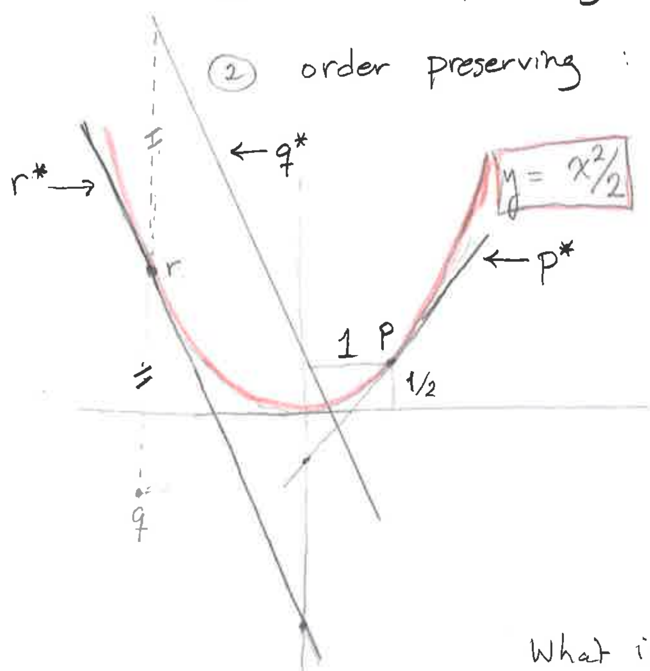
L14 Line Arrangements and Duality

Point - Line Duality

point	$\Leftrightarrow$	line
$p = (p_x, p_y)$	$\Rightarrow$	$p^*: y = p_x x - p_y$
$\ell^* = (m, -b)$	$\Leftarrow$	$\ell: y = mx + b$ (non-vertical)

For p in plane and nonvertical line ℓ in plane, duality transform $*$ is

1. incidence preserving : $p \in \ell$ iff $\ell^* \in p^*$
2. order preserving : p above ℓ iff ℓ^* above p^*

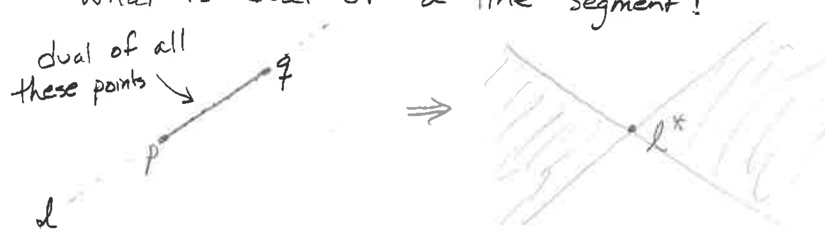


For point p on parabola $y = x^2/2$
p^* is tangent at p.

For point q not on parabola
every point with x-coord $= q_x$
has same slope as q

where is q^*?

What is dual of a line segment?

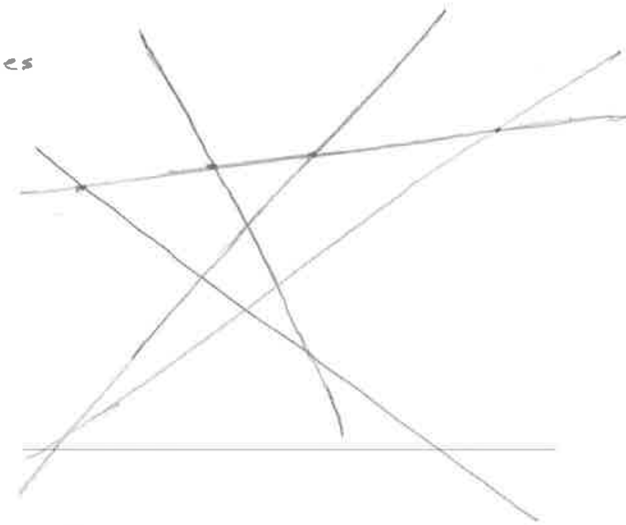


double wedge

Line arrangement

Set of n lines in the plane

edges 1d-cells
vertices 0d-cells
faces 2d-cells



Add a point at ∞ to get a planar graph.

no parallel lines
no 3 lines share a point

How many edges?

$n=1 \Rightarrow 1$
$n=2 \Rightarrow 4$

Claim 1. A simple arrangement of n lines has

$\binom{n}{2}$ vertices $\qquad \binom{n}{2}+n+1$ faces

n^2 edges

Proof
- Every pair of lines intersects in a distinct point $\Rightarrow \binom{n}{2}$ vertices
- If we add a line ℓ to an arrangement of $n-1$ lines
  - we split $n-1$ edges into two $\quad +(n-1)$
  - we introduce n new edges along ℓ $\quad +\,n$

$+2n-1$

$$\underbrace{(n-1)^2}_{\text{Ind. hyp. for } n-1} + 2n - 1 = \boxed{n^2}$$

- use Euler's formula for faces

point at ∞

$$f = 2 + e - v = 2 + n^2 - \left(\binom{n}{2} + 1\right) = \binom{n}{2} + n + 1$$

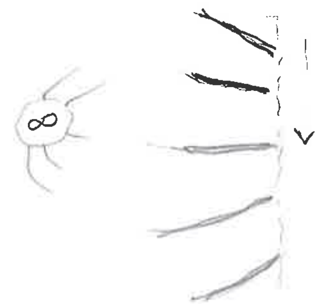
Construct a line arrangement in $O(n^2)$ time

What does this mean?

Incremental method

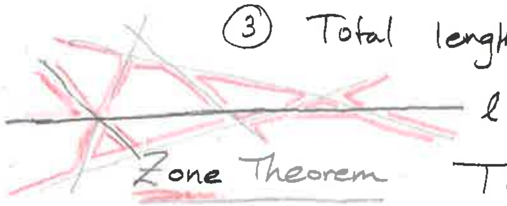
Given a $(i-1)$-line arrangement, add a new line l_i

1. Find leftmost face that contains l_i [takes $O(i)$ time.]



2. Walk boundary of this face until reach l_i's exit

3. Total length of all boundaries walked is certainly $O(i^2)$ but...



Zone Theorem The total number of edges in all faces whose closures intersect l in an arrangement of n lines is $\leq 10n$

l ← a new line

Proof Rotate the plane so that l is horizontal.

Each edge in $A(L)$ bounds two faces — arrangement of lines L

one to its left and one to its right

[for horizontal edges the face above is to its left]

So each face F has left bounding edges (those with F to right) and right bounding edges.