is empty for $u = v$, $u \stackrel{+}{\to} v$ induces a *proper* tree path implying $u \neq v$. We occasionally make use of straightforward generalizations to edges such as $(u,v) \stackrel{*}{\to} w$ in case $v \stackrel{*}{\to} w$.

If $v \stackrel{*}{\to} w$ ($v \stackrel{+}{\to} w$), v is said to be (strictly) *lower* than w, and w (strictly) *higher* than v. A vertex is *lowest* (*highest*) in a set of vertices, if no other member of that set is lower (higher). The *height* of a vertex v is its distance from the root. With these definitions we adopt the convention that the root is indeed the lowest vertex in a tree, and illustrations are drawn accordingly.

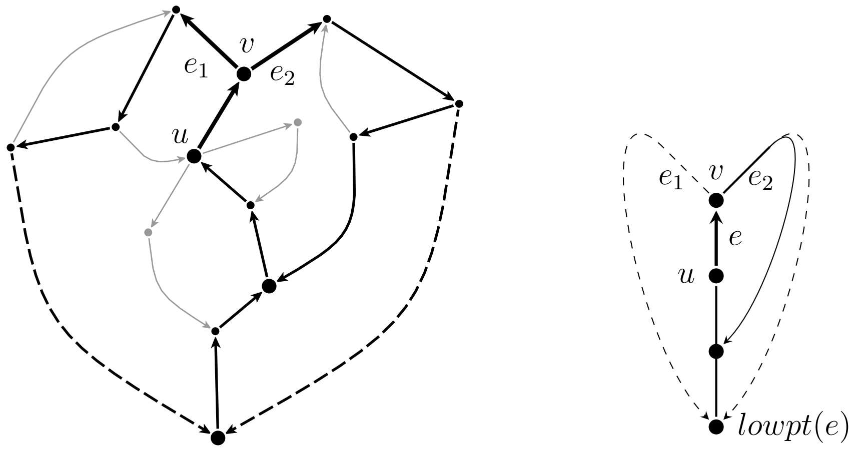
The characterizing property of DFS orientations is that the *target* w of every back edge $v \hookrightarrow w$ is a tree ancestor of (i.e., strictly below) its *source* v. Thus, each back edge $v \hookrightarrow w$ induces a *fundamental cycle* $C(v \hookrightarrow w) = w \stackrel{+}{\to} v \hookrightarrow w$, and these will be our primary objects of interest. Two cycles are called *overlapping*, if they share an edge, and it is the overlap of cycles that makes planarity testing challenging.

Lemma 3 *Let $G = (V, T \uplus B)$ be a DFS-oriented graph.*

(1) The fundamental cycles are exactly the simple directed cycles of G.
(2) Two distinct fundamental cycles are either disjoint, or their intersection forms a tree path.

PROOF.

(1) All fundamental cycles are simple and, because of DFS, directed. Now consider any simple directed cycle and let $v \in V$ be lowest on that cycle. Since every cycle contains at least one back edge, let $x \hookrightarrow u$ be the first back edge after v. Vertex v is lowest, so that u must be in $v \stackrel{*}{\to} x$. Since the cycle is simple, $u = v$ and there are no more edges.
(2) Let $w \stackrel{*}{\to} v \hookrightarrow w$ and $u \stackrel{*}{\to} x \hookrightarrow u$ be two fundamental cycles. Since they are distinct, $v \hookrightarrow w \neq x \hookrightarrow u$. Since there is exactly one path between any pair of vertices in a tree, two tree paths can join and fork at most



(a) original graph (b) sketch

Fig. 2. A fork with branching point v in the graph of Figure 1, and a sketched representation showing only those back edges that are return edges of $e = u \to v$. Note that edges to the lowpoint of e are dashed, and that e_2 is chordal but e_1 is not.

once. A non-empty intersection of $w \overset{*}{\to} v$ and $u \overset{*}{\to} x$ must, therefore, be a tree path itself. $\square$

For two overlapping cycles, the last edge $u \to v$ on the shared tree path together with the succeeding edges $e_1 = (v, w_1), e_2 = (v, w_2)$ on each cycle is called their *fork*, and v its *branching point*. We will see that finding a planar combinatorial embedding reduces to finding an appropriate ordering of all triplets of edges that form a fork. Since all forks at the same branching point share the incoming tree edge, it will be convenient to consider a *linearization* of the cyclic ordering of the outgoing edges around that vertex. It is defined by splitting the clockwise order restricted to outgoing edges at the incoming tree edge, or between any two consecutive outgoing edges if v is the root of a DFS tree.

In the next section, two simple observations help understand how cycle orientations impose fork orderings.

3.2 Orientation and nesting of fundamental cycles

Recall that there are two classes of simple directed cycles in a planar drawing, those oriented clockwise and those oriented counterclockwise. Since the intersection of overlapping fundamental cycles is a tree path containing at least one edge, the four possible configurations in Figure 3 can be summarized as follows, where two overlapping fundamental cycles are called *nested*, if the part

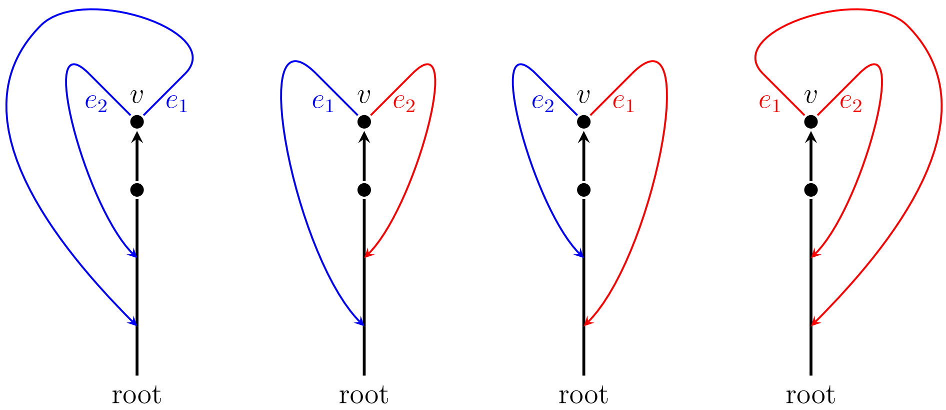


Fig. 3. In a planar drawing of a connected DFS-oriented graph, overlapping fundamental cycles are nested, if and only if they are oriented alike. If the root is incident to the outer face, the lowest vertex of their union is contained in the outer of the two cycles.

of one cycle that is not common to both is drawn completely inside the other cycle.

Observation 1 *In a planar drawing of a DFS-oriented graph $G = (V, T \uplus B)$, two overlapping cycles are nested, if and only if they are oriented alike.*

By assigning orientations we essentially determine whether the inside is to the left or to the right of a directed cycle, but the above observation does not specify which of two nested cycles is enclosed by the other.

For disambiguation we use the convention that the root of a DFS tree is incident to the outer face and define the nesting depth of a fundamental cycle using the following concepts.

The *return points* of a tree edge $v \to w \in T$ are the ancestors u of v with $u \overset{+}{\to} v \to w \overset{*}{\to} x \hookrightarrow u$ for some descendant x of w. A back edge $v \hookrightarrow w$ has exactly one return point, its target w. The return points of a vertex $v \in V$ are formed by the union of all return points of outgoing edges $(v, w) \in E^+(v) \subseteq T \uplus B$. A back edge $x \hookrightarrow u$ is a *return edge* for every tree edge $v \to w$ with $u \overset{+}{\to} v \to w \overset{*}{\to} x \hookrightarrow u$, and for itself.

The *lowpoint* of an edge is its lowest return point, if any, or its source if none exists. Note that the lowpoint of a back edge is also the lowest vertex of its fundamental cycle, and therefore called the *lowpoint of that cycle.*

Our second important observation establishes nesting constraints induced by lowpoints of cycles. It is justified by noting that if the root is on the outer face and there is a proper tree path from the lowpoint of one cycle to that of another cycle, this path can not be part of the inner of the two cycles.

Observation 2 *In a planar drawing of a connected DFS-oriented graph $G = (V, T \uplus B)$ with the root of the DFS tree on the outer face, overlapping fundamental cycles are nested according to their lowpoint order.*

3.3 Relation to planar embeddings

The above two observations about orientations have immediate consequences for planar embeddings which become evident by considering the single fork in each of the four configurations in Figure 3.

Considering the fork of a pair of differently oriented cycles, we see that the outgoing edge of the left cycle is before the outgoing edge of the right cycle in the linearized order at branching point v.

For a pair of cycles oriented alike, on the other hand, the order depends on their orientation. In case they are right cycles and one contains a vertex that is strictly lower than those in the other cycle, the lower cycle's outgoing edge (e_1 in Figure 3) comes first in the linearized order at branching point v. The converse is true when v is the branching point of left cycles.

A vertex may be the branching point for several pairs of overlapping cycles. Combining both observations yields a (partial) embedding at branching points: outgoing edge of left cycles need to be before those of right cycles, and the internal ordering in each subset is determined by lowpoints. Note that there may be ties, and that outgoing tree edges may be part of several, differently oriented cycles. We will have to resolve these ambiguities, but otherwise the approach rests entirely on Observations 1 and 2.

4 The Left-Right Planarity Criterion

With the above motivation in mind, we say that the *side* of a back edge in a planar drawing is *right*, if its fundamental cycle is oriented clockwise, and *left* otherwise. Assigning a side to a back edge thus corresponds to orienting a fundamental cycle, and this will be all that needs to be done.

The following, crucial definition summarizes all constraints resulting from sets of overlapping fundamental cycles in terms of their respective back edge. It is worth noting that all constraints are generated by a single type of configuration associated with forks.

Definition 4 (LR partition) *Let $G = (V, T \uplus B)$ be a DFS-oriented graph. A partition $B = L \uplus R$ of its back edges into two classes, referred to as* left

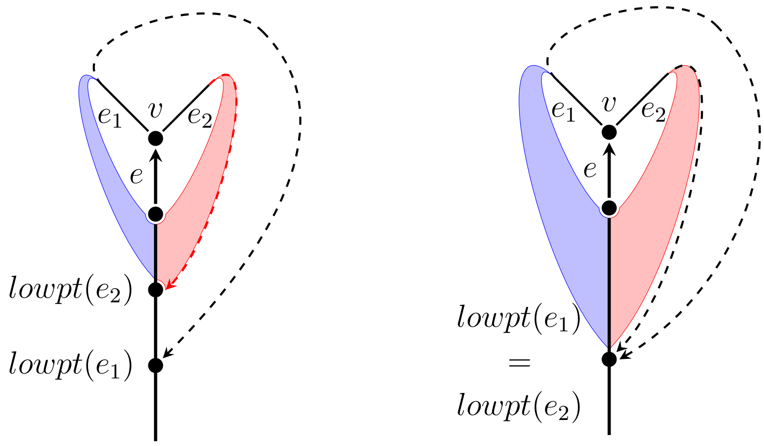


Fig. 4. LR constraints associated with $e = u \to v$.

and right*, is called* left-right partition*, or* LR partition *for short, if for every fork consisting of* $u \to v \in T$ *and* $e_1, e_2 \in E^+(v)$

(1) all return edges of e_1 *ending strictly higher than* $lowpt(e_2)$ *belong to one class and*
(2) all return edges of e_2 *ending strictly higher than* $lowpt(e_1)$ *to the other.*

The LR partition constraints are illustrated in Figure 4. They can be broken down into two sets of pairwise constraints: *same*-constraints forcing two back edges to be on the same side, and *different*-constraints forcing them to be on opposite sides; furthermore, each constraint is *associated* with a unique tree edge ($e = u \to v$ in Figure 4).

Note that two back edges are subject to a constraint only if their fundamental cycles overlap, and the minimal configurations inducing a constraint are characterized in Section 7. It is rather striking that the above partition constraints (based on an arbitrary DFS orientation) represent a condition equivalent to planarity.

Theorem 5 (Left-Right Planarity Criterion) *A graph is planar if and only if it admits an LR partition.*

While necessity of the LR constraints is straightforward, we prove sufficiency in the next section by constructing a planar embedding from a given LR partition. The construction is guided by those constraints that orientation and nesting of fundamental cycles impose on an embedding.

Removing the following ambiguity will simplify both, argumentation and algorithm. An LR partition is called *aligned*, if all return edges of a tree edge e that return to $lowpt(e)$ are on the same side.

Lemma 6 *Any LR partition can be turned into an aligned LR partition.*

PROOF. Consider two return edges b_1, b_2 of a tree edge $e = u \rightarrow v$ that end at $lowpt(e)$. If one of them is involved in any LR constraint as specified in Definition 4, this constraint is associated with a tree edge $e' = u' \rightarrow v'$ such that $v' \overset{*}{\rightarrow} v$ and $lowpt(e')$ is strictly lower than $lowpt(e)$. Since b_1, b_2 originate from a common subtree entered by e and have the same return point, actually both are involved in this constraint and even required to be on the same side. Thus, alignments do not lead to contradictions. $\square$

4.1 Combinatorial embedding

Consider again Figure 3, and recall that the orientation of overlapping fundamental cycles induces a partial ordering of edges around forks.

As motivated Section 3, the clockwise cyclic orderings of edges around non-root vertices are linearized by starting from the unique incoming tree edge. Then, outgoing edges belonging to a counterclockwise cycle need to appear before those belonging to a clockwise cycle. Moreover, outgoing edges of clockwise (counterclockwise) cycles must be ordered outside in (inside out) around their branching point.

Given a DFS-oriented graph $G = (V, T \uplus B)$ together with an LR partition of all back edges, we show that a planar embedding can be obtained by extending the partition to cover tree edges as well and defining a linear nesting order on the outgoing edges of each vertex. If the root is incident to the outer face, the order determines an outside-in nesting of the cycles. The order is used without modification as the embedding order for right outgoing edges, but reversed for left outgoing edges by flipping them to appear before any right edges. In an implementation, this can be realized by assigning each edge its rank in the nesting order, changing the sign of left edges to minus, and sorting the edges according to the signed ranks.

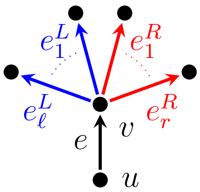
Extension of LR partitions to tree edges is straightforward. If a tree edge has any return edges (i.e., its source is neither the root nor a cut vertex), it is assigned to the same side as one of its return edges ending at the highest return point (i.e., according to an innermost fundamental cycle it is part of). Otherwise, the side is arbitrary.

To determine the partial *nesting order* $\prec$, assume for a moment that all edges are on the right side and consider a fork consisting of $u \rightarrow v$ and outgoing edges e_1, e_2 of v. If both have return edges, v is a branching point of overlapping fundamental cycles sharing $u \rightarrow v$. Since both cycles are clockwise for now, we must properly nest them to avoid edge crossings. Since we fixed the root of the DFS tree to be part of the outer face, we have to define $e_1 \prec e_2$ if and only if the lowpoint of e_1 is strictly lower than that of e_2. If both have the same lowpoint, but, say, only e_2 has another return point, we say that e_2 is *chordal* and let $e_1 \prec e_2$, because cycles containing e_2 and a return edge ending higher than $lowpt(e_2)$ can only lie inside of cycles containing e_1 and a return edge ending at $lowpt(e_1) = lowpt(e_2)$. If both e_1 and e_2 are chordal, the tie is broken arbitrarily, because eventually these two edges must be on different sides anyway.

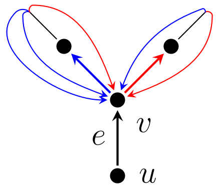
In the planarity testing algorithm, $\prec$ will be mimicked by defining the *nesting depth* of an edge e to be twice the height of the lowest lowpoint of any cycle containing e, plus one if e is chordal.

The partial nesting order $\prec$ is extended to a combinatorial embedding by *LR ordering*, i.e. by flip-reversing left edges before right ones and placing incoming back edges on the appropriate side of the tree edge leading to them. Some care is needed to avoid crossings of back edges, but we will see that, algorithmically, this embedding is almost trivial to obtain.

Definition 7 (LR Ordering) *Given an LR partition, let $e_1^L \prec \cdots \prec e_\ell^L$ be the left outgoing edges of a vertex v, and $e_1^R \prec \cdots \prec e_r^R$ its right outgoing edges. If v is not the root, let u be its parent. The clockwise* left-right ordering, *or* LR ordering *for short, of the edges around v is defined as follows:*



$$
\begin{array}{l}
(u, v), \\
L(e_\ell^L), e_\ell^L, R(e_\ell^L), \ldots, L(e_1^L), e_1^L, R(e_1^L), \\
L(e_1^R), e_1^R, R(e_1^R), \ldots, L(e_r^R), e_r^R, R(e_r^R)
\end{array}
$$

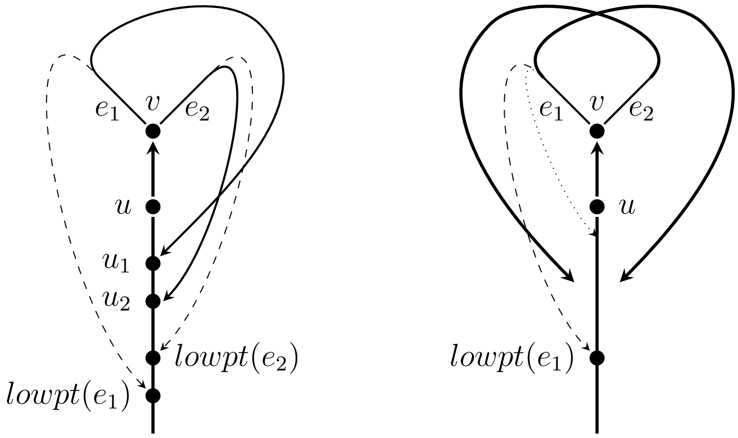


where (u, v) is absent if v is the root, and $L(e)$ and $R(e)$ denote the left and right incoming back edges whose cycles share e. For two back edges $b_1 = x_1 \hookrightarrow v, b_2 = x_2 \hookrightarrow v \in R(e)$ let $z \to x, (x, y_1), (x, y_2)$ be the fork of $C(b_1)$ and $C(b_2)$. Then, b_1 comes after b_2 in $R(e)$ if and only if $(x, y_1) \prec (x, y_2)$. If $b_1, b_2 \in L(e)$, the order is reversed.

Lemma 8 *Given an LR partition, LR ordering yields a planar embedding.*

PROOF. Let $G = (V, T \uplus B)$ be a DFS-oriented graph with an LR partition $B = L \uplus R$. We assume that the partition is aligned and extend it to cover also the tree edges as described above. Now consider the embedding defined by LR ordering the edges around each vertex.

Since a graph with a spanning tree can always be drawn in such a way that a given embedding is respected, no two edges cross more than once, and none



(a) same side (b) different sides

Fig. 5. Two types of crossings in proof of Lemma 8.

of the crossings involves a tree edge, the embedding is either planar, or any such drawing yields a simple crossing of two back edges (a crossing of more than two edges can be resolved into pairwise crossings). Only two cases are possible.

Case 1: (crossing back edges in same class)

Assume $x_1 \hookrightarrow u_1$, $x_2 \hookrightarrow u_2 \in R$ cross (the other case is symmetric). If $u_1 = u_2$, the crossings contradicts our definition of LR ordering the edges around that vertex.

W.l.o.g. we may therefore assume that u_1 is strictly higher than u_2, and u_2 therefore outside of the clockwise cycle $u_1 \overset{+}{\to} x_1 \hookrightarrow u_1$. Since the crossing is simple, x_2 in turn must be inside this cycle, and $u_1 \overset{+}{\to} x_1$ and $u_2 \overset{+}{\to} x_2$ cannot be disjoint (because we must enter the cycle somewhere along $u_2 \overset{+}{\to} x_2$). Let v be their highest common vertex, and e_1, e_2 the first edges on $v \overset{*}{\to} x_1$ and $v \overset{*}{\to} x_2$.

Since x_2 is inside of the clockwise cycle, e_1 comes before e_2 in the order around v. On the other hand, the LR partition requires that all return edges of e_1 ending higher than u_2 are on the same side as $x_1 \hookrightarrow u_1$, so that also e_1 is a right edge. LR ordering at v then implies that e_2 must be a right edge as well with $e_1 \prec e_2$.

By definition of $\prec$, either $lowpt(e_1)$ is strictly lower than u_2, or $lowpt(e_1) = u_2 = lowpt(e_2)$ and e_2 is chordal as well. In the former case, $x_1 \hookrightarrow u_1$ and $x_2 \hookrightarrow u_2$ had to be assigned different sides. In the latter case, the highest ending return edge of e_2 is right as is e_2, but conflicting with $x_1 \hookrightarrow u_1$ which is also right. In either case a contradiction.

Case 2: (crossing back edges in different classes)

Assume $x_1 \hookrightarrow u_1 \in R$ and $x_2 \hookrightarrow u_2 \in L$ (the other case is symmetric). Since the crossing is simple, the tree paths $u_1 \overset{+}{\to} x_1$ and $u_2 \overset{+}{\to} x_2$ cannot be disjoint and we define v, e_1, e_2 as in Case 1.

Again, e_1 must be before e_2 in the LR ordering of v for the back edges to cross. If $u_1 = lowpt(e_1) = lowpt(e_2) = u_2$, the LR partition is not aligned.

Otherwise, we may assume that $lowpt(e_1)$ is strictly lower than u_2 (the case that $lowpt(e_2)$ is strictly lower than u_1 is symmetric). Then, all return edges of e_2 ending at u_2 or higher must be on the same side as $x_2 \hookrightarrow u_2 \in L$, so that e_2 is left as well. Since e_1 comes before e_2, it must also be left and $e_2 \prec e_1$.

Due to the way we define sides for tree edges, e_1 is left only if it has a left return edge ending strictly higher than $lowpt(e_1)$ (because it must end at least as high as $x_1 \hookrightarrow u_1 \in R$ and the LR partition is aligned). On the other hand, $e_2 \prec e_1$ implies that $lowpt(e_2)$ is lower than or equal to $lowpt(e_1)$. This is a contradiction, since the LR constraints rule out that e_1 and e_2 have return edges ending strictly higher than $lowpt(e_2)$ and $lowpt(e_1)$ that are both on the left.

Since both types of crossings contradict our assumptions, the embedding is planar. □

We have thus proved constructively the non-obvious implication of the left-right planarity criterion (Theorem 5).

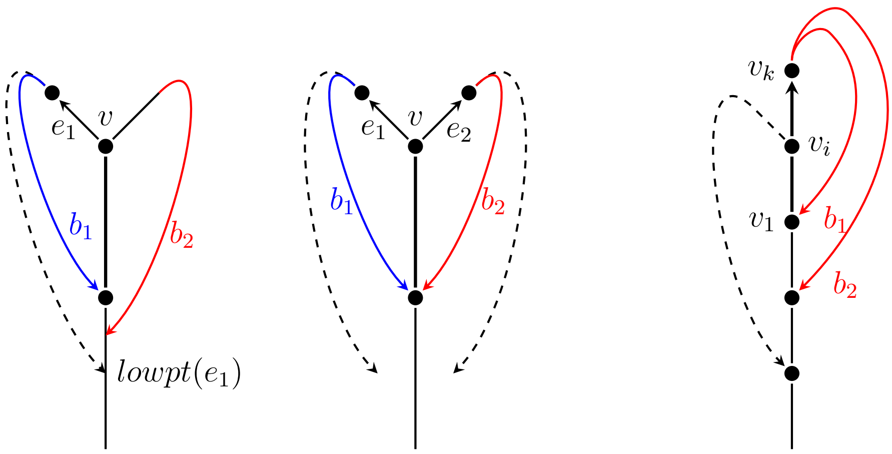
5 Straightforward Algorithm

As an intermediate exercise, we derive a polynomial-time planarity test directly from the characterization in the previous section. It mainly serves to build a better intuition for the subsequent linear-time algorithms.

Let $G = (V, T \uplus B)$ be a DFS-oriented graph. According to Theorem 5, testing planarity amounts to testing for the existence of an LR partition $B = L \uplus R$ of its back edges. Such a partition exists, if and only if the LR constraints of all forks can be satisfied simultaneously. This can be tested using the following immediate consequence of Definition 4, which is illustrated in Figure 6.

Corollary 9 *Let $G = (V, T \uplus B)$ be a DFS-oriented graph. For a pair of back edges $b_1, b_2 \in B$ with overlapping fundamental cycles, let $v_1 \to \cdots \to v_k$ be the tree path of their intersection and $(v_{k-1}, v_k), e_1, e_2$ the corresponding fork with $e_1 \overset{*}{\to} b_1$ and $e_2 \overset{*}{\to} b_2$. Then, b_1 and b_2 are subject to*

- *a* different-*constraint, if and only if $lowpt(e_2) < lowpt(b_1)$ and $lowpt(e_1) < lowpt(b_2)$.*
- *a* same-*constraint, if and only if $lowpt(e') < \min\{lowpt(b_1), lowpt(b_2)\}$ for some $e' = (v_i, w) \in T \uplus B$, $1 < i < k$, $w \neq v_{i+1}$.*



(a) *different*-constraint (b) *same*-constraint

Fig. 6. The constraints between pairs of back edges b_1, b_2 summarized in the definition of LR constraints are induced by three types of minimal configurations (de Fraysseix and Rosenstiehl 1985; cf. Corollary 9).

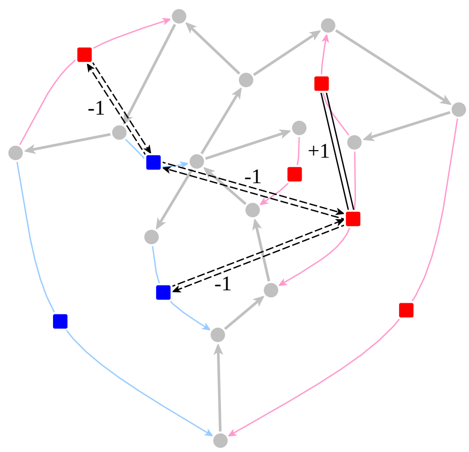


Fig. 7. The constraint graph for the example from Figure 3 consists of eight (square) vertices, one *same*-constraint, and three *different*-constraints. Note that the LR partition is not unique because, e.g., the lower isolates are not aligned.

With this observation and precomputed lowpoints, we can test whether two given back edges are subject to a constraint by traversing their fundamental cycles, determining the branching point in case they overlap, and comparing a few lowpoints (possibly including those of edges incident to vertices in the intersection). Note that a *different*-constraint can be associated with only one fork, whereas a *same*-constraint may be induced repeatedly.

Pairwise constraints can be represented in a graph that has back edges as its vertices and an edge between two of them, if they are subject to a constraint. To distinguish the type of constraint, we use *signed edges* that carry labels "+1" or "−1" as indicated in Figure 7.

Definition 10 *Let $G = (V, T \uplus B)$ be a DFS-oriented graph such that each pair of back edges $b_1, b_2 \in B$ is subject to at most one type of constraint. The signed graph $\mathcal{C}(G) = (B, E(\mathcal{C}); \sigma : E(\mathcal{C}) \to \{-1, +1\})$ with*

$$\sigma(b_1, b_2) = \begin{cases} -1 & \textit{if } b_1, b_2 \in B \textit{ are subject to a } \text{different-}\textit{constraint} \\ +1 & \textit{if } b_1, b_2 \in B \textit{ are subject to a } \text{same-}\textit{constraint} \end{cases}$$

is called constraint graph *of G.*

If any pair of back edges is subject to both a *same*-constraint and a *different*-constraint, no LR partition can exist and hence the graph is non-planar. This is noticed during the construction of $\mathcal{C}(G)$, and we may therefore assume in the following that each pair of back edges is subject to at most one type of constraint.

Finding an LR partition that satisfies all LR constraints is then equivalent to testing whether the constraint graph is *balanced* (Harary and Cartwright, 1956), i.e. whether there is a bipartition such that each edge labeled "+1" connects two vertices in the same set, and each edge labeled "−1" connects vertices in different sets. Balancedness of signed graphs is equivalent to the absence of cycles with an odd number of edges labeled "−1," and can hence be tested in linear time using a variant of breadth-first search (Harary and Kabell, 1980). The reader is encouraged to fill in the details.

6 Linear-Time Algorithm

The straightforward partition approach of the previous section can be refined into a linear-time algorithm for planarity testing and embedding. Extraction of a minimal non-planar subgraph is treated only in the next section. After a high-level description of its three main phases shown in Algorithm 1, full implementation details are provided for all operations but those concerning the specific data structure representing the graph and its embedding.

Orientation. The algorithm is based on the left-right planarity criterion and therefore starts with a depth-first search to orient the input graph. For each connected component, the root of its spanning DFS tree is appended to a list, *Roots*. The tree-path distance of a vertex from the root of its component is stored in an array *height*, so that roots of unexplored components are identified by still having the initial value ∞. Different from most other planarity algorithms, there is no need to worry about biconnected components.

During DFS, lowpoints of edges are computed and the partial nesting order $\prec$

variable	**type**	**interpretation**	**initially**
height	integer node array	tree-path distance from root	∞
lowpt	integer edge array	*height* of lowest return point	n.a.
$lowpt2$	integer edge array	*height* of next-to-lowest return point (tree edges only)	n.a.
nesting_depth	integer edge array	proxy for nesting order $\prec$ given by twice *lowpt* (plus 1 if chordal)	n.a.

(a) orientation phase

variable	**type**	**interpretation**	**initially**
ref	edge array of edges	edge relative to which side is defined	$\perp$
side	edge array of signs $\{-1, 1\}$	side of edge, or modifier for side of reference edge	1
$I = [low, high]$	pair of edges	interval of return edges represented by its two edges with extremal lowpoints	n.a.
$P = (L, R)$	pair of intervals	intervals with conflicting edges, i.e., a conflict pair	n.a.
S	stack of conflict pairs	conflict pairs consisting of current return edges	$\emptyset$
stack_bottom	edge array of conflict pairs	top of stack S when traversing the edge (tree edges only)	n.a.
lowpt_edge	edge array of edges	next back edge in traversal (i.e. with lowest return point)	n.a.

(b) testing phase

variable	**type**	**interpretation**
leftRef	vertex array of edges	leftmost back edge from current DFS subtree (i.e. after next incoming left back edge)
rightRef	vertex array of edges	tree edge leading into current DFS subtree (i.e. before next incoming right back edge)

(c) embedding phase

Fig. 8. Main variables used in the algorithm.

Algorithm 1: Left-Right Planarity Algorithm

input: simple, undirected graph $G = (V, E)$
output: planar embedding (halts if graph is not planar)

if $|V| > 2$ **and** $|E| > 3|V| - 6$ **then HALT: not planar**

▼ **orientation**
  for $s \in V$ **do**
    if $height[s] = \infty$ **then**
      $height[s] \leftarrow 0$; append $Roots \leftarrow s$
      DFS1(s) `/* see Algorithm 2 */`

▼ **testing**
  sort adjacency lists according to non-decreasing $nesting\_depth$
  for $s \in Roots$ **do** DFS2(s) `/* see Algorithm 3 */`

▼ **embedding**
  for $e \in E$ **do** $nesting\_depth[e] = \underline{\text{sign}}(e) \cdot nesting\_depth[e]$
  sort adjacency lists according to non-decreasing $nesting\_depth$
  for $s \in Roots$ **do** DFS3(s) `/* see Algorithm 6 */`

where
integer sign(edge *e*)
  if $ref[e] \neq \perp$ **then**
    $side[e] \leftarrow side[e] \cdot \underline{\text{sign}}(ref[e])$
    $ref[e] \leftarrow \perp$
  return $side[e]$

is determined by assigning to each edge an integer value $nesting\_depth$ such that $e_1 \prec e_2$ implies $nesting\_depth[e_1] < nesting\_depth[e_2]$.

Testing. To determine whether there exists an aligned LR partition, the DFS trees are traversed for a second time. The traversal order is modified, however, by visiting outgoing edges in the order given by $nesting\_depth$. This second traversal halts if the graph is not planar, and we discuss in Section 7 how to extract a Kuratowski subgraph in that case.

The tentative side of edges may change often during the test, so that the bipartition is maintained only implicitly for efficiency reasons. An edge array ref specifies for each edge a reference edge relative to which its side is defined, and in an edge array $side$ a value of $+1$ or -1 indicates whether the side of the edge is the same as, or different from, the side of its reference edge. If the reference edge of e is undefined, i.e. $ref[e] = \perp$, the value of $side[e]$ specifies the side directly, where -1 is for left and $+1$ is for right.

Embedding. Given an LR partition, flip-reversal of left edges is performed by sorting the outgoing edges in all adjacency lists once again according to their nesting order, though now modified by the signs in *side*. Since the multiplication of *nesting_depth* with *side* only changes the sign of left edges to negative, they are effectively placed before all right edges, and in reverse order. To complete the LR ordering, incoming edges are rearranged during a third traversal of the DFS forest that is guided once again by the new order of outgoing edges.

For each of the three main phases, we provide detailed pseudo-code with ample comments in the subsequent sections.

6.1 Orientation

Algorithm 2: Phase 1 – DFS orientation and nesting order

```
DFS1(vertex v)
  e ← parent_edge[v]
  while there exists some non-oriented {v, w} ∈ E do
    orient {v, w} as (v, w)
    lowpt[(v, w)] ← height[v];   lowpt2[(v, w)] ← height[v]
    if height[w] = ∞ then /* tree edge */
      parent_edge[w] ← (v, w)
      height[w] ← height[v] + 1
      DFS1(w)
    else /* back edge */
      lowpt[(v, w)] ← height[w]
    ▼ determine nesting depth
      nesting_depth[(v, w)] ← 2 · lowpt[(v, w)]
      if lowpt2[(v, w)] < height[v] then /* chordal */
        nesting_depth[(v, w)] ← nesting_depth[(v, w)] + 1
    ▼ update lowpoints of parent edge e
      if e ≠ ⊥ then
        if lowpt[(v, w)] < lowpt[e] then
          lowpt2[e] ← min{lowpt[e], lowpt2[(v, w)]}
          lowpt[e] ← lowpt[(v, w)]
        else if lowpt[(v, w)] > lowpt[e] then
          lowpt2[e] ← min{lowpt2[e], lowpt[(v, w)]}
        else
          lowpt2[e] ← min{lowpt2[e], lowpt2[(v, w)]}
```

The purpose of the first DFS is to orient the graph, and to determine lowpoints and nesting order $\prec$. It is therefore a standard DFS computing the auxiliary variables listed in Table 8(a). Except for *height*, all of them are determined during backtracking.

Our use of lowpoints is slightly unusual in two ways. Firstly, we determine lowpoints for edges rather than vertices, and, secondly, we do not assign DFS numbers, but heights. The latter induce the same ordering of ancestors as do DFS numbers, but are related to the tree more intuitively, and in general result in a smaller range of values which may in turn speed up the subsequent sorting of adjacency lists according to *nesting_depth*.

Second lowpoints stored in $lowpt2$ only serve to determine whether an edge has more than one return point (i.e., it is chordal), and are not needed by themselves.

The rationale for representing $\prec$ via *nesting_depth* is two-fold: firstly, we can sort the edges in linear time using, e.g., bucket sort, because the range of values is linear in the size of the graph, and secondly, flip-reversal of left edges after the second phase can be performed by changing their sign and sorting again.

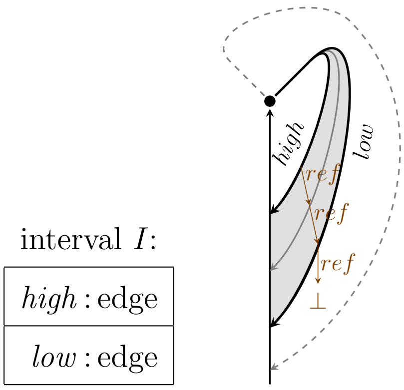
6.2 Testing

The second phase is the working horse of the algorithm. It determines an aligned LR partition including all tree edges, if one exists. With this, LR ordering can be carried out as described in Section 6.3, otherwise the code can be augmented to identify fundamental cycles whose union yields a Kuratowski subgraph as described in Section 7.

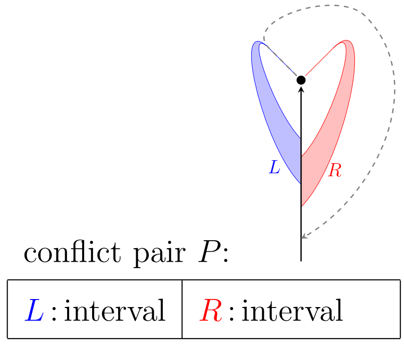
Our strategy will be to implement the straightforward algorithm of Section 5 without constructing the constraint graph explicitly. Recall that constraints are associated with a tree edge and that there are only two types of constraints: according to Definition 4, a pair of back edges with overlapping fundamental cycles can be required to be placed either on the same side or on different sides.

Clearly, we cannot afford to detect the edges of the signed constraint graph individually, because their number may already be quadratic in the size of the original graph. Since our actual goal is a bipartition certifying that the constraint graph is balanced, we will eagerly maintain bipartitions of its connected components and represent constraints only implicitly to test for contradictions.

To represent a bipartition it is sufficient to have a signed spanning forest of



(a) Return edges forced to be on the same side are represented in a list linked by ref-pointers and ordered by height of return point



(b) Two intervals containing one or more pairs of conflicting edges are stored together

Fig. 9. The main data structure is a stack S storing conflicting pairs of intervals with consecutively returning back edges.

the constraint graph available. We will therefore construct a rooted tree for each component using a reference pointer ref for every edge. Such a pointer is stored not only for back edges, but also for tree edges in the DFS orientation, since in the extended LR partition required for LR ordering, their sides are determined by reference to a return edge ending at the highest return point, anyway.

A second array, *side*, is used to store the side of all edges that are roots in our spanning forest of the constraint graph. For all other edges the array holds the sign of the unique outgoing constraint-graph edge linking them to their corresponding reference edge. As indicated earlier, values $+1$ and -1 will therefore be interpreted either as *right* and *left*, or as *same* and *different*.

To grow the partial bipartition, we need to keep track of all constraints encountered during an ordered examination of all forks, but instead of storing constraints individually, a compact data structure is used to represent their transitive closure. Observe that the *same*-constraints induced by a fork $u \to v$, $e_1, e_2 \in E^+(v)$ in Definition 4 involve two sets of return edges with a simple structure. For, say, e_1 let $h = x_h \hookrightarrow u_h$ and $\ell = x_\ell \hookrightarrow u_\ell$ be the two (possibly equal) return edges ending at the highest and lowest return point of e_1 that is also a return point of $u \to v$ (i.e., $u_h \neq u$) and strictly higher than $lowpt(e_2)$. Then we know that h_ℓ and all return edges $x' \hookrightarrow u'$ of e_1 with a return point in $u_\ell \xrightarrow{*} u_h$ are in the same group of *same*-constraints. This *interval* of edges can thus be represented by its two bounding members, h and ℓ, as shown in Figure 9(a). Return edges belonging to an interval are maintained in a singly-linked list, from highest to lowest return point, using the ref-array.

The closure of *different*-constraints can be summarized similarly, because by transitivity it always involves all pairs of edges in a pair of intervals. A *conflict pair* therefore consists of two intervals of edges subject to at least one *different*-constraint as shown in Figure 9(b). It represents their tentative assignment to the left and right, and thus a partial bipartition.

The second DFS traversal is designed to build an extended LR partition of edges incrementally by merging conflict pairs. Its main data structure is a stack S of conflict pairs representing all constraints associated with a tree edge that has been traversed, but not yet backtracked over. Note that these constraints involve only back edges that have already been traversed, but return to a vertex below the current one. In other words, each back edge in the stack is a return edge for at least one tree edge in the current DFS path.

By processing the DFS trees bottom-up, the constraints associated with an edge can be determined by merging those associated with its outgoing edges. Two main invariants are maintained. Clearly, we can not prove them before the algorithm is described, but since they provide an orientation for understanding the implementation, they are stated already here and the reader is encouraged to check that they are maintained. The first invariant eventually yields correctness of the implementation,

> **Partitioning Invariant:** The additional conflict pairs accumulated at the top of the stack between traversing a tree edge and backtracking over it represent a partial bipartition satisfying all non-crossing constraints associated with that edge.

and the second one ensures that constraint merging can be carried out efficiently.

> **Ordering Invariant:** For any two conflict pairs P, Q where P is above Q in the stack, no edge in P has a return point below that of an edge in Q. Each interval in a conflict pair is represented as a singly-linked list of return edges that is ordered from highest to lowest return point as well.

6.2.1 Ordered traversal

Pseudo-code for the second DFS is given in Algorithms 3–5. All edges have been oriented during the first DFS, and they are traversed again in the same direction. The traversal order differs, though, since adjacency lists have been rearranged according to *nesting_depth*, so that outgoing edges with lower lowpoints are traversed first. This reordering is crucial for the ordering invariant.

When visiting a vertex v during the DFS traversal, the high-level task is to

Algorithm 3: Phase 2 – Testing for LR partition

```
DFS2(vertex v)
    e ← parent_edge[v]
    for e_i ∈ E⁺(v) = ⟨e_1, ..., e_d⟩ do /* ordered by nesting_depth */
        stack_bottom[e_i] ← top(S)
        if e_i = parent_edge[target(e_i)] then /* tree edge */
            DFS2(target(e_i))
        else /* back edge */
            lowpt_edge[e_i] ← e_i;   push (∅, [e_i, e_i]) → S
        ▼ integrate new return edges
            if lowpt[e_i] < height[v] then /* e_i has return edge */
                if e_i = e_1 then
                    lowpt_edge[e] ← lowpt_edge[e_1]
                else
                    ▶ add constraints of e_i (Algorithm 4)

    ▼ remove back edges returning to parent
        if e ≠ ⊥ then /* v is not root */
            u ← source(e)
            ▶ trim back edges ending at parent u (Algorithm 5)
            ▼ side of e is side of a highest return edge
                if lowpt[e] < height[u] then /* e has return edge */
                    h_L ← top(S).L.high;   h_R ← top(S).R.high
                    if h_L ≠ ⊥ and (h_R = ⊥ or lowpt[h_L] > lowpt[h_R]) then
                        ref[e] ← h_L
                    else
                        ref[e] ← h_R
```

recursively determine the constraints for all outgoing edges and integrate them into those associated with parent edge $e = u \to v$ (if v is not a DFS root).

Before traversing an outgoing edge $e_i \in E^+(v)$, we therefore remember the top conflict pair $stack\_bottom[e_i]$ on S (where $top(S) = \perp$ if S is empty). If e_i was a tree edge in the first traversal, all constraints associated with e_i are recursively determined and pushed onto S. If e_i is a back edge, it is pushed onto S in a conflict pair of its own because it may be involved in later constraints. Recall that our goal is to determine an *aligned* LR partition. We therefore store in an edge array *lowpt_edge* the first back edge not traversed earlier. For edges that have return edges, this is the first return edge to their lowpoint and can thus be used as a reference for other return edges that have to be assigned to the same side to meet the consistency requirement. A back edge e_i is its own unique return edge to its lowpoint so that we let $lowpt\_edge[e_i] = e_i$.

From the partitioning invariant we known that when returning from the traversal of e_i, the conflict pairs above $stack\_bottom[e_i]$ represent a partial LR partition of all return edges of e_i. While processing the first outgoing edge e_1 we simply leave them on the stack, if any, and pass on $lowpt\_edge[e_1]$ to $lowpt\_edge[e]$. Note that, since e_1 has a return edge, $parent\_edge[v] = e \neq \perp$, i.e. v is not a root. For each of the other outgoing edges $e_i = e_2, \ldots, e_d \in E^+(v)$, the constraints above $stack\_bottom[e_i]$ are merged into those which have already been accumulated for e and are directly beneath in S. Constraint integration is the most essential step and described separately in Algorithm 4 and below.

After all outgoing edges have been traversed, we trim all those back edges from the top of S that are return edges of some $e_i \in E^+(v)$, but not of e, i.e. which end at u. This requires some annoyingly lengthy but simple case distinctions given in Algorithm 5 and explained below. Observe that, if v is a DFS root, then there is no parent edge $e = u \to v$, but there are also no remaining constraint pairs on S, since a DFS root does not have outgoing back edges and there is more than one outgoing tree edge only if each leads into a different biconnected component.

If existent, parent edge e is finally assigned to the side of a back ending at the highest return point as suggested by the LR ordering procedure of Section 4.1. By the ordering invariant, this edge is the highest return edge in one of the two intervals in the top conflict pair, and we have already removed all non-return edges. Observe that the stack cannot be empty if there is a return edge.

6.2.2 Adding constraints associated with the next outgoing edge

We have to merge all constraints associated with the next outgoing edge, e_i, with those already accumulated from $e_1, \ldots, e_{i-1}$. The involved intervals are therefore gathered one by one in an initially empty conflict pair P as illustrated in Figure 10.

Merge return edges of e_i into right interval. All return edges of e_i have been traversed since traversing e_i, and they are represented in the top conflict pairs on stack S down to, but not including, $stack\_bottom[e_i]$. All of these intervals have to be merged on one side because of the *same*-constraints induced by the fundamental cycle of $lowpt\_edge[e]$ according to Definition 4. If there is a conflict pair with two non-empty intervals, merging on one side violates an earlier constraint and the graph is not planar.

There is at least one conflict pair above $stack\_bottom[e_i]$ for otherwise we would not have entered this section. The non-empty interval of each such pair is merged in the right interval $P.R$ of P without changing their order by

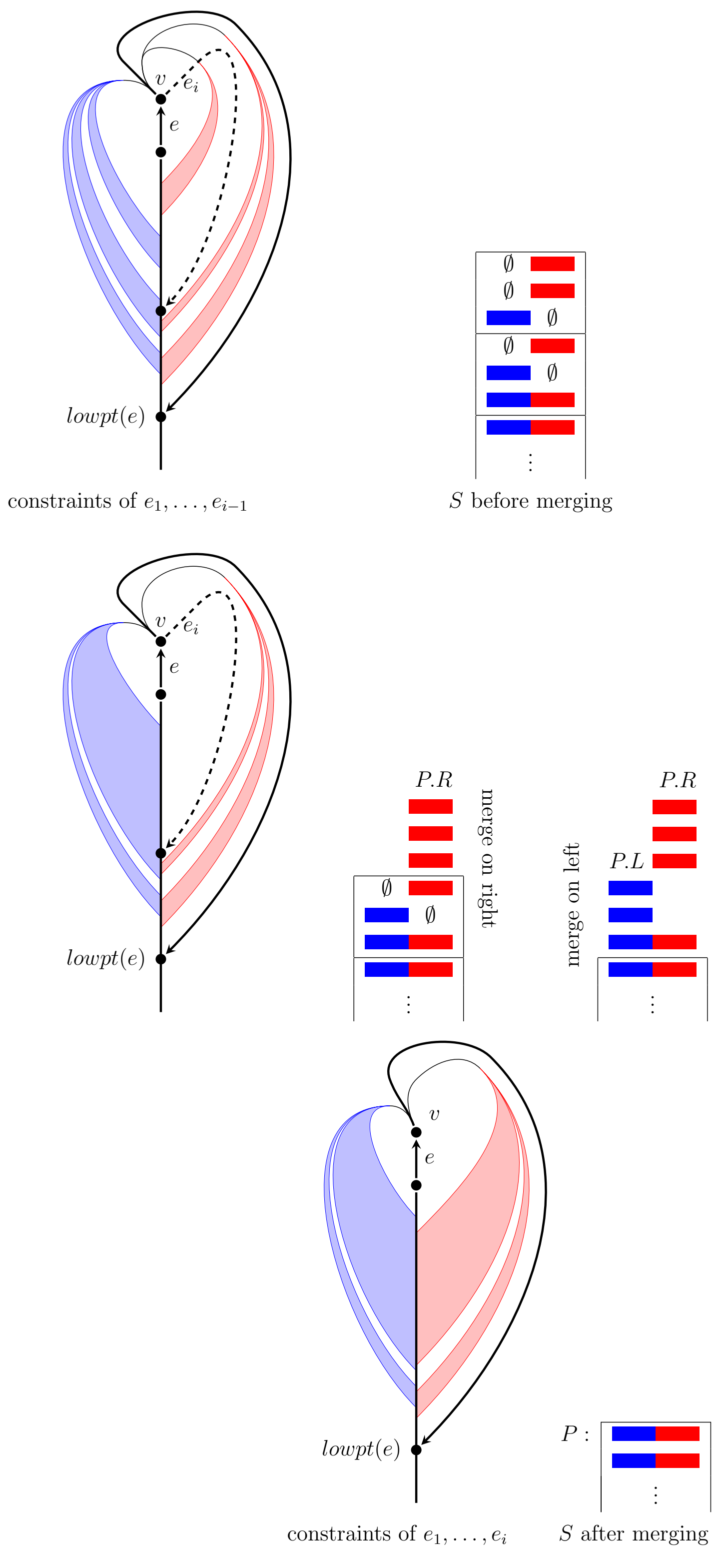


Fig. 10. In the core step of the algorithm, the constraints of e_i are merged into those of $e_1, \ldots, e_{i-1}$. Horizontal lines indicate where the top of stack S is divided by $stack\_bottom[e_i]$ and the topmost pair not in conflict with $lowpt\_edge[e_i]$. If $lowpt(e_i) = lowpt(e)$, the pair containing only $lowpt\_edge[e_i]$ is not merged into $P.R$, but the bipartition is aligned by assigning $ref[lowpt(e_i)] \leftarrow lowpt(e)$.

Algorithm 4: Adding constraints associated with e_i (part of Alg. 3)

```
▼ add constraints of e_i
    P ← (∅, ∅)
    ▼ merge return edges of e_i into P.R
        repeat
            Q ← pop(S)
            if Q.L ≠ ∅ then swap Q.L, Q.R
            if Q.L ≠ ∅ then
                HALT: not planar
            else
                if lowpt[Q.R.low] > lowpt[e] then /* merge intervals */
                    if P.R = ∅ then /* topmost interval */
                        P.R.high ← Q.R.high
                    else
                        ref[P.R.low] ← Q.R.high
                    P.R.low ← Q.R.low
                else /* align */
                    ref[Q.R.low] ← lowpt_edge[e]
        until top(S) = stack_bottom[e_i]
    ▼ merge conflicting return edges of e_1, ..., e_{i-1} into P.L
        while conflicting(top(S).L, e_i) or conflicting(top(S).R, e_i) do
            Q ← pop(S)
            if conflicting(Q.R, e_i) then swap Q.L, Q.R
            if conflicting(Q.R, e_i) then
                HALT: not planar
            else /* merge interval below lowpt(e_i) into P.R */
                ref[P.R.low] ← Q.R.high
                if Q.R.low ≠ ⊥ then P.R.low ← Q.R.low
            if P.L = ∅ then /* topmost interval */
                P.L.high ← Q.L.high
            else
                ref[P.L.low] ← Q.L.high
            P.L.low ← Q.L.low
    if P ≠ (∅, ∅) then push P → S

where
boolean conflicting(interval I, edge b)
    return (I ≠ ∅ and lowpt[I.high] > lowpt[b])
```

having the lowest edge of $P.R$ refer to the highest edge of the next conflict pair and replacing it accordingly. An exception is the interval containing a return edge to the lowpoint of e; to align the LR partition, we make it refer to the *lowpt_edge* directly.

Merge conflicting return edges of $e_1, \ldots, e_{i-1}$ into left interval. Return edges of $e_1, \ldots, e_{i-1}$ with lowpoints higher than $lowpt[e_i]$ are subject to pairwise *same*-constraints and to a *different*-constraint with respect to some return edge of e_i. (If $lowpt[e_i] = lowpt[e]$ this is not $lowpt\_edge[e_i]$ but, e.g., a back edge returning to $lowpt2[e_i]$ which must exist, because apparently $lowpt2[e_{i-1}]$ exists as well by the way outgoing edges are ordered).

So while there are conflict pairs on the stack that contain return edges with lowpoints higher than $lowpt[e_i]$, these have to be merged on one side. If such a pair contains two intervals ending above $lowpt[e_i]$, we again have a contradiction with a previous constraint and thus non-planarity. If only one side ends above $lowpt[e_i]$, we merge the other into $P.R$ (effectively closing these constraints under transitivity).

The actual merging of intervals is performed in the same way as above, and the final pair can be placed on the stack.

6.2.3 Trimming back edges

The purpose of Algorithm 5 is to remove all those back edges from conflict pairs on the stack that have the parent of the current tree edge $e = u \rightarrow v$ as their lowpoint, because they are no return edges of e or any lower tree edge, and therefore not subject to any constraint associated with a tree edge still to be processed.

Dropping entire conflict pairs. If the lowest lowpoint on either side of a conflict pair P is the source of the current tree edge $u \rightarrow v$, all lowpoints of back edges in P are the same and the edges will not be involved in any future constraints. The pair is finalized by assigning the lowest back edge of the left interval to the left side. Since *side* is initialized with 1, the lowest back edge in the right interval $P.R$ is already assigned correctly to the right side, and all other back edges b in P to the same side as $ref[b]$.

Trimming a left interval. Since back edges in an interval are concatenated by ref-pointers in an order monotonic in the *height* of their lowpoints, we can simply remove back edges from the upper end of the left interval until the highest lowpoint is no longer u, or the interval has become empty. In the latter case the lower end of the interval is still defined and made to refer to an edge on the other side, setting its *side* to -1 accordingly. Note that the right interval cannot be empty for otherwise the entire conflict pair had been removed in the first while loop. All other removed back edges still refer to a

Algorithm 5: Removing back edges ending at parent u (part of Alg. 3)

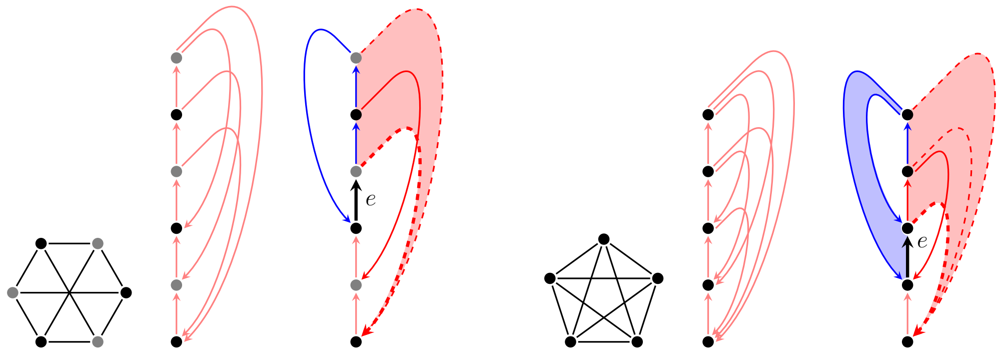
```
▼ trim back edges ending at parent u
    ▼ drop entire conflict pairs
        while S ≠ ∅ and lowest(top(S)) = height[u] do
            P ← pop(S)
            if P.L.low ≠ ⊥ then side[P.L.low] ← −1
    if S ≠ ∅ then /* one more conflict pair to consider */
        P ← pop(S)
        ▼ trim left interval
            while P.L.high ≠ ⊥ and target(P.L.high) = u do
                P.L.high ← ref[P.L.high]
            if P.L.high = ⊥ and P.L.low ≠ ⊥ then /* just emptied */
                ref[P.L.low] ← P.R.low;   side[P.L.low] ← −1
                P.L.low ← ⊥
        ▶ trim right interval
        push P → S

where
integer lowest(conflictpair P)
    if P.L = ∅ then return lowpt[P.R.low]
    if P.R = ∅ then return lowpt[P.L.low]
    return min{lowpt[P.L.low], lowpt[P.R.low]}
```

back edge on the same side, so that the initial 1 of their *side*-entry must not be changed.

Trimming a right interval. This is symmetric to the previous operation. Note, however, that the assigned *side* in case the right interval becomes empty is -1 as well, because this indicates that the side of the lowest back edge is different from the side of the lowest back edge in the left interval. Again, the left interval cannot be empty.

Assigning a side to a tree edge. After trimming all back edges ending at source u of the current tree edge $e = u \to v$ in Algorithm 3, the side of e is determined by reference to a highest return edge. There is a return edge only if $lowpt[e] < height[u]$. Otherwise, u is a cutvertex or root and it does not matter which side e is assigned to. Since the existence of a return edge renders S non-empty, the ordering invariant asserts that the highest return point is found by comparing lowpoints of the highest return edges in the two intervals of the top constraint pair on S (checking for existence).



(a) $K_{3,3}$ (b) K_5

Fig. 11. The algorithm testing $K_{3,3}$ and K_5 for planarity. For both cases, the status before starting the second DFS is depicted in the middle, and the algorithm halts in the configuration on the right while processing e.

At the end of the testing phase, a non-crossing LR partition is given implicitly by edge arrays ref and *side*, if and only if the graph is planar. These define the side of an edge e relative to another, where *side*$[e]$ indicates whether the side is the same or different from that of $ref[e]$. Since $ref[e]$ always has a strictly lower target than e, the referrals are acyclic and form a rooted spanning forest of the constraint graph. The roots of that forest refer to $\perp$, and their side is determined explicitly by *side*. After dereferencing all referrals at the beginning of the embedding phase, the LR partition is known explicitly.

Two small examples are shown in Figure 11. Even though both graphs are non-planar, the workings of the algorithm are nicely illustrated, since coloring and embedding correspond to the current (implicitly represented) bipartition and LR ordering.

6.3 Embedding

Compared to other planarity algorithms, the embedding phase is extremely simple. LR ordering the outgoing edges of the DFS-oriented graph is achieved by sorting them according to their *nesting_depth* on both sides. Such an embedding of outgoing edges is already sufficient for a planar combinatorial embedding (see, e.g., Cai 1993), but for completeness we provide full details in Algorithm 6.

The DFS forest is traversed for the third time. Since, after sorting, outgoing edges are already ordered in the desired way, back edges are encountered exactly as required in the definition of LR ordering. As described in Table 8(c) we therefore maintain, for each vertex v, the two positions next to which the

Algorithm 6: Phase 3 – Embedding

DFS3(vertex v)
  for $e_i \in E^+(v) = \langle e_1, \ldots, e_d \rangle$ **do**
    $w \leftarrow \text{target}(e_i)$
    if $e_i = parent\_edge[w]$ **then** `/* tree edge */`
      make e_i first edge in adjacency list of w
      $leftRef[v] \leftarrow e_i$; $rightRef[v] \leftarrow e_i$
      DFS3(w)
    else `/* back edge */`
      if $side[e_i] = 1$ **then**
        place e_i directly after $rightRef[w]$ in adjacency list of w
      else
        place e_i directly before $leftRef[w]$ in adjacency list of w
        $leftRef[w] \leftarrow e_i$

next left or right incoming back edge is to be inserted.

Observe that incoming back edges from the same subtree actually appear in in counterclockwise order. If the data structure available for embedded graphs does not provide a constant-time method for direct neighbor insertion, we may therefore use the now obsolete array ref to build a singly-linked list of all edges incident to a vertex in counterclockwise order instead.

6.4 Running time and implementation

Theorem 11 *Algorithm 1 can be implemented to test in $\mathcal{O}(n)$ time whether a graph is planar and return a planar combinatorial embedding if it is.*

PROOF. We have argued throughout this section that the algorithm correctly yields an LR ordering if the graph admits an LR partition. Hence, correctness is established by the left-right planarity criterion (Theorem 5). Recall that the initial test is justified by Corollary 2, and we may hence assume that $m \in \mathcal{O}(n)$.

The algorithm performs three DFS traversals, and rearranges the edges twice in between. Both rearrangements are obtained from sorting the edges according to *nesting_depth*, which can be done in linear time using, e.g., bucket sort because all entries are integers with absolute value less than $2n$.

The first DFS clearly requires constant time per edge traversal and backtracking step, and hence linear time overall.

During the second traversal, every back edge is pushed onto the stack exactly once (when it is traversed), so that the number of newly generated constraint pairs is bounded by the number of back edges. If more than a constant number of constraint pairs is inspected during the addition of constraints, a corresponding number of them is merged. Since also the total time spent on trimming back edges that return to the parent is linear in the number of edges, the overall running time is linear.

Dereferencing ref-pointers takes linear time, because it is performed only once before the third DFS traversal, which also requires linear time if the graph data structure provides a constant-time operation to move an edge next to another in the embedding order. If it does not, the algorithm can be altered to re-use ref-pointers for the embedding as described in Section 6.3. $\square$

The left-right approach can be implemented as described above and our experiences with its performance essentially confirm the favorable results of Boyer, Cortese, Patrignani, and Di Battista (2004). A special edge numbering scheme used in the PIGALE implementation (de Fraysseix and Ossona de Mendez, 2002) serves to avoid repeated DFS traversals, but it seems that most of the running time in our implementations is actually spend on the sorting of adjacency lists.

Note, however, that both sorting and DFS traversal can be avoided during the testing phase by splitting the stack into singly-linked lists associated with edges and processing edges (i.e., merging their final list of constraints into that of another edge) in the order given by *nesting_depth*. This order is established by creating two buckets for each *height* and adding an edge to its respective bucket when its *lowpt* is known during the initial DFS, i.e. when it is backtracked over. Since *lowpt* is determined bottom-up, edges added to the same bucket end up being in the desired order.

7 Non-Planarity

[This section is under revision]

8 Discussion

We have reviewed the left-right planarity criterion (Theorem 5) and described a simple linear-time algorithm (Algorithm 1) based on it. While this is not a review of graph planarity, and many important references and developments are left out, some notes on closely related work seem in place.

8.1 Characterization

In Section 7 we made use of a characterization of planar graphs in terms of forbidden subgraphs (Kuratowski, 1930). This characterization can be reinterpreted as identifying the overlapping cycle structures of $K_{3,3}$ and K_5 as the two minimal configurations that can not be drawn planarly.

Therefore, among the various later characterizations, the criterion due to Mac Lane (1937) appears to be related most closely, because it is also formulated in terms of a representative set of cycles. Consider the set of all undirected cycles of a graph, and define the sum of two cycles as the symmetric difference of their edge sets. These together form a vector space, called *cycle space*. A *basis* of the cycle space is a minimum-cardinality set of cycles such that every cycle is the sum of some basis cycles.

Theorem 12 (Mac Lane's Planarity Criterion) *A graph is planar, if and only if it has a cycle basis in which every edge appears at most twice.*

For a better intuition, consider a planar drawing of a connected planar graph. Traversing each face in the drawing (say, inner faces clockwise, the outer face counterclockwise) yields the set of (directed) *facial cycles* forming a basis of the cycle space. As required, every edge is traversed exactly twice (once in each direction).

Any cycle basis for a graph G has cardinality $\mu(G) = m - n + \kappa(G)$, where $\kappa(G)$ is the number of connected components of G and $\mu(G)$ is called the *cyclomatic number* of G. This is exactly the number of non-tree edges of a spanning forest and, in fact, the fundamental cycles of any spanning forest induce a cycle basis.

The left-right criterion thus also asks for a cycle basis with a special property, namely that its elements, the (directed) fundamental cycles of a DFS orientation, can be bipartitioned such that all constraints associated with forks are satisfied.

The cycle bases considered in these criteria are therefore maximally distinct. While the basis cycles in Mac Lane's criterion are as different as possible (with each edge in at most two cycles), the basis cycles in the left-right criterion are as concentrated as possible (with their overlap forming a spanning forest).

8.2 Development

The earliest precursor of the left-right approach is a planarity characterization of Wu (1955), which states that a graph is planar, if and only if a certain system of linear equations has a solution. It was complemented by the concept of crossing chains in Tutte (1970), and refined to Boolean variables and fewer equations in the 1970s (see Wu 1985, 1986; Liu 1990). The variables in this smaller system are associated with the edges, and the equations represent constraints generated from configurations of overlapping cycles obtained from a spanning DFS forest. An alternative interpretation of the existence of a solution is that of balancing a constraint graph as in Section 6.2. Rosenstiehl (1980) gives an algebraic proof for this characterization.

This work was further developed in several papers, but the descriptions are rather incomplete, in particular with respect to linear-time implementation (de Fraysseix and Rosenstiehl, 1982, 1985; Xu, 1989; Cai, Han, and Tarjan, 1993).

Finally, de Fraysseix, Ossona de Mendez, and Rosenstiehl (2006) simplified the approach even further by concentrating on the single constraint-inducing configuration of Definition 4. While this paper is still incomplete and difficult to read, the linear-time implementation is described in just enough detail to provide a basis for replication. Among the differences to the present description is the maintenance and merging of constraints, since intervals are described as stacks rather than their extreme pairs of edges and there is a constraint stack for each edge rather than our global stack S. It turns out, however, that the most recent implementation in PIGALE (de Fraysseix and Ossona de Mendez, 2002) uses a similar representation.

The characterization of Kuratowski subgraphs in terms of configurations induced by a DFS spanning tree given in de Fraysseix and Ossona de Mendez (2003) and de Fraysseix (2008) led to a linear-time extraction algorithm associated with the left-right approach. Because of principal commonalities it is likely that similarities can be unveiled, but the extraction algorithm given here is original and more intimately related to the left-right characterization.

8.3 Algorithms

The first published polynomial-time planarity testing algorithm is due to Auslander and Parter (1961). It is based on an observation already noted above, namely that in a planar drawing of a graph every simple cycle forms a closed curve partitioning the plane into an inside and an outside region. Consider the graph obtained by removing the edges of some simple cycle, but retaining copies of vertices on the cycle for every incident non-cycle edge. These vertices are called *attachments* and the connected components of the resulting graph are called *segments*. Clearly, each segment must be drawn completely inside or completely outside of the removed cycle, but a pair of segments must not be placed in the same region if their attachments interleave on the cycle. Planarity can thus be tested by recursively choosing cycles and sides.

The related algorithm of Demoucron, Malgrange, and Pertuiset (1964) also starts from a simple cycle, but then iteratively chooses a path that can be added into one of the current faces. The algorithm is not only simple, but also has the unusual property to eagerly maintain a partial embedding that is not changed later on. Both algorithms require $\Omega(n^2)$ time, though.

In a graph-algorithmic milestone, the first linear-time planarity test was presented by Hopcroft and Tarjan (1974). Their approach is called *path-addition* because it refines that of Auslander and Parter (1961) by adding paths rather than segments, and in an order determined from a depth-first search of the graph. It took many years, though, until finally Mehlhorn and Mutzel (1996) complemented the algorithm with an $\mathcal{O}(n)$ embedding phase.

Recall how we observed in Section 3 that likewise-oriented cycles are nested if they overlap. Maybe because de Fraysseix and Rosenstiehl (1982) phrased the notions of left and right in terms of angles with the DFS tree rather than orientations of fundamental cycles, it has gone almost unnoticed that the left-right approach is yet another refinement of Auslander and Parter (1961) and Hopcroft and Tarjan (1974), progressing from segments to paths to edges. Together with Canfield and Williamson (1990) and Haeupler and Tarjan (2008) this observation instills hope that there may be a useful and elegant unification of path- and vertex-addition approaches including the two most efficient versions of de Fraysseix, Ossona de Mendez, and Rosenstiehl (2006) and Boyer and Myrvold (2004).

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