(L16) Crossing Number Inequality

Crossing number of a graph G is the lowest number of edge crossings of a drawing of G in the plane.

Increasing edges beyond $3n-6$ means $cr(G) \geq 1$ (for $n \geq 3$)

In fact $cr(G) \geq e - 3n$

proof Removing one edge from each crossing results with planar graph. So $e - cr(G) \leq 3n - 6$ by Euler

Thm $cr(G) \geq \dfrac{e^3}{64n^2}$ for $e > 4n$

proof Choose a random induced subgraph H of G (and its drawing with $cr(G)$ crossings) by independently adding each vertex of G to H with probability p ← to be chosen later

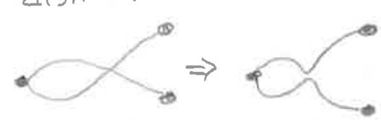
Let N_H = # vertices in H X_H = # crossings in H

E_H = # edges in H

$\boxed{X_H \geq cr(H) \geq E_H - 3N_H}$

$E[N_H] = pn$ $E[X_H] = p^4 cr(G)$ ← each crossing involves four distinct vertices

$E[E_H] = p^2 e$

else  would reduce $cr(G)$

$p^4 cr(G) \geq p^2 e - 3pn$

Let $p = 4n/e < 1$ since $e > 4n$

$cr(G) \geq e^3/64n^2$ ☒

Line Arrangement and Duality

Ham Sandwich Theorem

__Thm__ Given any two sets A and B of points in the plane there exists a line that cuts both sets in half. (special cases for odd point sets and co-linear points)

[In d-dimensions:

Any d measurable objects can be cut in half by a single (d-1) dimensional hyper plane]

__proof__ Consider the arrangement of dual lines for A and B
↑ red ↑ blue

Trace the $|A|/2$ level $\bar{A}$ in A^*
and the $|B|/2$ level $\bar{B}$ in B^* from left to right

[These are the dual points for lines that bisect A and lines that bisect B]

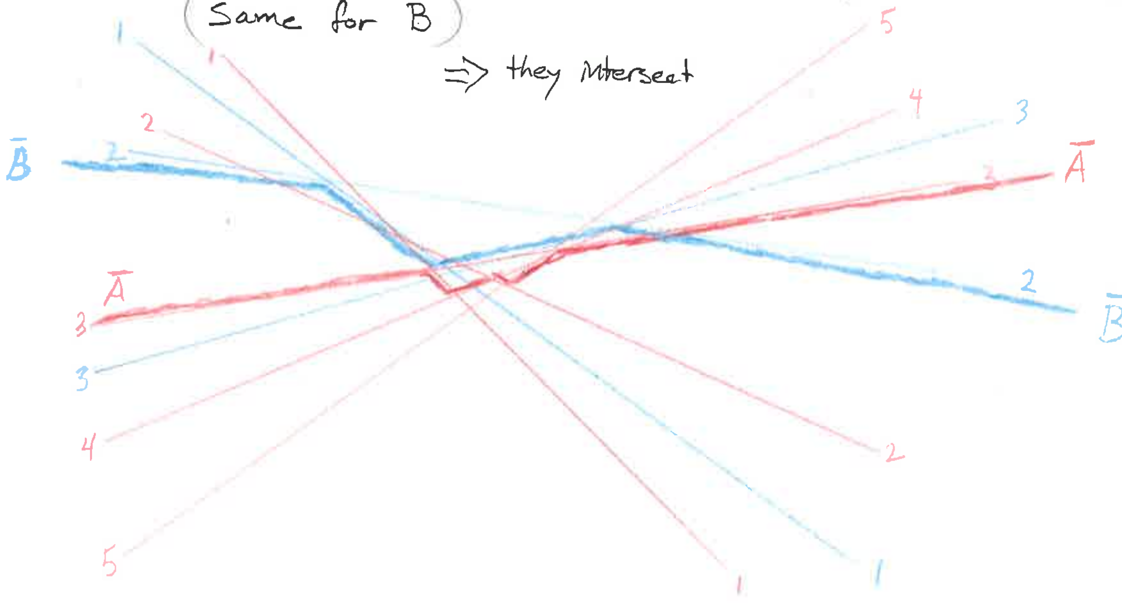
If these two levels intersect, their intersection point is the dual of a line that bisects both.

__Claim__ they do intersect.

__proof__ Every line in $A^* \cup B^*$ that is above / below $\bar{A}$ on the left is below / above $\bar{A}$ on the right

(Same for $\bar{B}$)

⇒ they intersect



Problem Given n points test if there are three that are colinear: (3 Colinear)

$\Theta(n^2)$ time solution

Seems like overkill → Build arrangement A of dual lines; check for vertex in A of degree ≥ 6

How do we solve this?

3 SUM problem Given n numbers are there three that sum to zero?

Find a $O(n^2)$ alg. for 3SUM

A problem P is 3SUM-hard if an algorithm for P that runs in subquadratic* time implies a subquadratic* time algorithm for 3SUM.

[like NP-hard: using a fast alg. for P and subquadratic* additional time we can solve 3SUM]

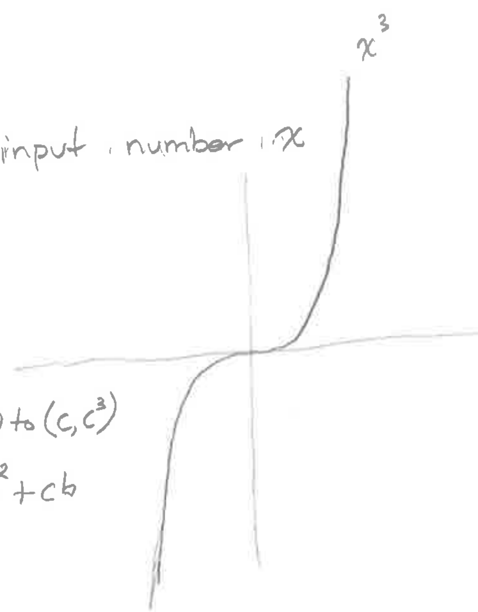
Subquadratic $\equiv o(n^2)$

Subquadratic* $\equiv O(n^{2-\Omega(1)})$

History: 3SUM was widely beleived to require $\Omega(n^2)$ time but Grønlund + Pettie gave an $O(n^2/(\log n/\log\log n)^{2/3})$ time alg. in 2014. [$O(n^2(\log\log n)^{O(1)}/\log^2 n)$ by Chen in 2018]

Thm 3 Colinear is 3SUM-hard

proof Given an input to 3SUM, for each input number x Create the point (x, x^3)



Claim $(a,a^3)\ (b,b^3)\ (c,c^3)$ are colinear iff $a+b+c=\emptyset$

Colinear $\iff$ slope (a,a^3) to (b,b^3) = slope (b,b^3) to (c,c^3)

$\iff \dfrac{b^3-a^3}{b-a} = \dfrac{c^3-b^3}{c-b} \iff b^2+a^2+ab = c^2+b^2+cb$

$\iff b(a-c) = c^2-a^2 \iff b = \dfrac{c^2-a^2}{a-c}$

$\iff b = -(a+c) \iff a+b+c=0$