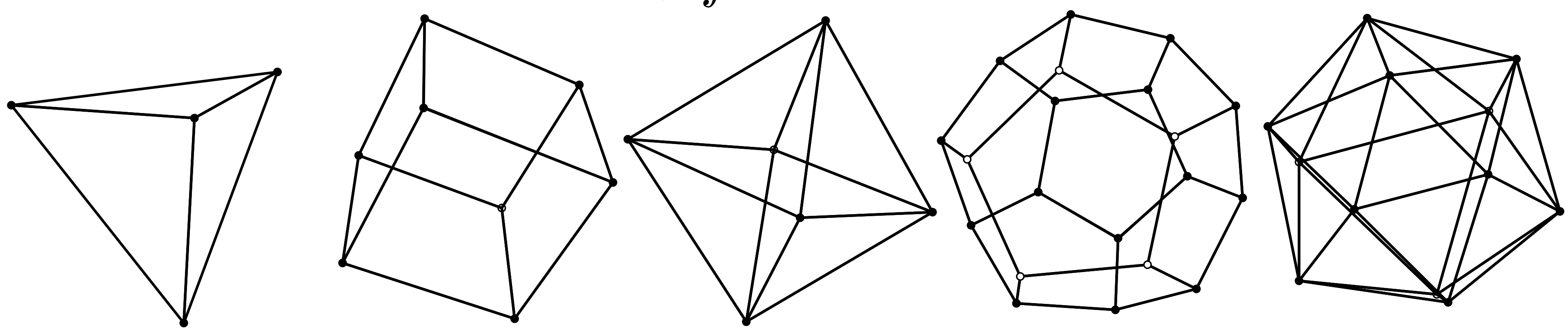
Euler's formula $n - e + f = 2$ and Platonic solids



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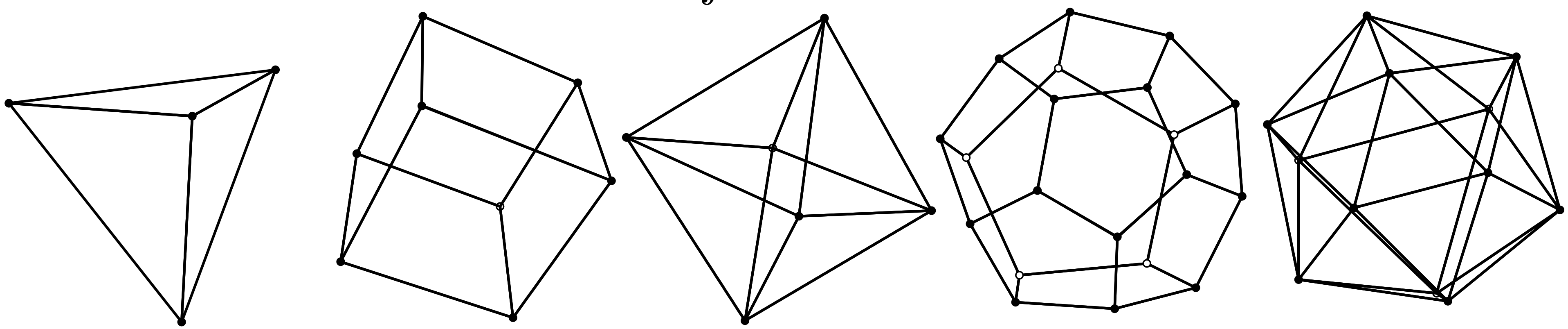

spherical projection makes these planar graphs

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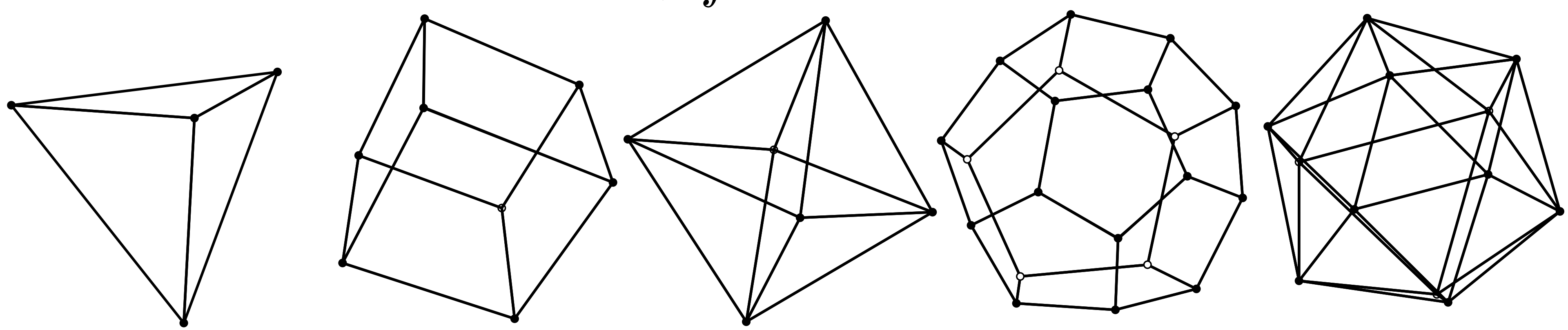


spherical projection makes these planar graphs

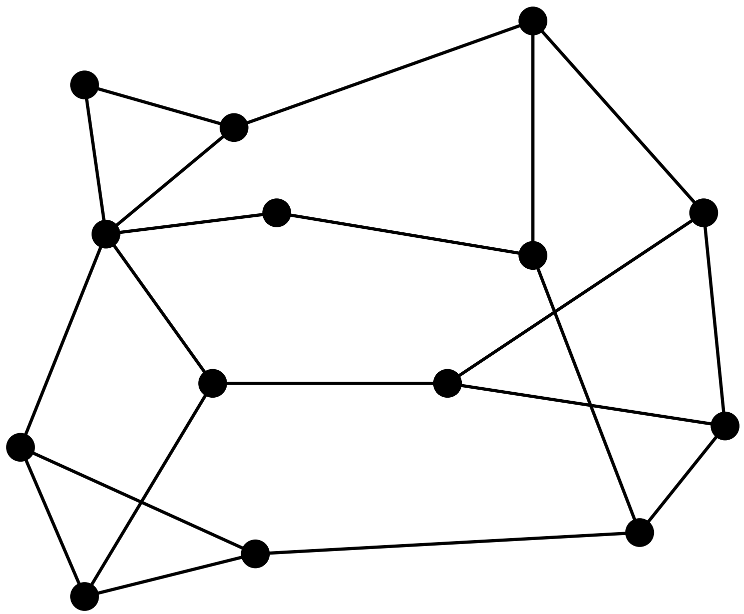
$p =$ vertex degree ≥ 3, $q =$ face degree ≥ 3

$pn = 2e$ $\qquad qf = 2e$ $\qquad$ Euler implies $\frac{2e}{p} - e + \frac{2e}{q} = 2$

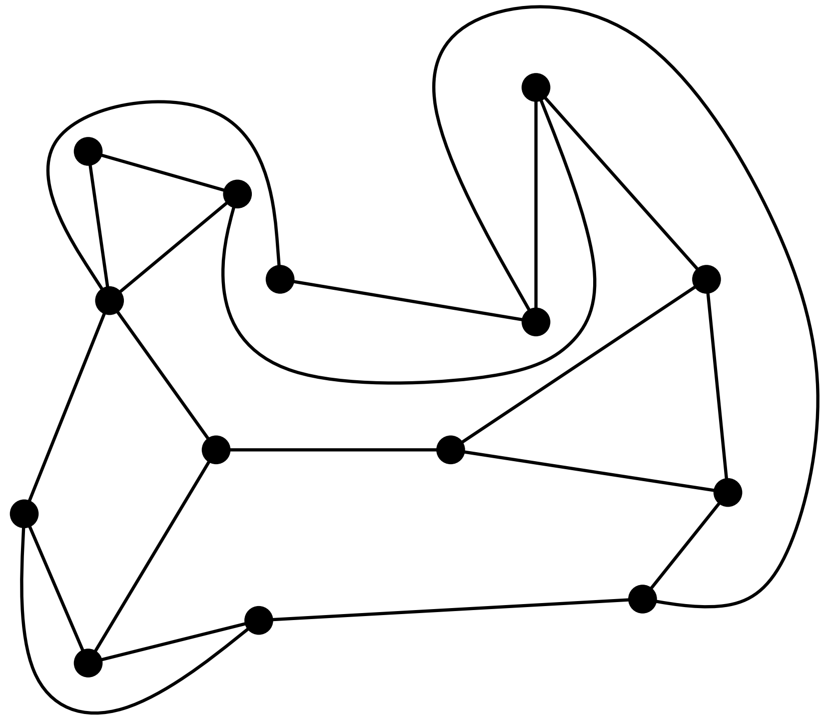
so $\frac{2}{p} + \frac{2}{q} > 1$

Thus $(p, q) \in \{(3,3), (3,4), (4,3), (3,5), (5,3)\}$

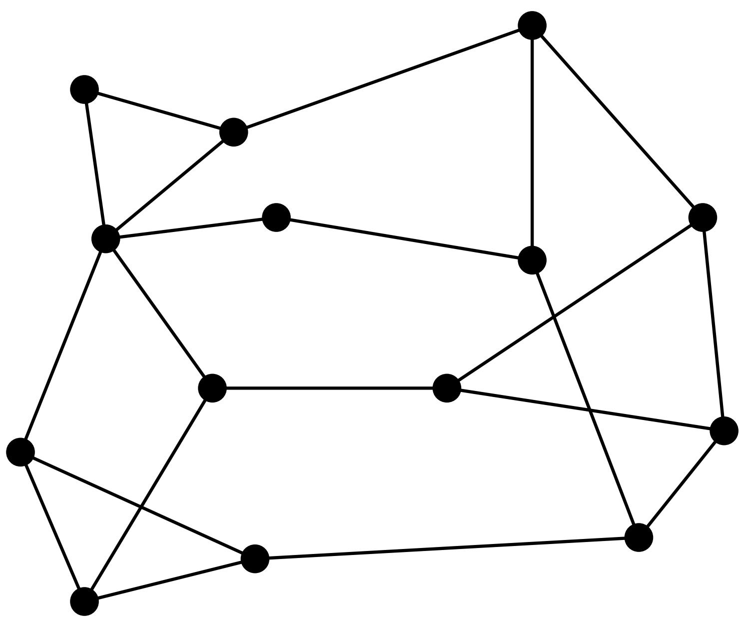
Planar graph



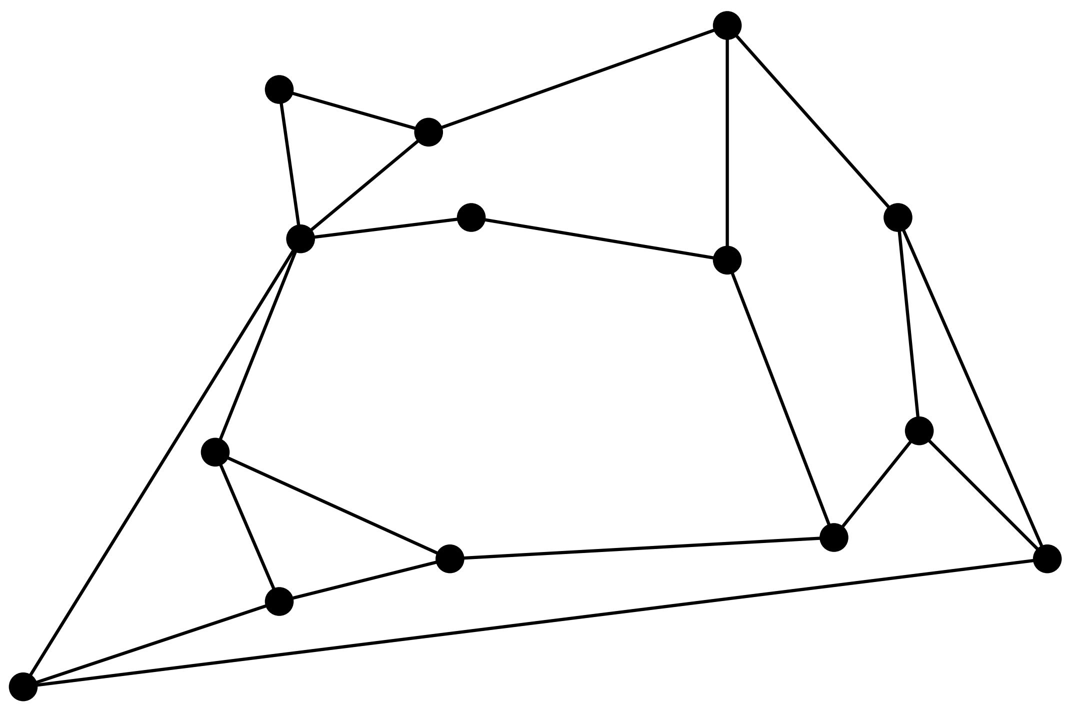
Planar drawing



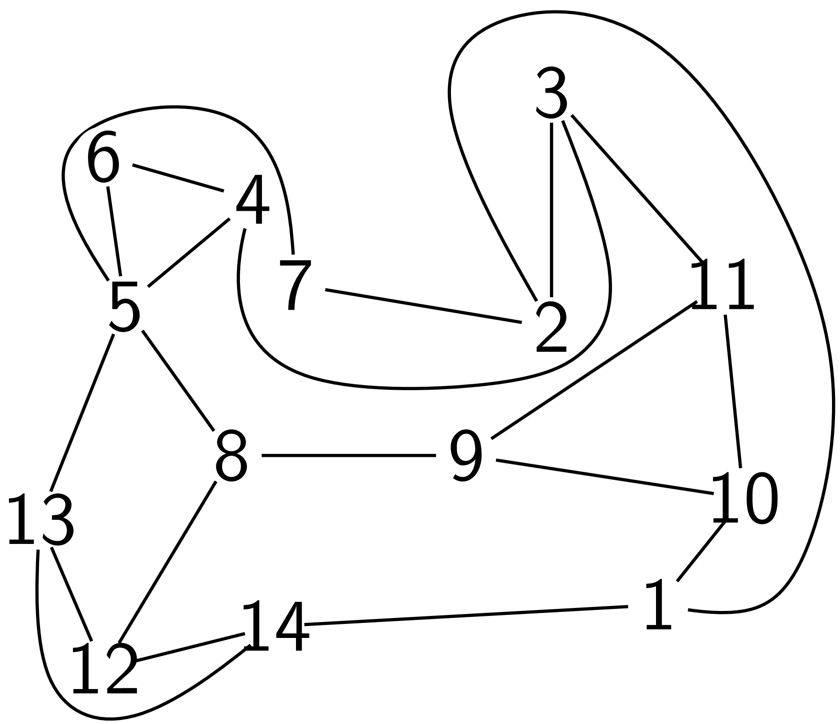
Planar graph



Planar (straight-line) drawing



Planar drawing

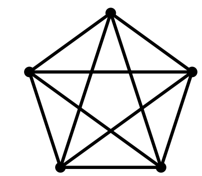
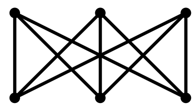


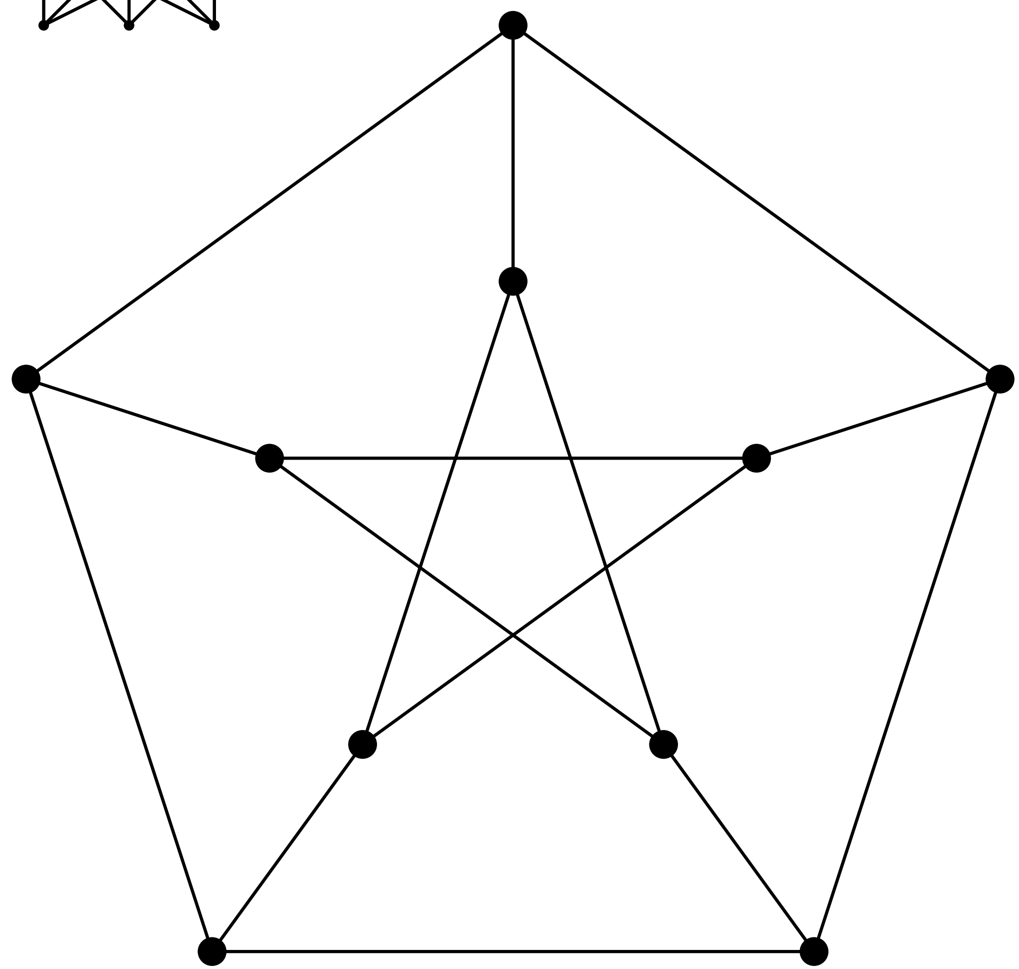
Planar Embedding

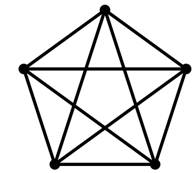
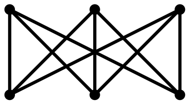
1: 2,14,10
2: 1,3,7
3: 2,11,4
4: 3,5,6
5: 4,8,13,7,6
6: 4,5
7: 2,5
8: 5,9,12
9: 8,11,10
10: 1,9,11
11: 3,10,9
12: 8,14,13
13: 5,12,14
14: 1,13,12

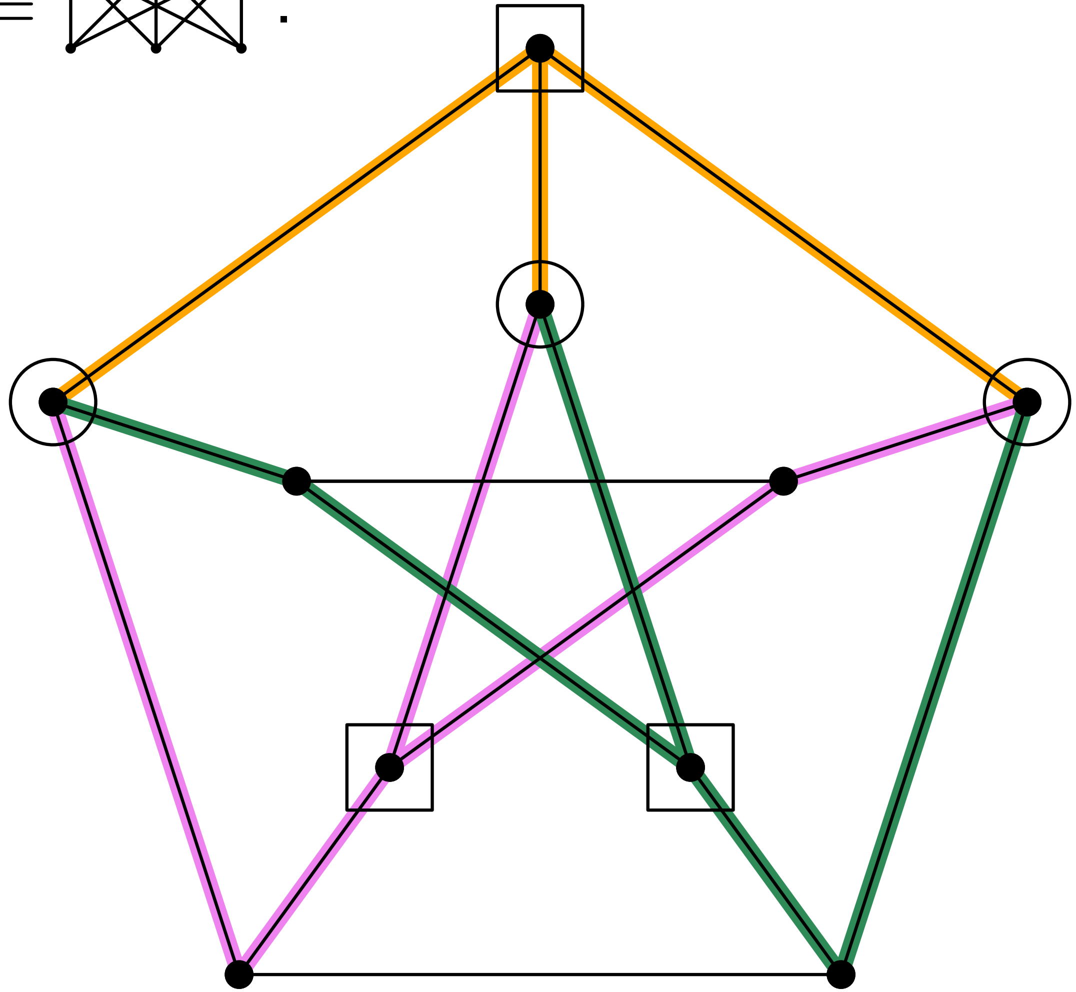
Outer face: 1,14,13,5,7,2

(Clockwise order of neighbors.)

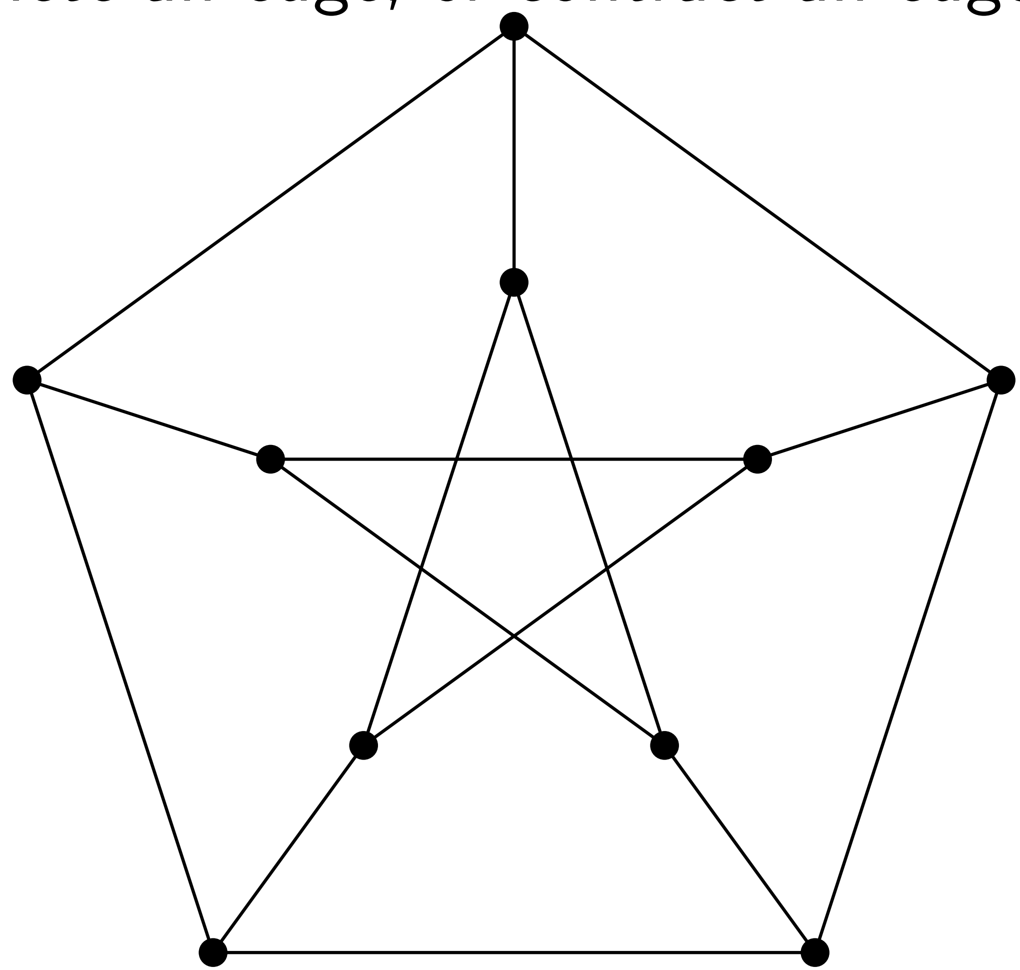
Kuratowski's Theorem G is planar iff it doesn't contain a subgraph that is a subdivision of $K_5 \equiv$  or $K_{3,3} \equiv$  .



Kuratowski's Theorem G is planar iff it doesn't contain a subgraph that is a subdivision of $K_5 \equiv$  or $K_{3,3} \equiv$  .



Wagner's Theorem G is planar iff it doesn't have K_5 or $K_{3,3}$ as a *minor* (to get a minor, repeatedly delete a vertex, delete an edge, or contract an edge).



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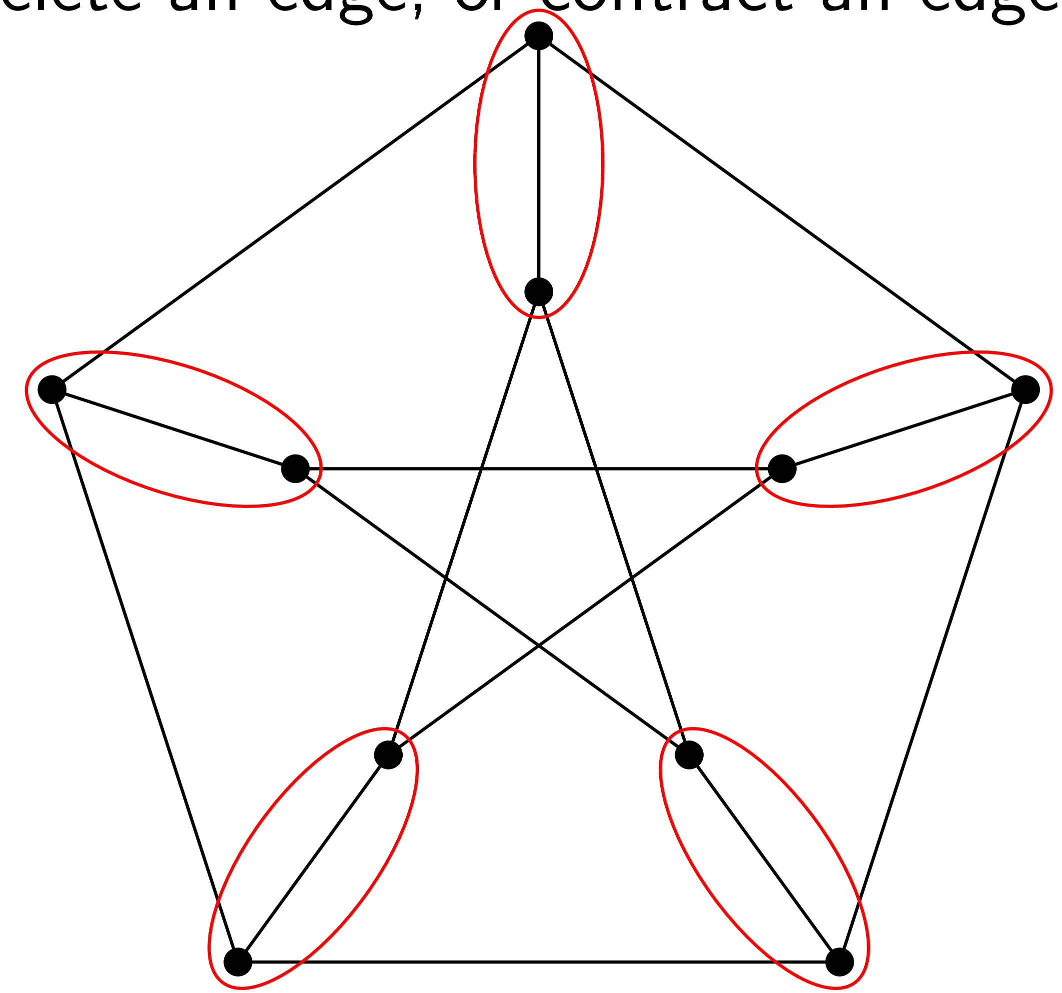
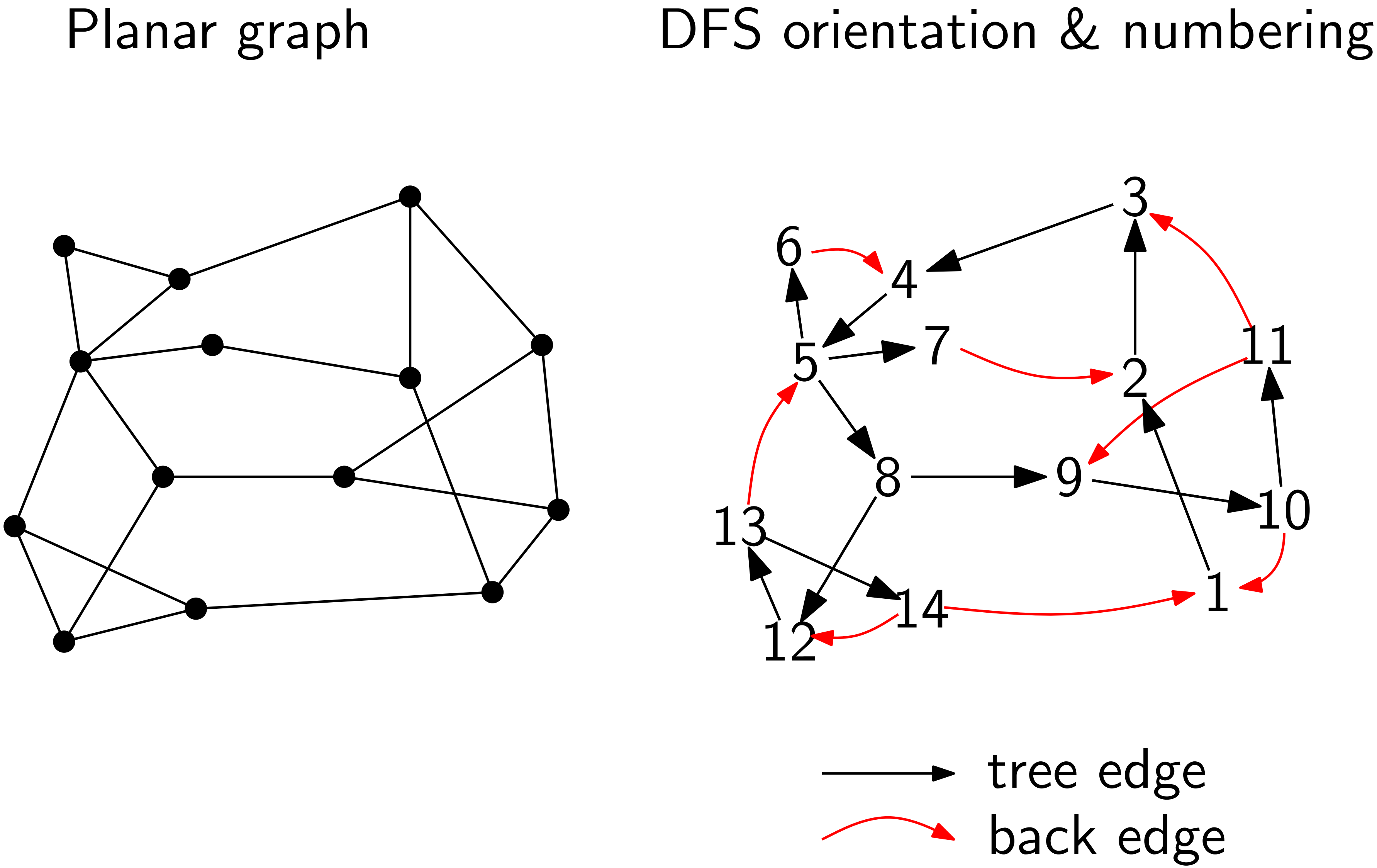
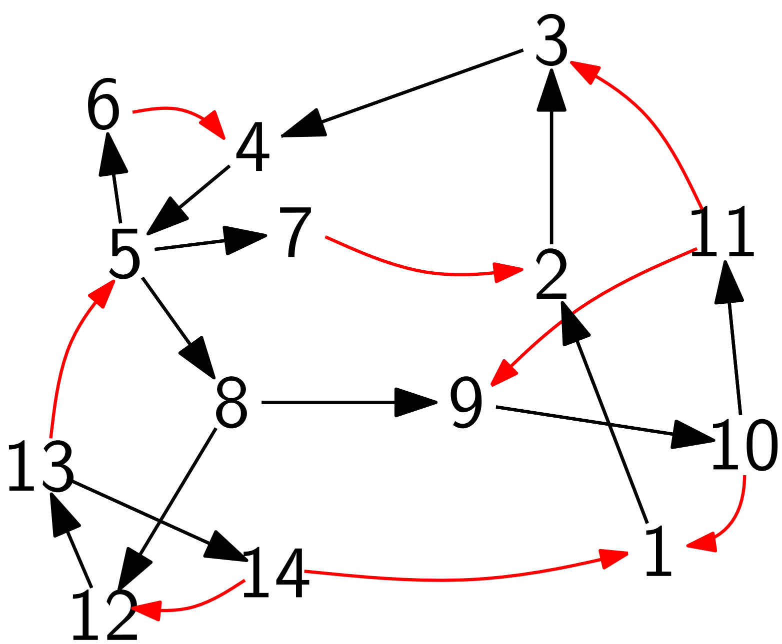


Kuratowski's and Wagner's conditions do not imply a fast algorithm to check planarity, but there are linear time planarity testing algorithms,

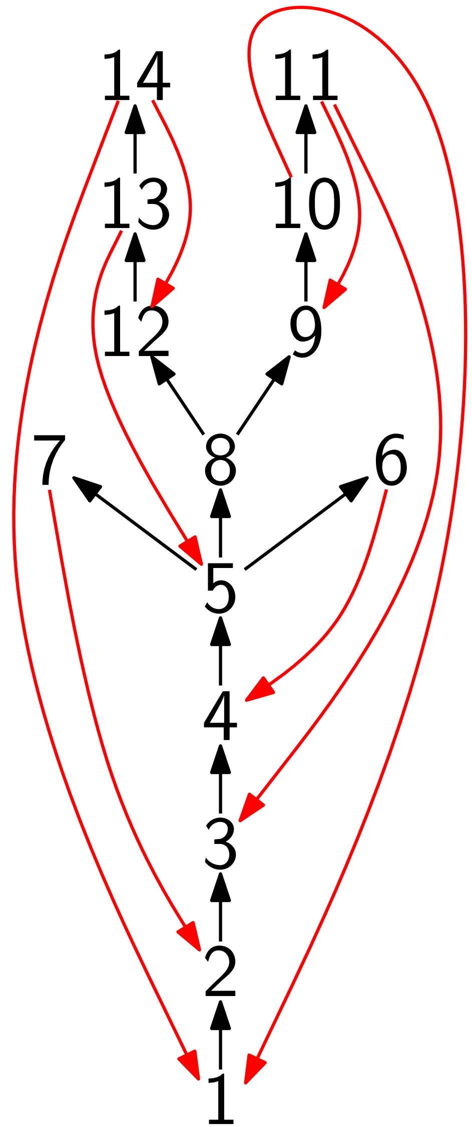


Planar graph
DFS orientation & numbering
3
6
4
5
7
2
11
8
9
13
10
1
14
12
tree edge
back edge

DFS orientation & numbering

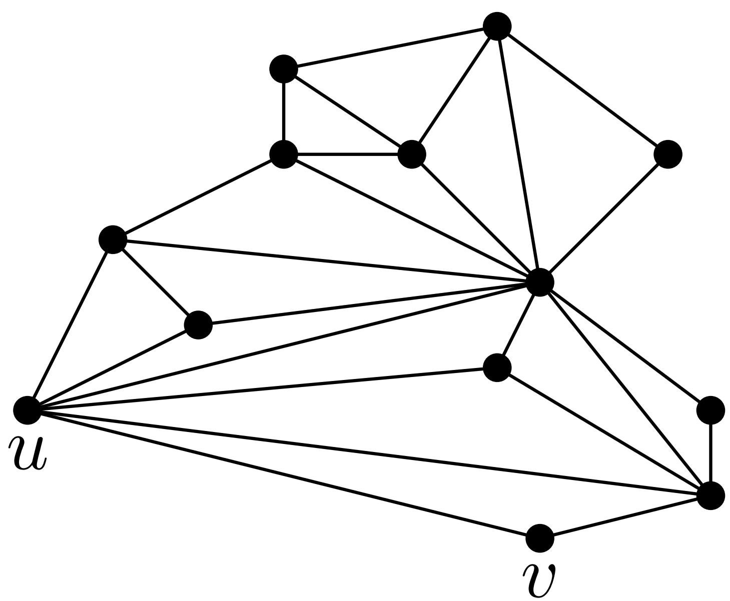


LR partition [Brandes 11]



Drawing a Planar Graph on a Grid [de Fraysseix, Pach, Pollack 90]

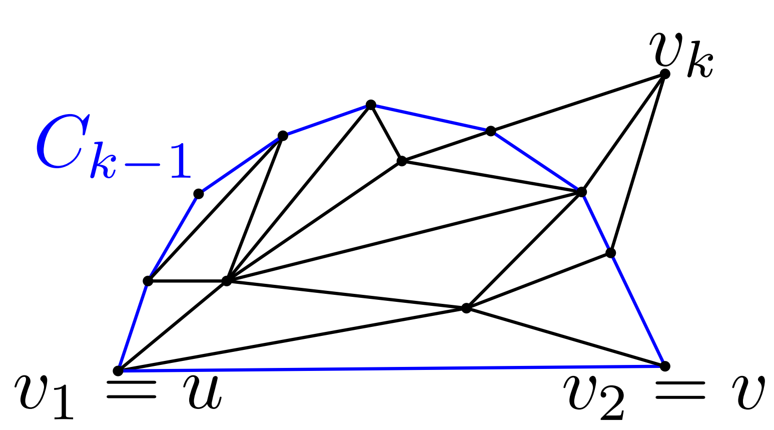
Lemma. Let G be a simple planar graph with an embedding and $u = u_1, u_2, \ldots, u_k = v$ be a cycle of G. Then there exists a vertex w on the cycle, different from u and v and not adjacent to any inside chord.

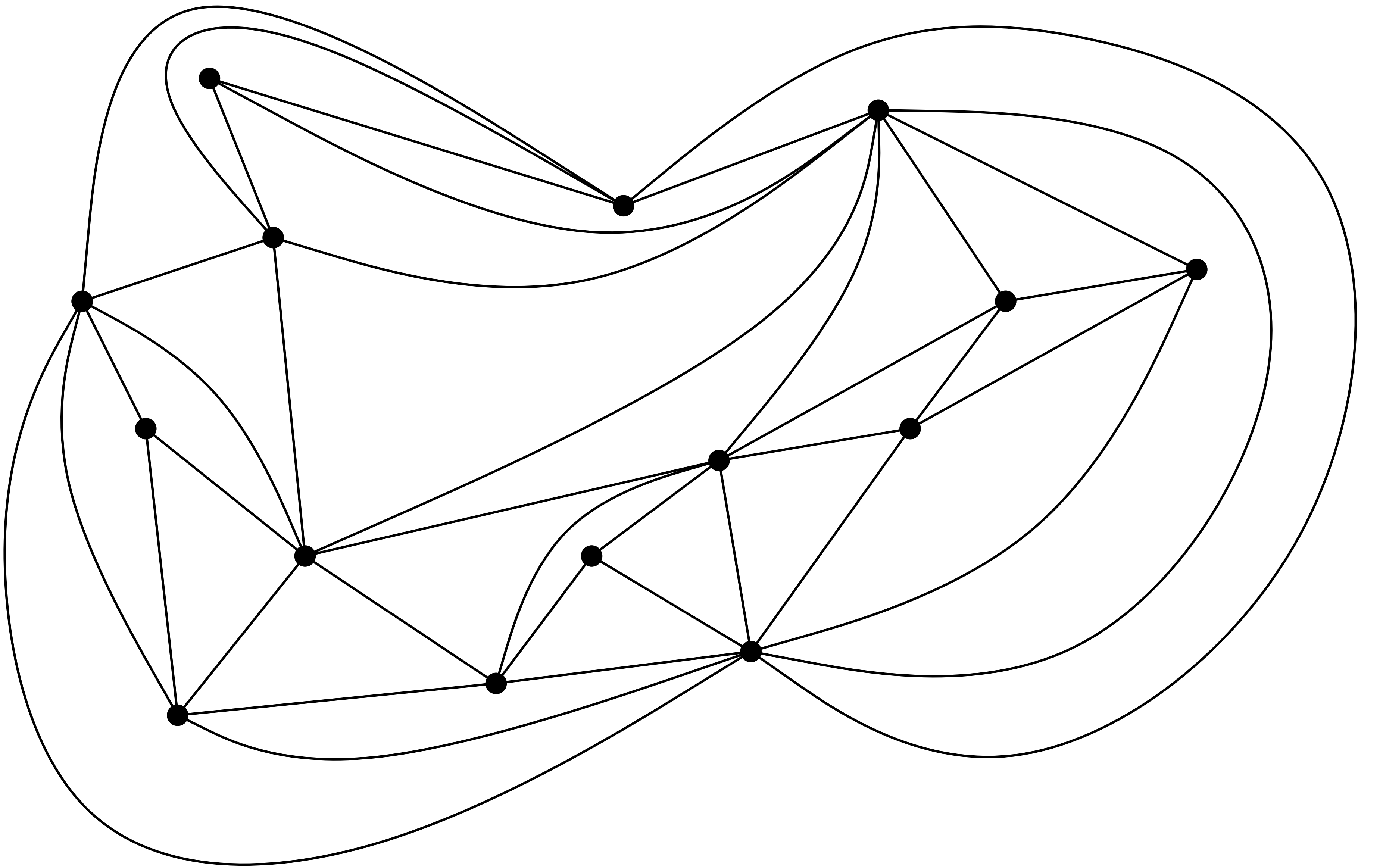


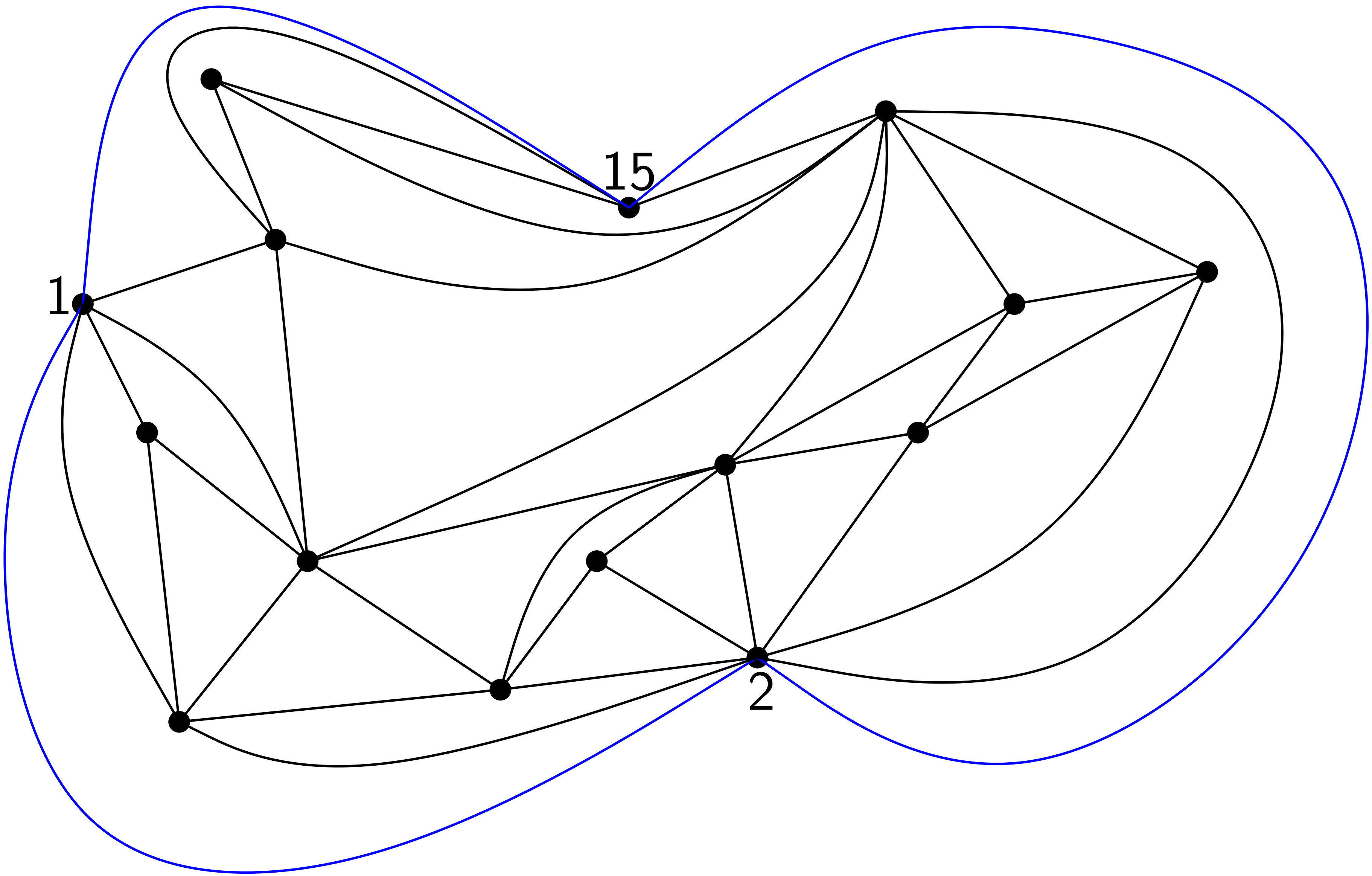
Canonical Ordering. If $G = (V, E, \mathcal{E})$ is a maximal plane graph with exterior face u, v, w then there exists an ordering of V, $v_1 = u, v_2 = v, v_3, \ldots, v_n = w$ such that for all $4 \leq k \leq n$:

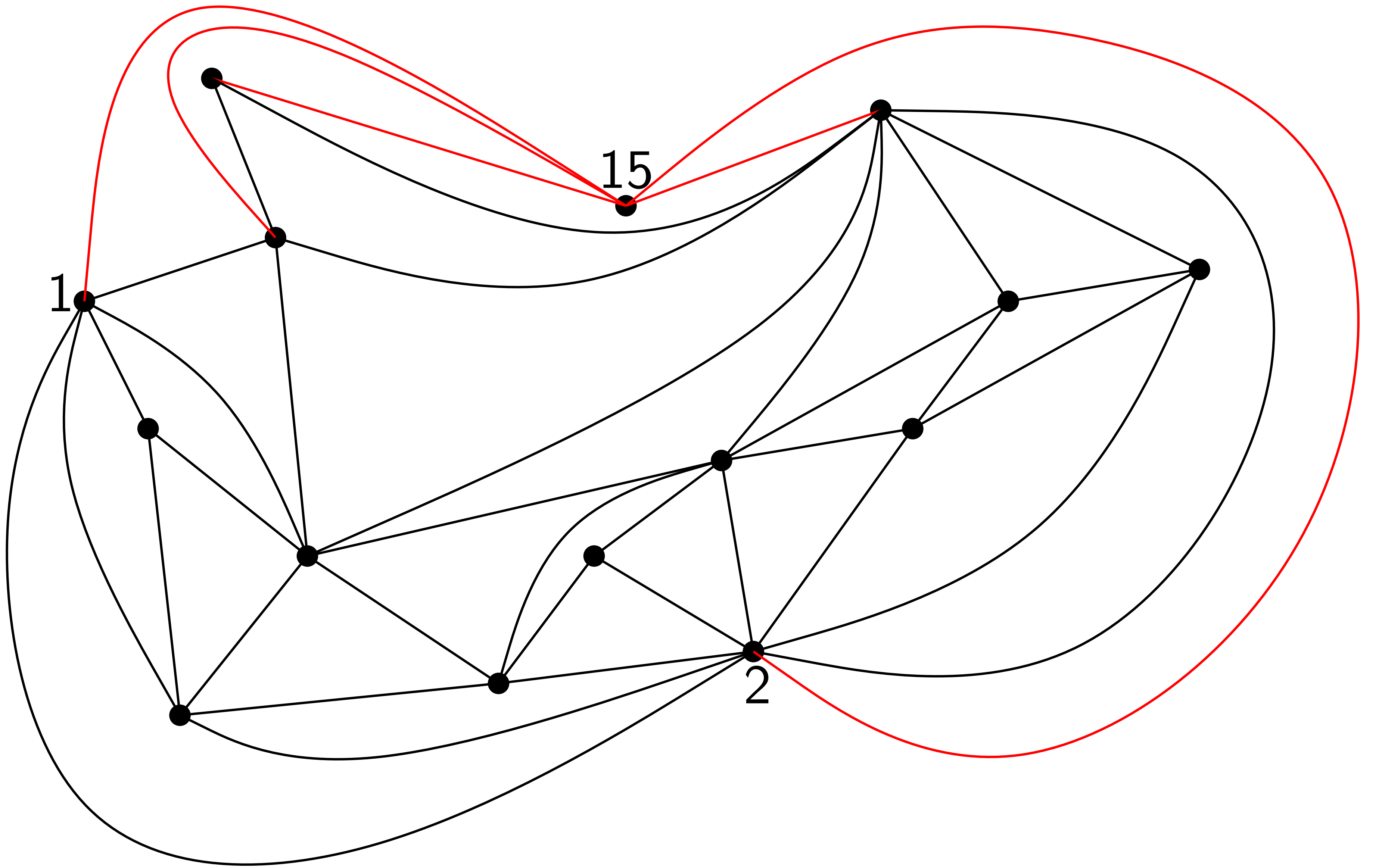
(i) The subgraph G_{k-1} induced by $v_1, v_2, \ldots, v_{k-1}$ is 2-connected with uv on its exterior cycle C_{k-1};

(ii) v_k is in the exterior face of G_{k-1}, and its neighbors in G_{k-1} are a length ≥ 1 subpath of the path $C_{k-1} - uv$.

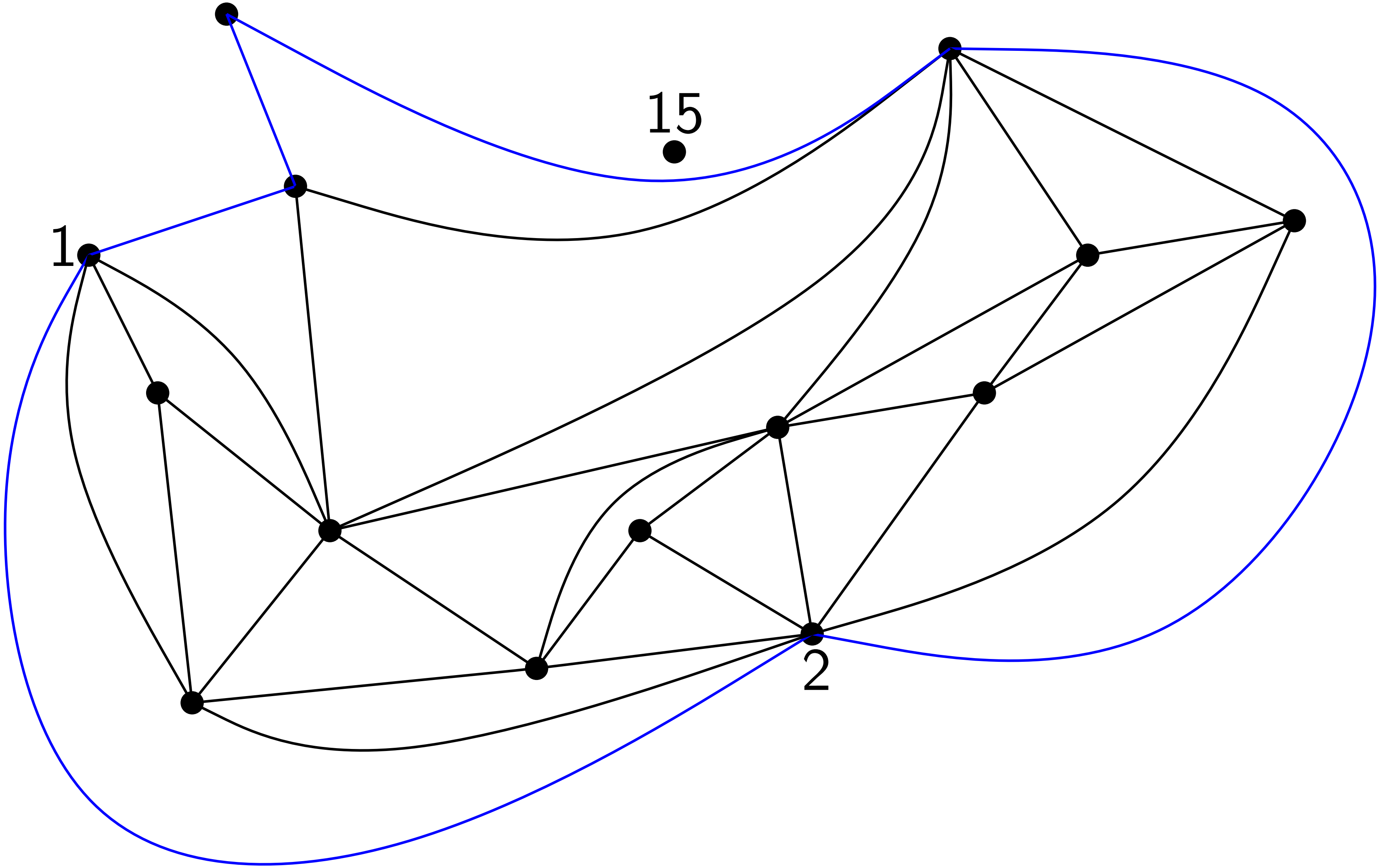





15
1
2


15
1
2

Use Lemma to find next vertex.



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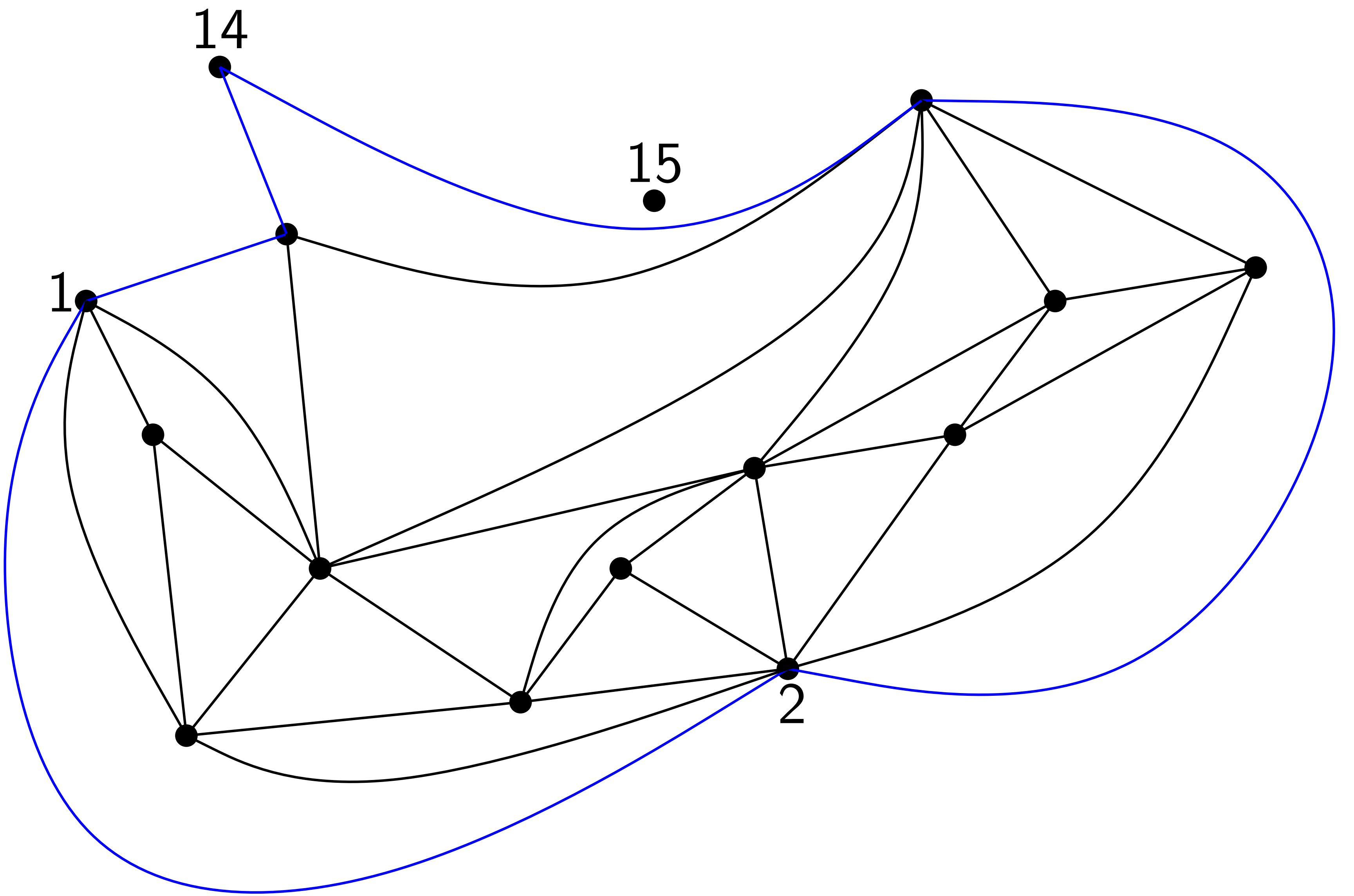
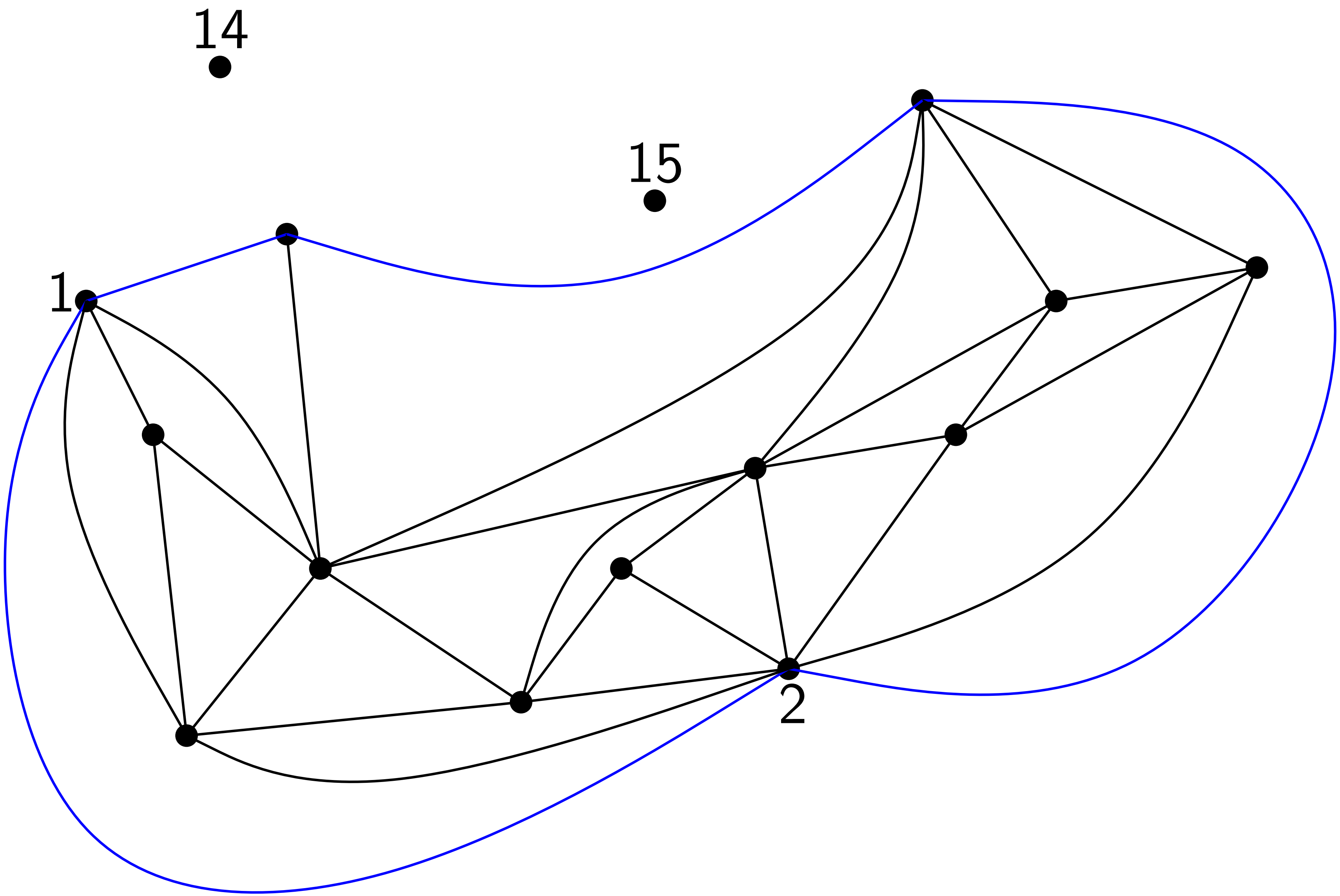
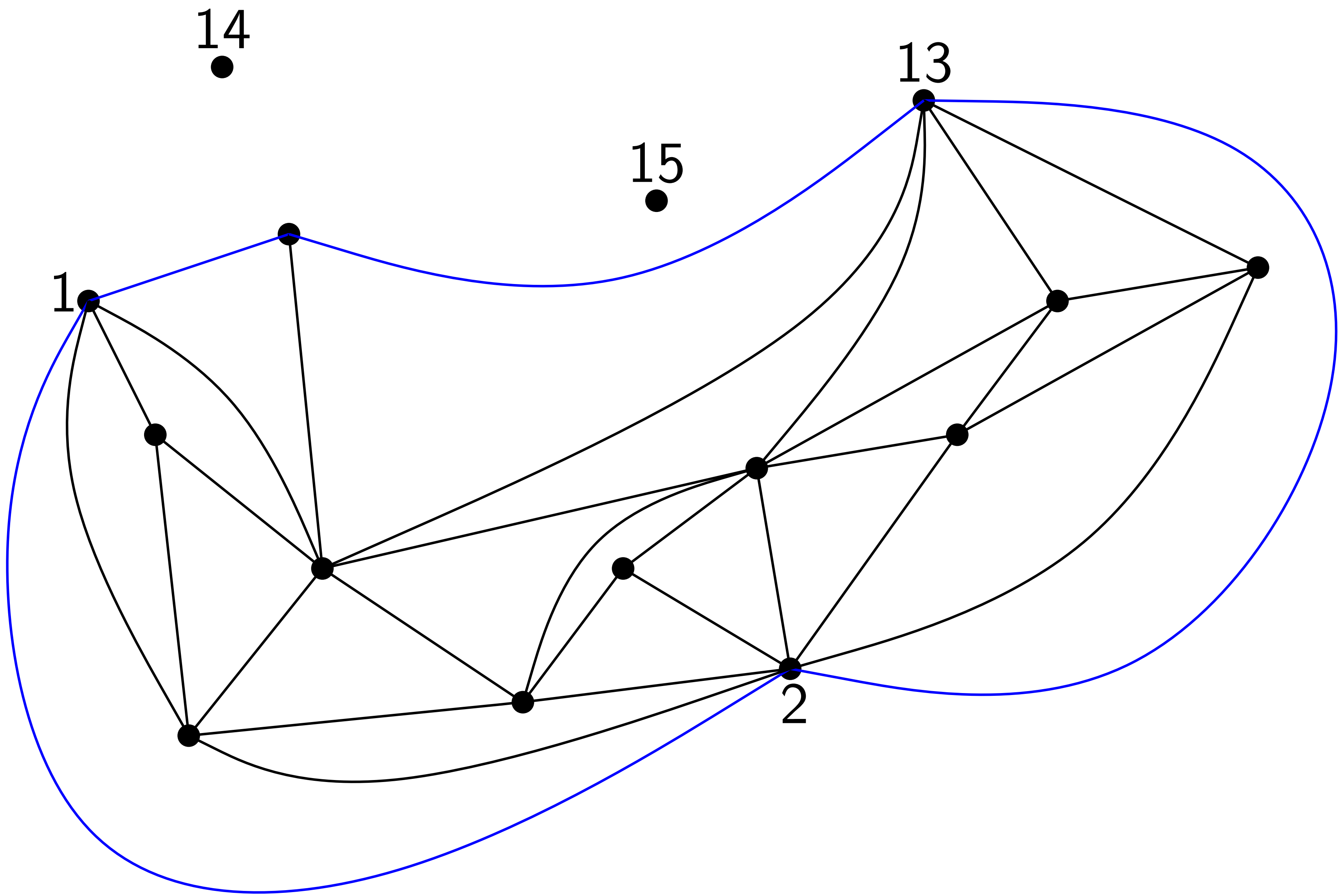
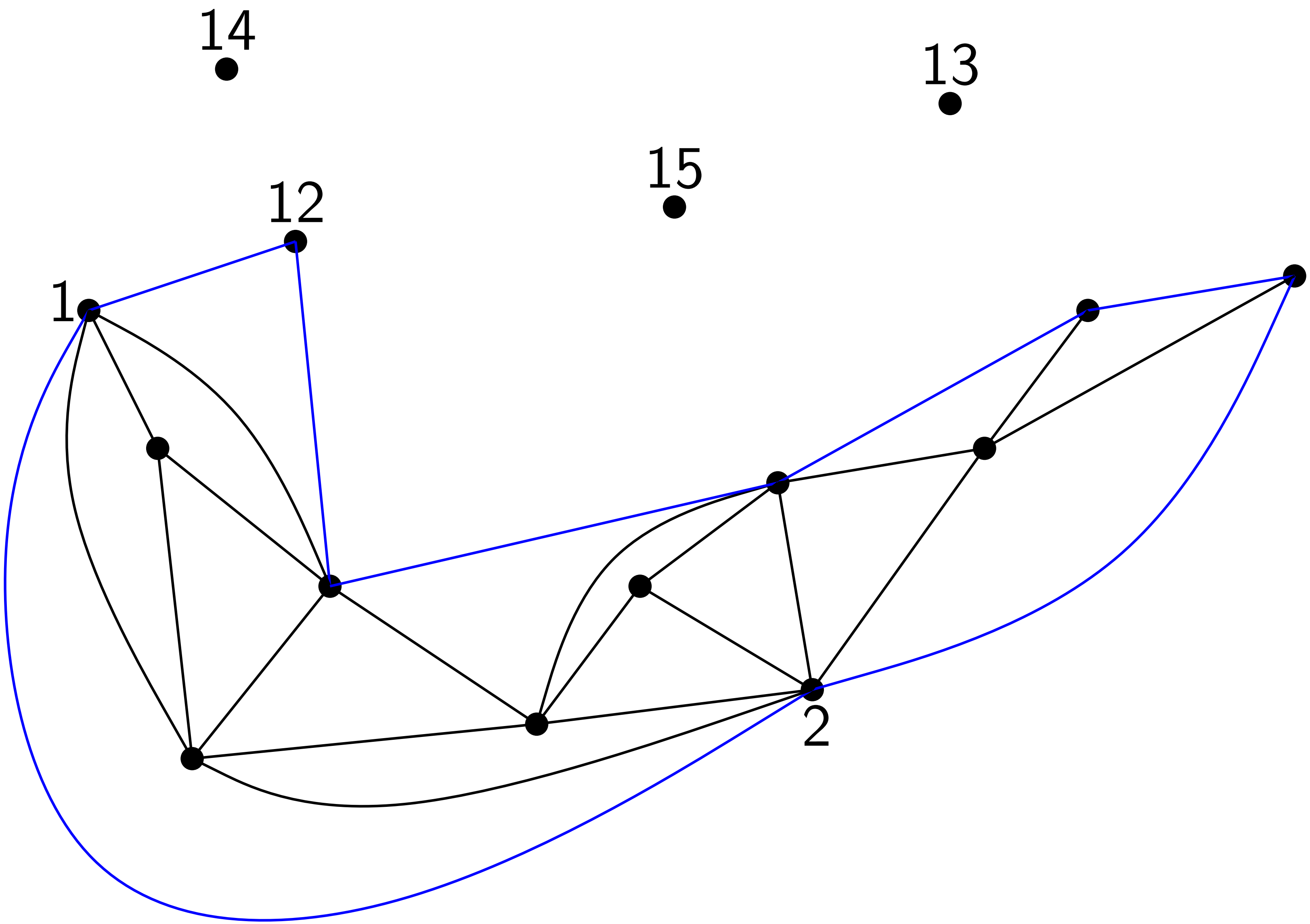


Repeat




14
13
15
1
2


14
13
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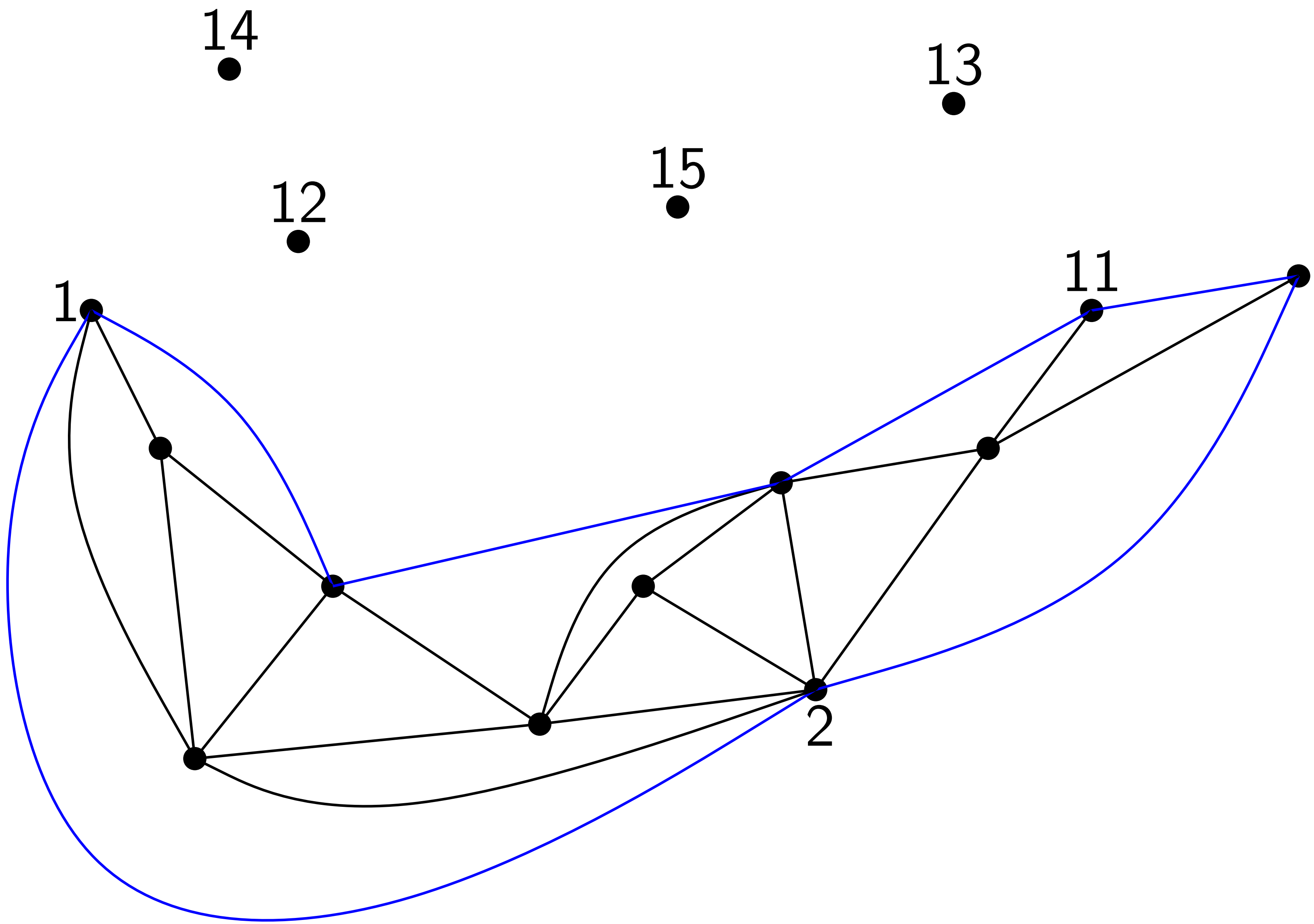
12
1
2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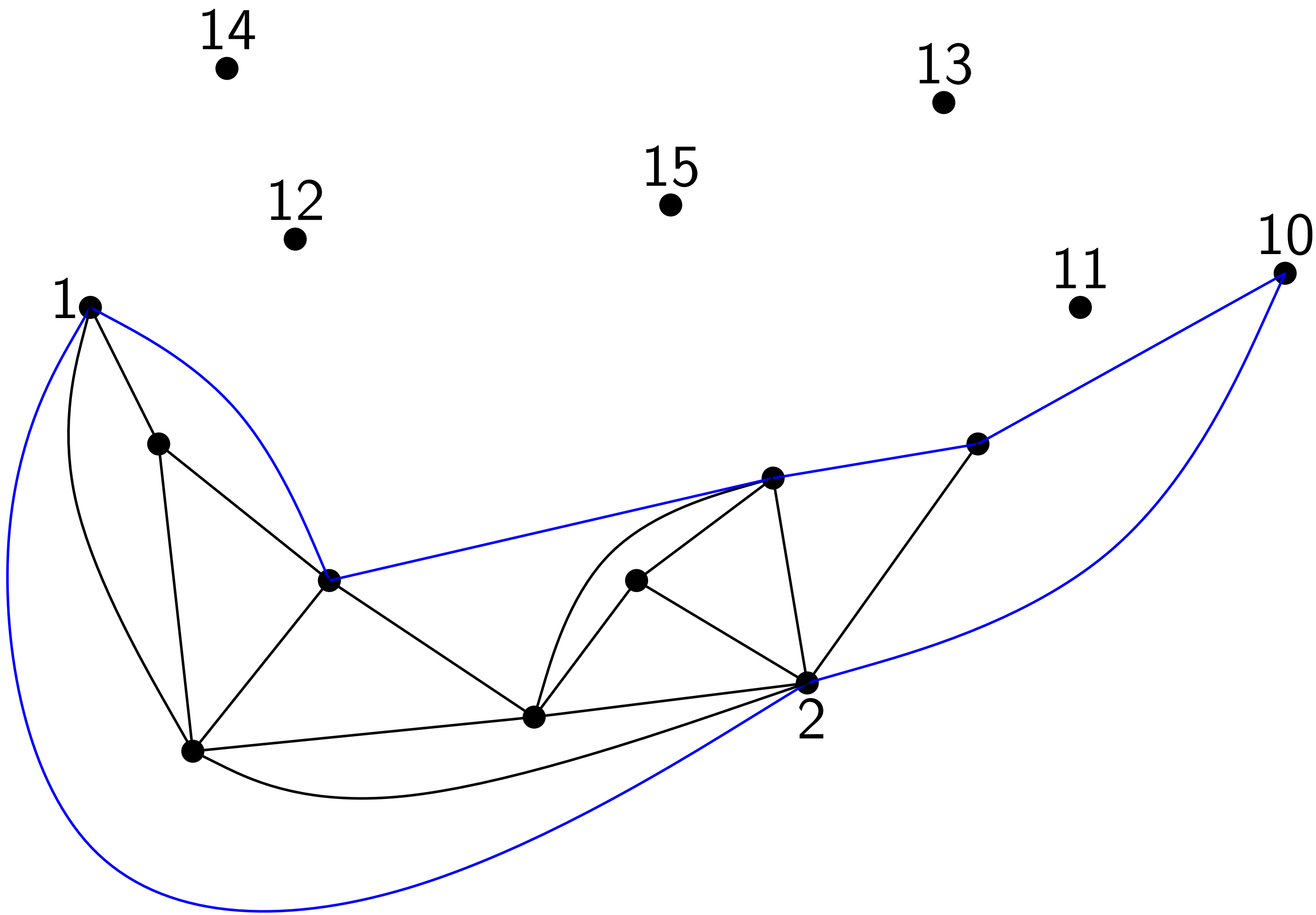
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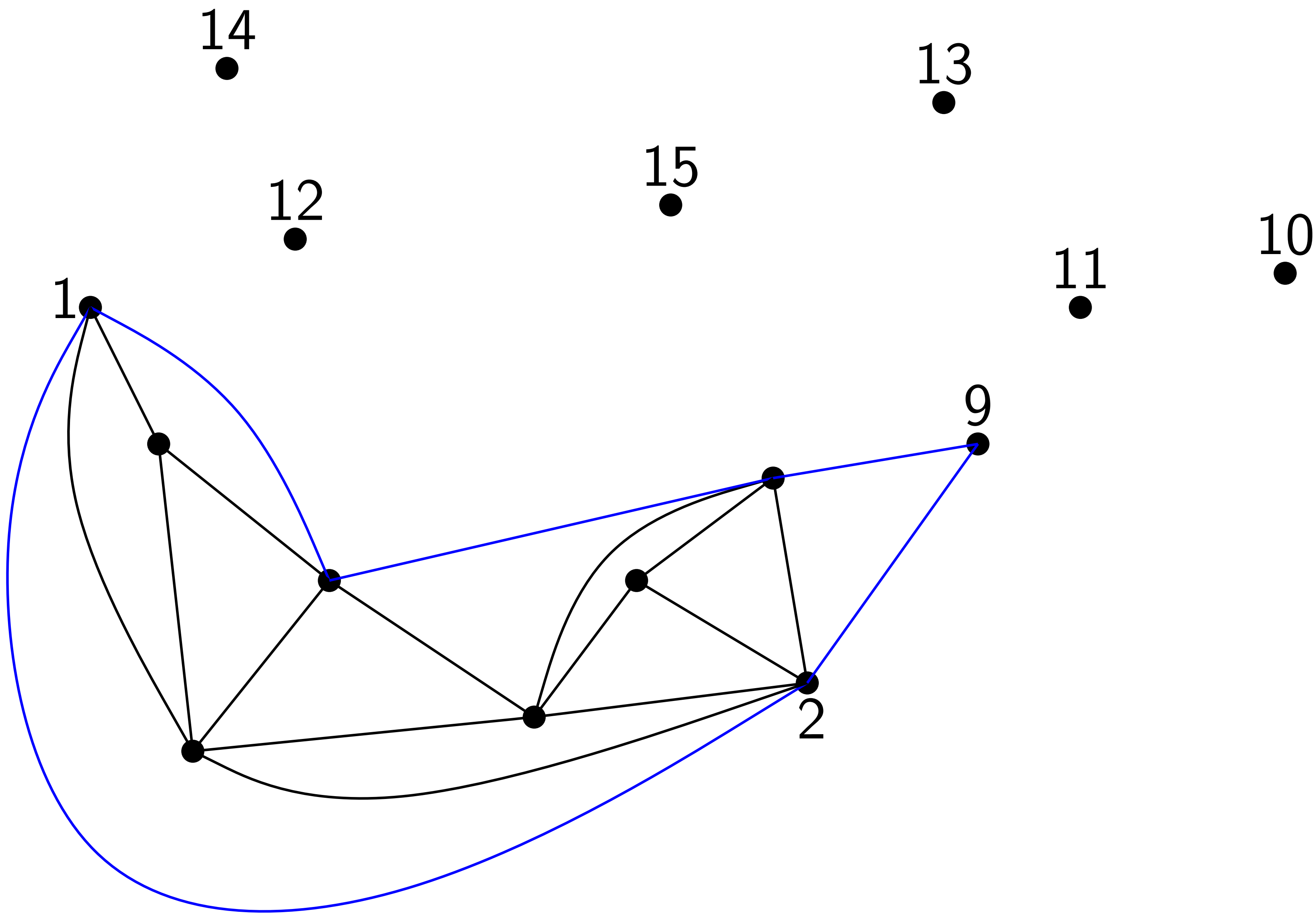
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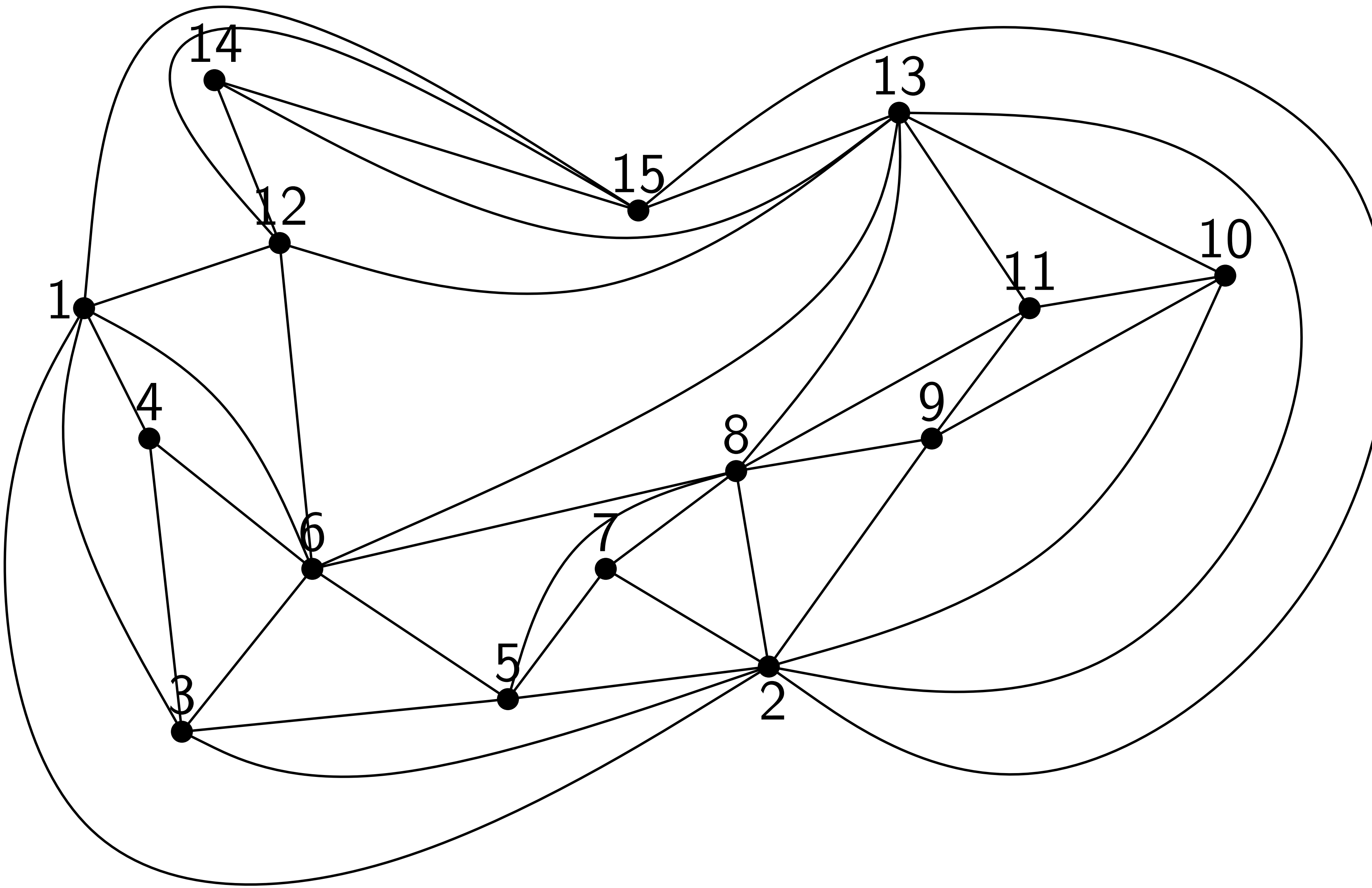
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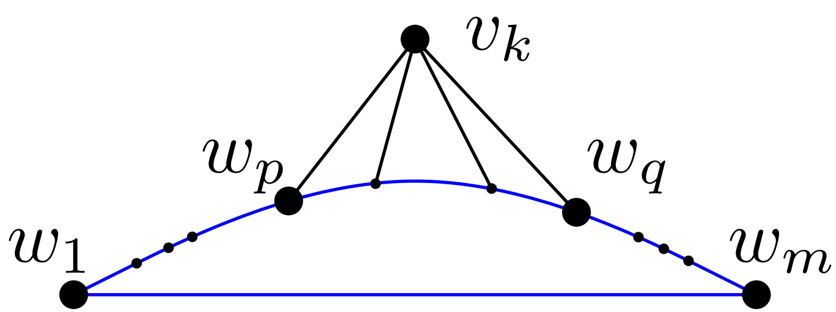

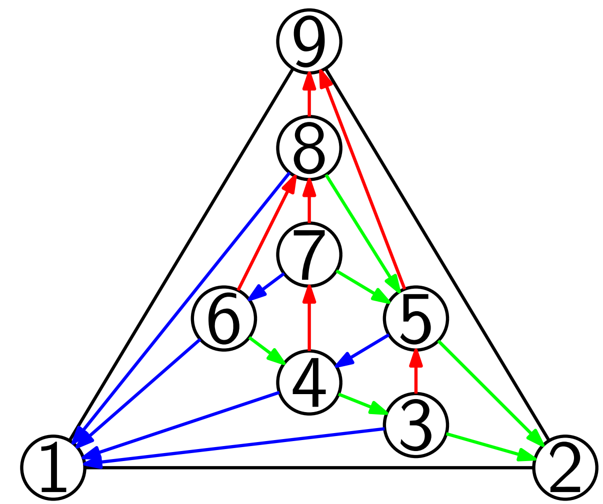

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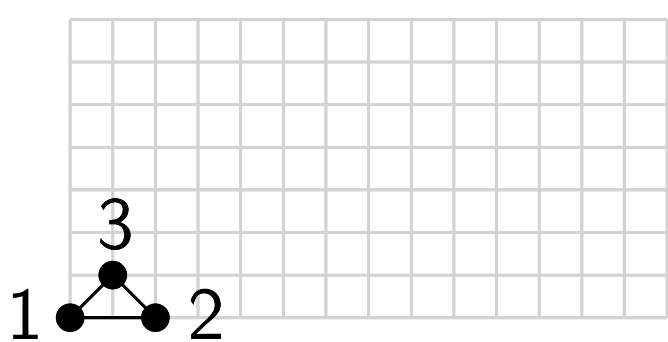
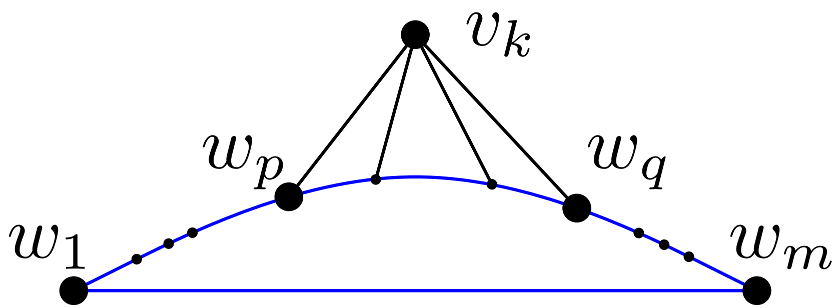
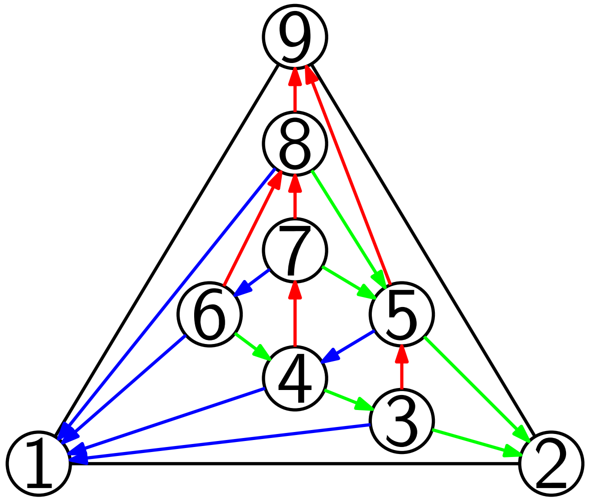
$L(v_k) = \{v_k\} \cup \bigcup_{i=p+1}^{q-1} L(w_i)$.







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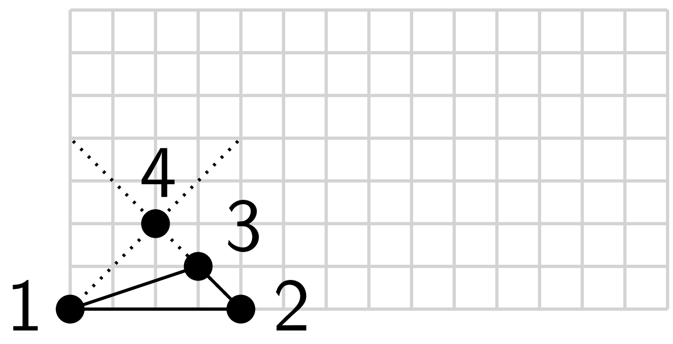

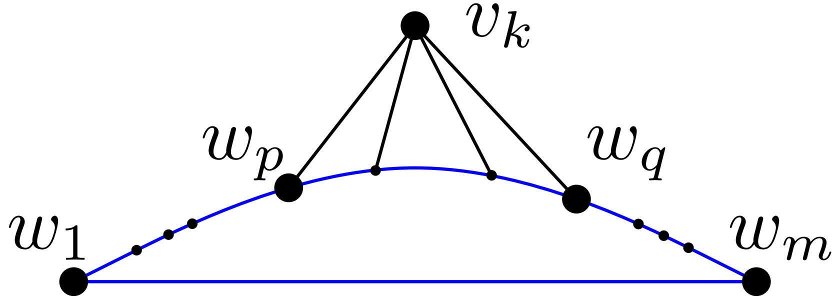

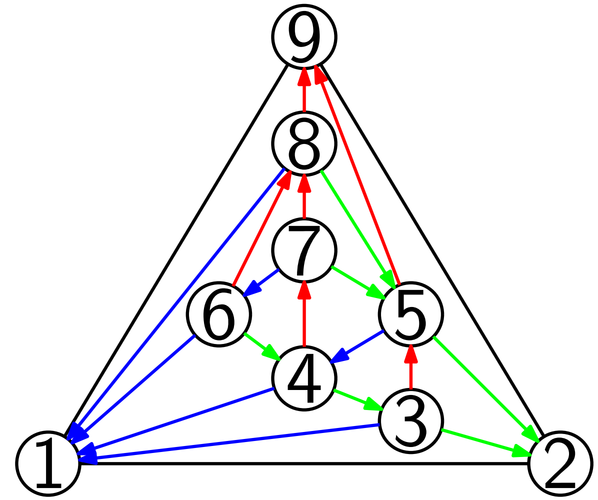

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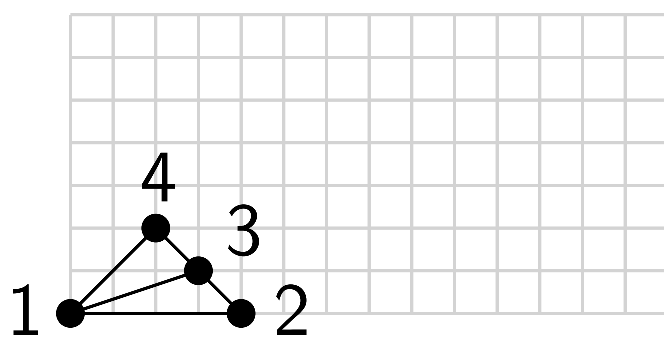

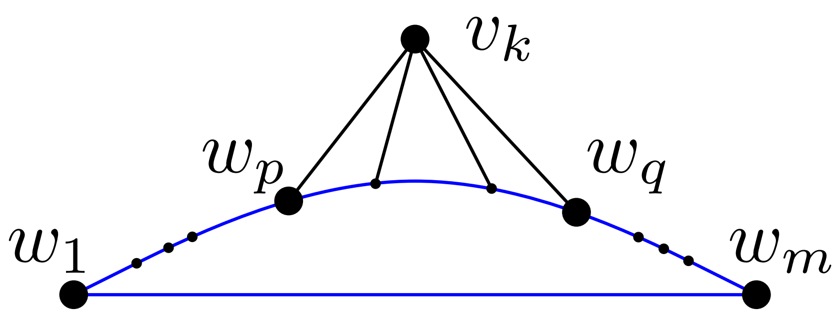

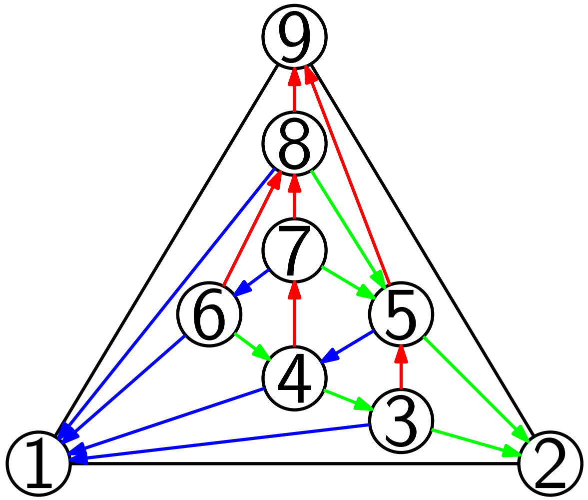

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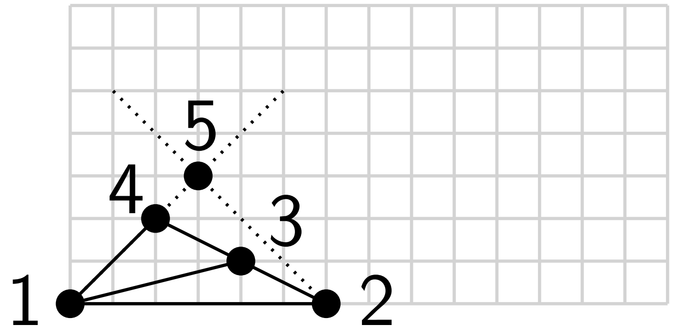

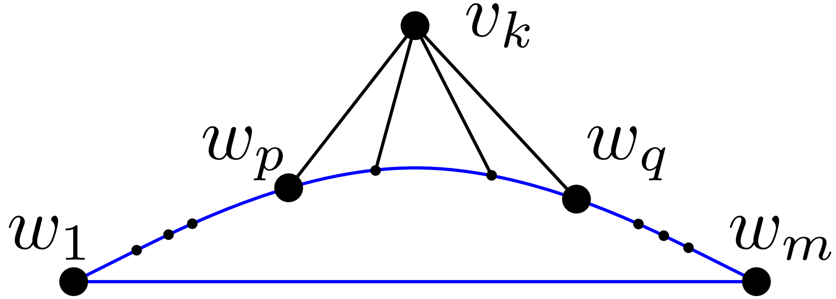

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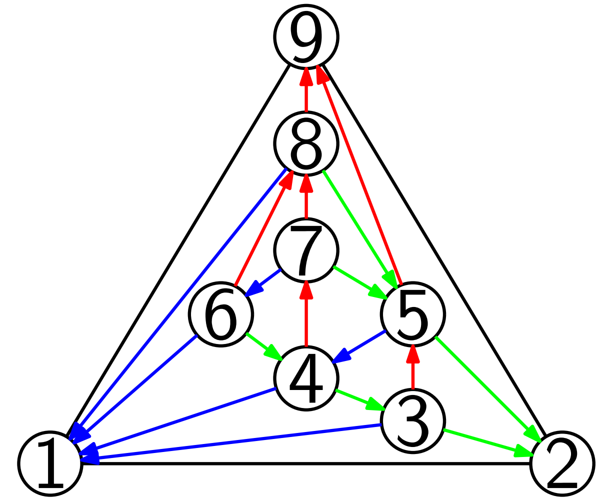

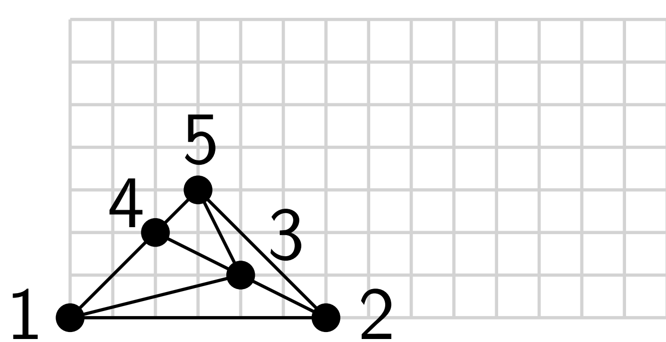

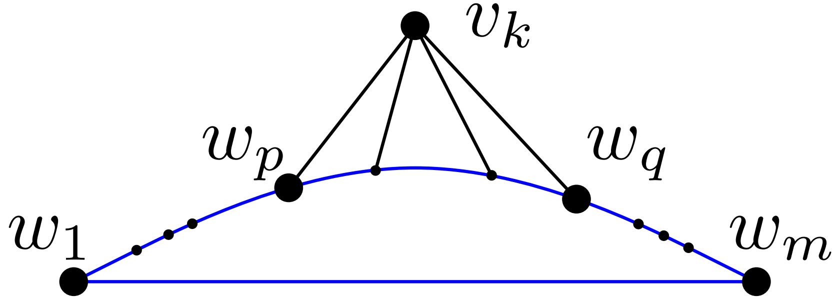

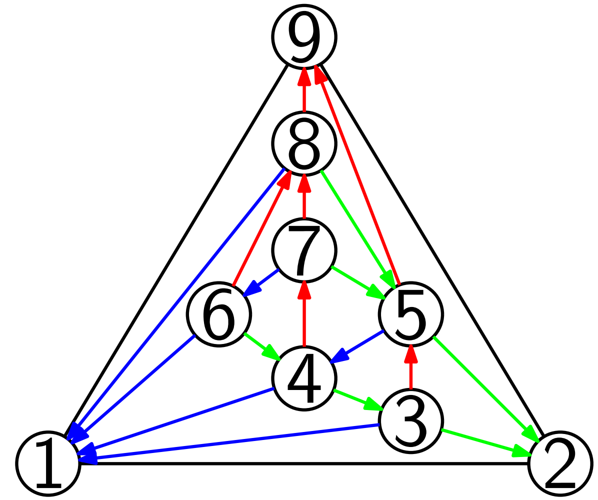

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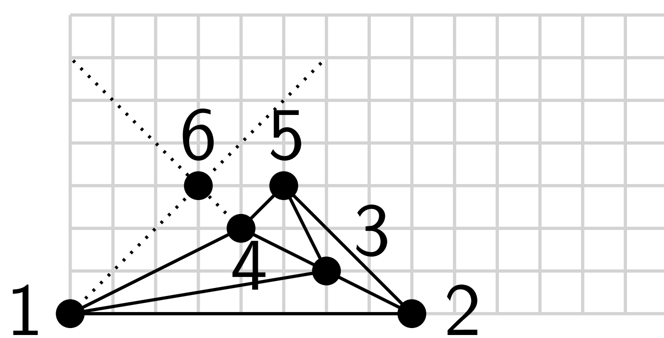

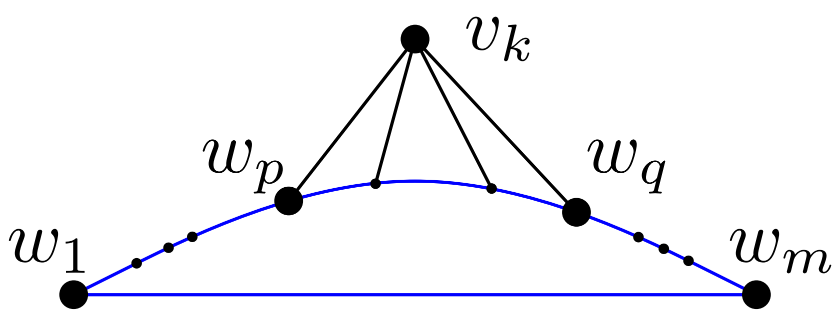

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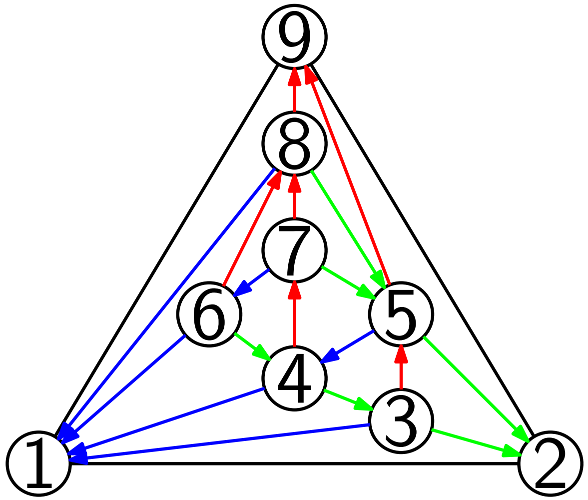

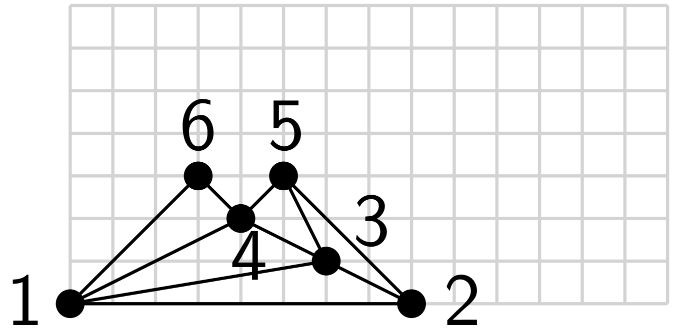

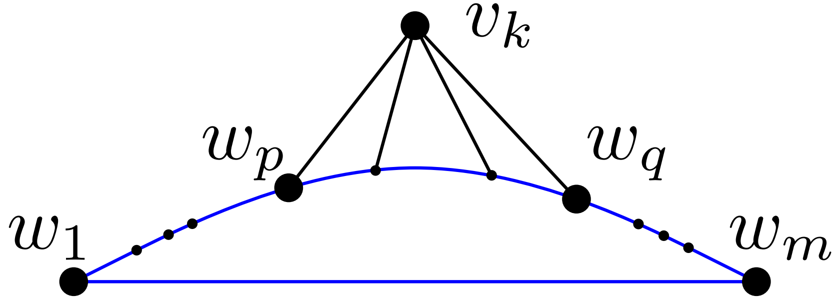

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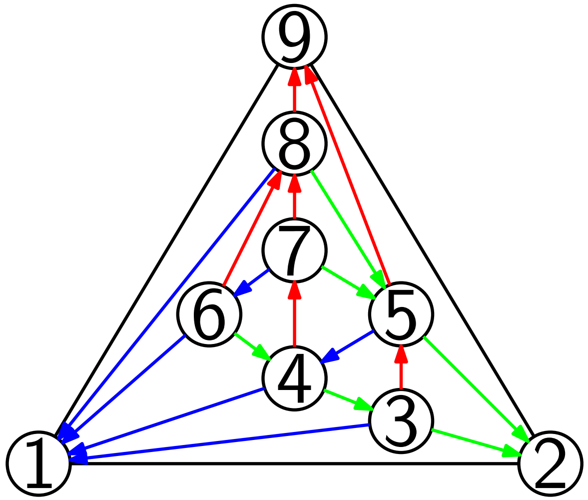
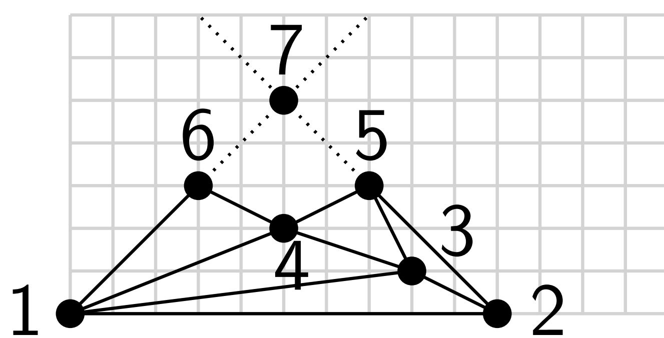
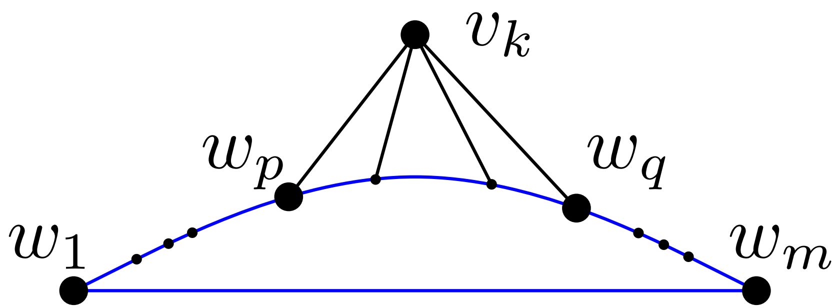
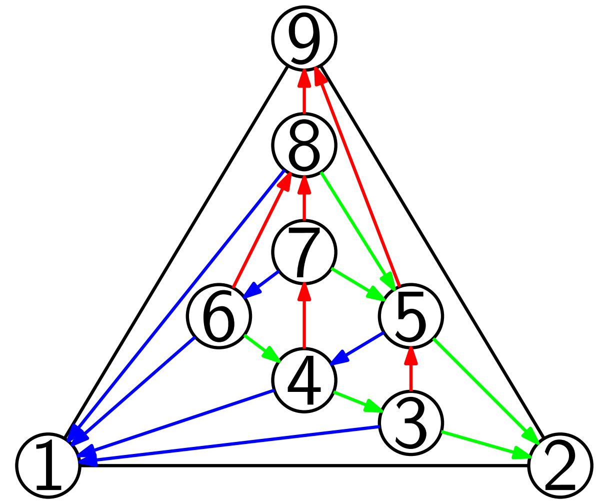




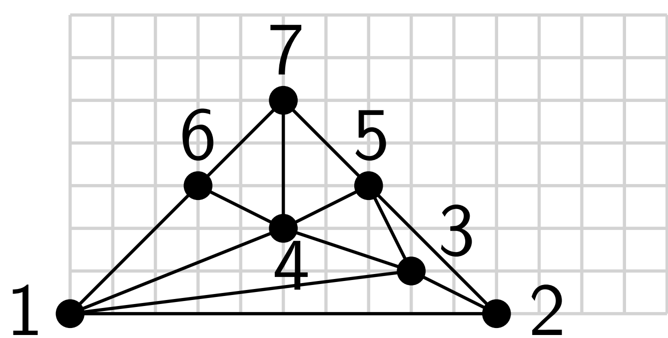


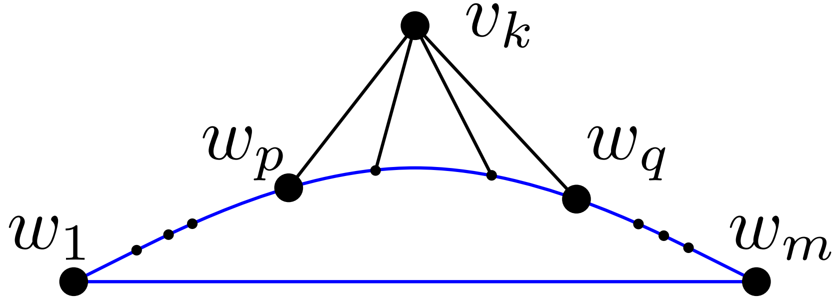


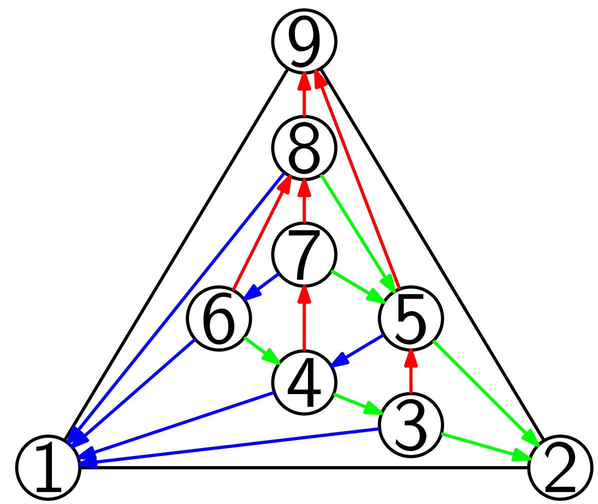
If $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ then $x(v) {+}{=}\, 1$.

If $v \in \bigcup_{i=q}^{m} L(w_i)$ then $x(v) {+}{=}\, 2$.

$L(v_k) = \{v_k\} \cup \bigcup_{i=p+1}^{q-1} L(w_i)$.



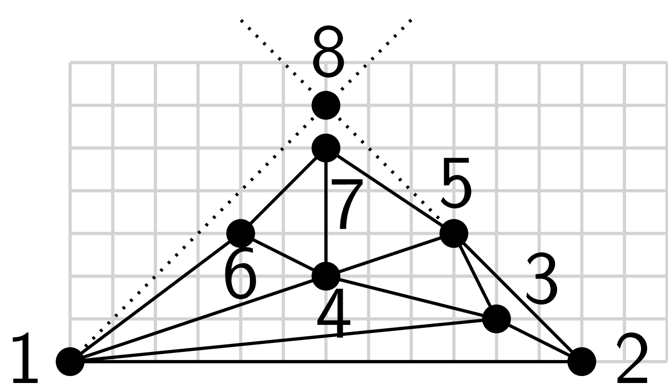


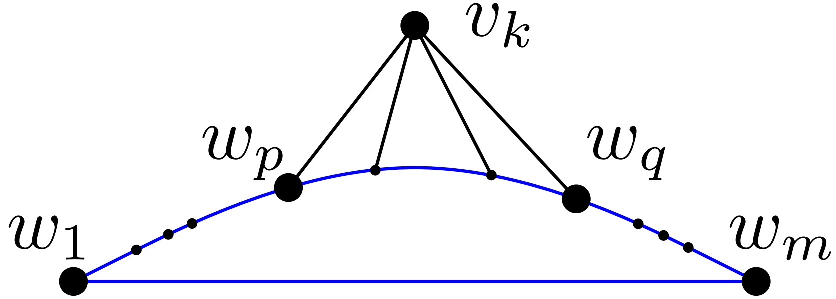


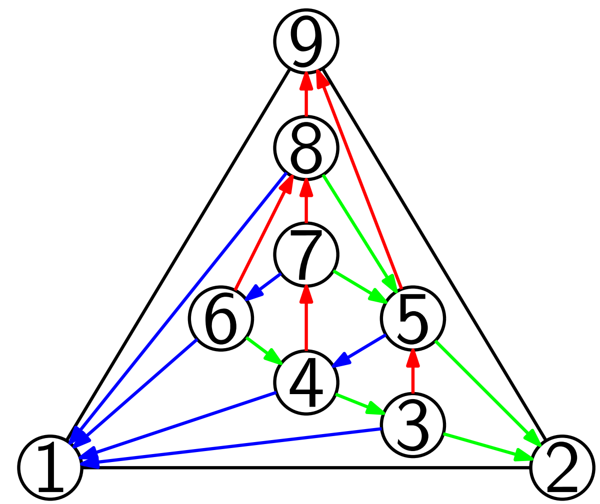
If $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ then $x(v) += 1$.

If $v \in \bigcup_{i=q}^{m} L(w_i)$ then $x(v) += 2$.

$L(v_k) = \{v_k\} \cup \bigcup_{i=p+1}^{q-1} L(w_i)$.



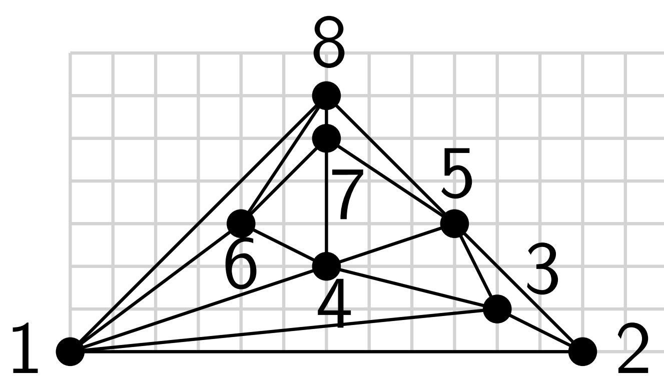


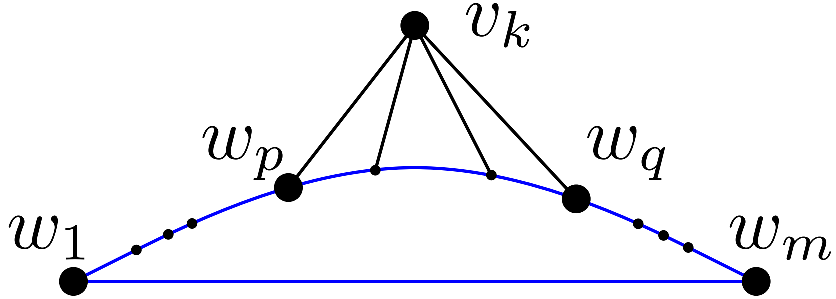


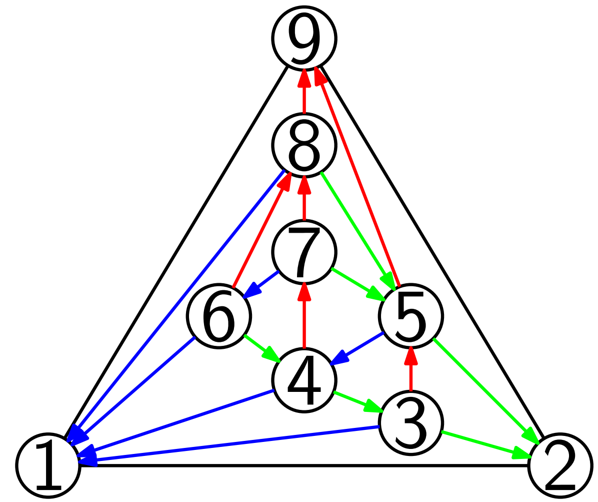
If $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ then $x(v) {+}{=} 1$.

If $v \in \bigcup_{i=q}^{m} L(w_i)$ then $x(v) {+}{=} 2$.

$L(v_k) = \{v_k\} \cup \bigcup_{i=p+1}^{q-1} L(w_i)$.



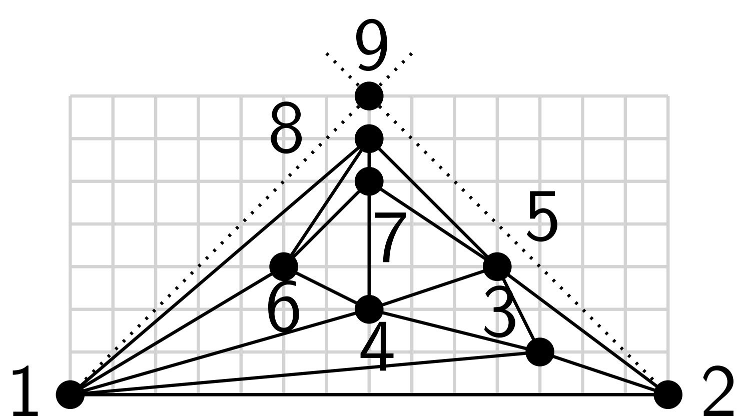


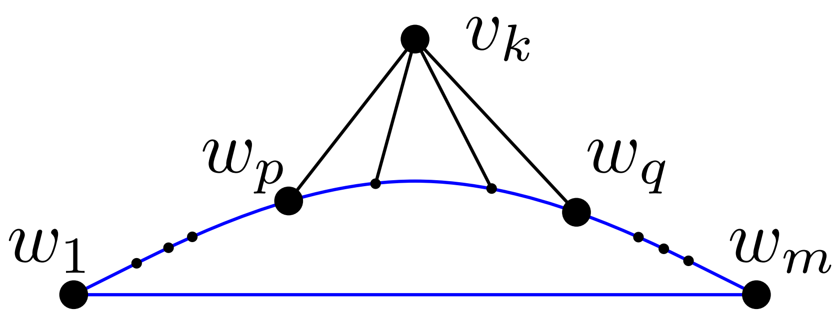


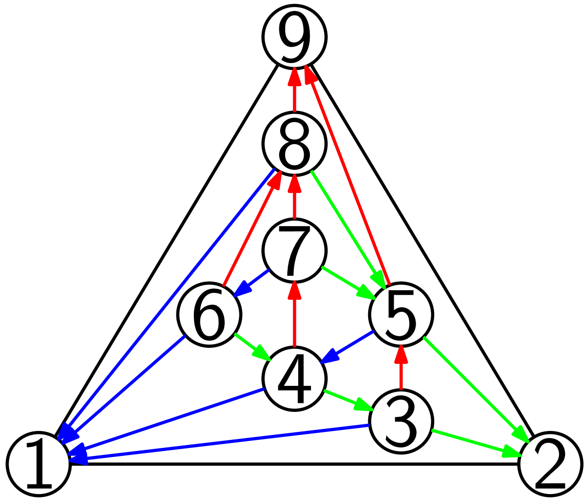
If $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ then $x(v) \mathrel{+}= 1$.

If $v \in \bigcup_{i=q}^{m} L(w_i)$ then $x(v) \mathrel{+}= 2$.

$L(v_k) = \{v_k\} \cup \bigcup_{i=p+1}^{q-1} L(w_i)$.







If $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ then $x(v) {+}{=} 1$.

If $v \in \bigcup_{i=q}^{m} L(w_i)$ then $x(v) {+}{=} 2$.

$$L(v_k) = \{v_k\} \cup \bigcup_{i=p+1}^{q-1} L(w_i).$$