CPSC 516 Homework 3 Solutions

Dylan Brown

1. Given $n-1$ real numbers, $x_1 < x_2 < \ldots < x_{n-1}$, describe a linear time algorithm to test if these numbers could be the x-coordinates of the boundaries between the Voronoi regions of some set of n sites on the x-axis.

Note that there must be one vertex between each pair of midpoints, and one on each end, and that the vertex's position is only constrained by the vertices on either side of it. This allows us to perform a single sweep to determine if the constraints are met.

Algorithm. We will use a_i, b_i to represent the endpoints of the possible interval for vertex i to sit in whilst still satisfying all constraints of smaller i. Initialize $a_1 = -\infty, b_1 = x_1$ because that's the only constraint on v_1 when you only consider that vertex and the first midpoint. As a result, v_2 is only constrained to the range (x_1, x_2), so set $a_2 = x_1, b_2 = x_2$.

Now for each $i \in [3, n]$ set $a_i = x_{i-1} + |x_{i-1} - b_{i-1}|$, and $b_i = \min(x_i, x_{i-1} + |x_{i-1} - a_{i-1}|)$. Essentially this mirrors the previous interval (a_{i-1}, b_{i-1}) across the point x_{i-1}, except it cuts off any end of the interval which goes past the next midpoint x_i. This is because the position of v_i must be the same distance to x_{i-1} as v_{i-1}. If we have $a_i > b_i$ or $a_i = x_i$ or $b_i = x_{i-1}$, then we can immediately return false, as there is no good position for v_i.

If we get to the end without failing out, we clearly can return true. Simply picking $v_n = \frac{a_n + b_n}{2}$ and then cascading the spacings back through will generate a valid solution, because any point in each (a, b) interval does not break the constraints of any previous variable.

Question 2

Aryan Tajmir Riahi

(a) First assume $P \in S$ has a non-empty furthest point Voronoi diagram. This means that there is a point O which $\overline{OQ} < \overline{OP} (\forall Q \in S)$. Thus the circle centered at O and passing through P (named C) has all points of S within itself. Now define the line l as the tangent line to C at P. Clearly l is passing through P and has all other points within C (and as a result all point of S) on one side of it. Thus P is on the convex hull of S.

For the other direction assume $P \in S$ is on the convex hull of S. Therefore there is a line l passing through P that has all other points of S on one side of it. Assume l' is the ray perpendicular to l and passing through P and is on the side of l where all other point of S are. Assume $Q \in S$ has distance x_0 from l and y_0 from l'. Also assume P' is a point on l' where $\overline{PP'} = x_1$. Then $\overline{P'Q}^2 = (x_1 - x_0)^2 + y_0^2 = x_1^2 - 2x_1x_0 + x_0^2 + y_0^2$. As x_0, y_0 are independent of choice of P' as P' is getting further (x_1 is increased), at some point $2x_1x_0$ would get bigger than $x_0^2 + y_0^2$ thus $\overline{P'Q}$ will get smaller than $\overline{PP'}$ at some point and after that. As there are finite points for x_1 large enough P' will be closer to any point other than P. i.e. will be in the furthest point Voronoi region of P.

Yibo Jiao

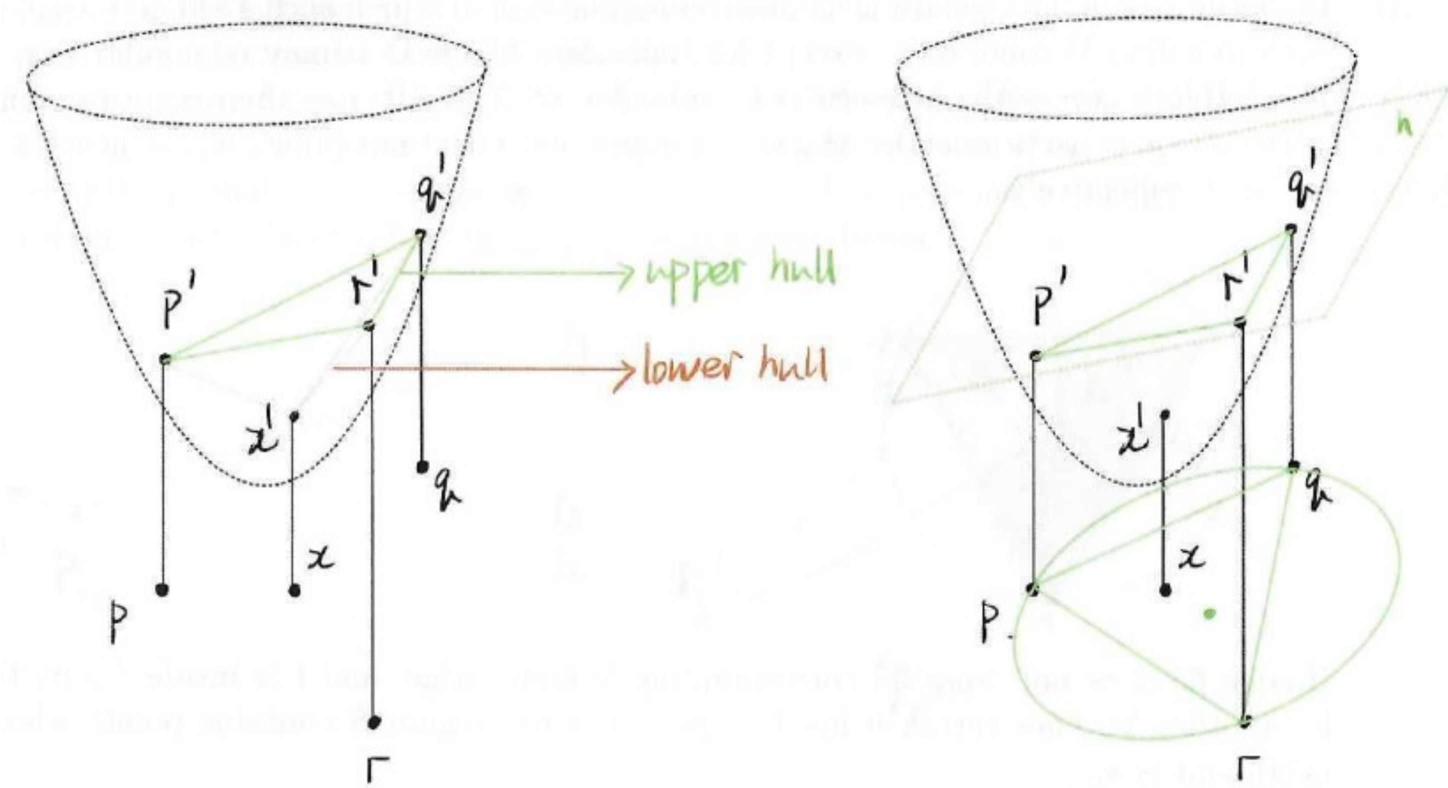
(b) *Statement:* For any farthest point Delaunay triangle with vertices $\triangle pqr$, the circle $\bigcirc pqr$ is a enclosing circle of S.

proof: A farthest point Delaunay edge connects p, q only if $\exists x, d(x,p) = d(x,q) \geq d(x,s), \forall s \in S$. Thus for a farthest Delaunay triangle $\triangle pqr$, the circle $\bigcirc pqr$, centring at x has: $d(x,p) = d(x,q) = d(x,r) \geq d(x,s), \forall s \in S$. Therefore, $\bigcirc pqr$ is a enclosing circle of S.

Statement: The triangulation of the upper convex hull of the 3D parabolic projection of S, projecting back to 2D forms a farthest point Delaunay triangulation.

proof: We prove by showing that any triangle $\triangle p'q'r'$ on the triangulation of the upper convex hull projected back to 2D $\triangle pqr$ forms a circle $\bigcirc pqr$ that encloses S. For all sites in S, project sites to parabolic $z = x^2 + y^2$, after projection $s' = (s_x, s_y, s_x^2 + s_y^2)$.

Consider points p', q', r' on a triangle of the triangulation of the upper convex hull of the projection. We prove that for any other point x in S, lies inside of $\bigcirc pqr$. Since $\triangle p'q'r'$ is a triangle on the upper convex hull, then for plane h passing through $p'q'r'$, x' is on the lower side of h. Similar to proving the Delaunay triangulation using parabolic lifting: compute the plane equation of h and the center, radius of $\bigcirc pqr$, we prove that x always lies inside of $\bigcirc pqr$. In addition, the upper convex hull does not include any points that are on the convex hull of S because such points are always lower than vertices on the convex hull on the parabolic.



Pseudo-code:

Parabolic lift S onto 3D $O(n)$

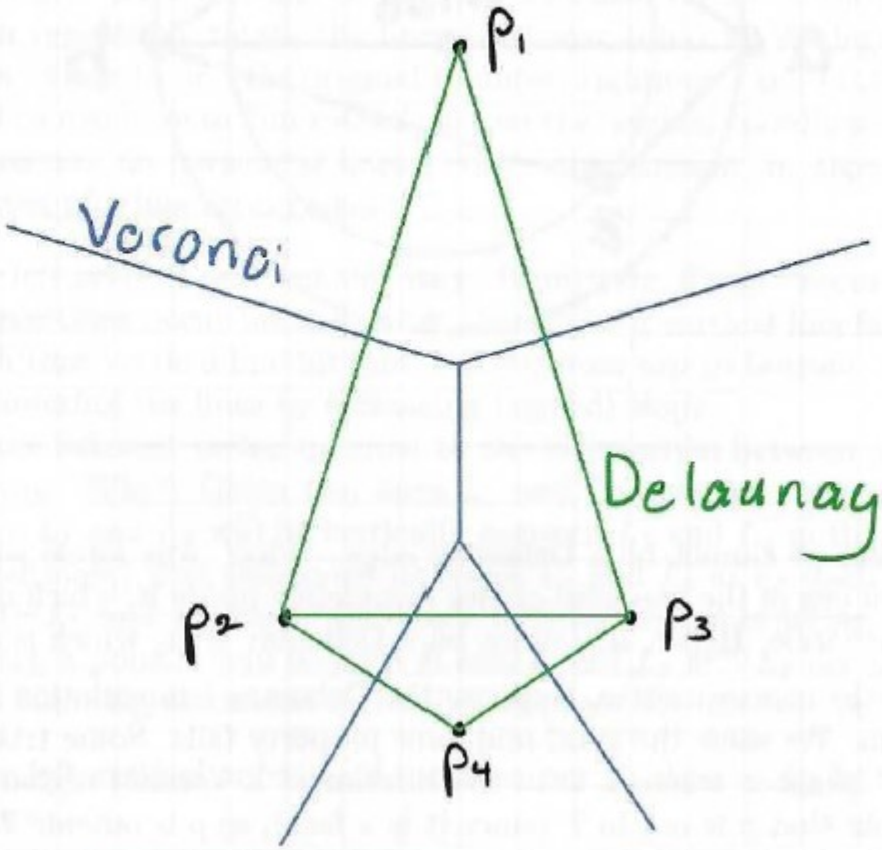
Compute 3D convex hull of the 3D polyhedron. $O(n \log n)$

Return upper hull triangulation.

Exercise 3.

Joseph Poremba

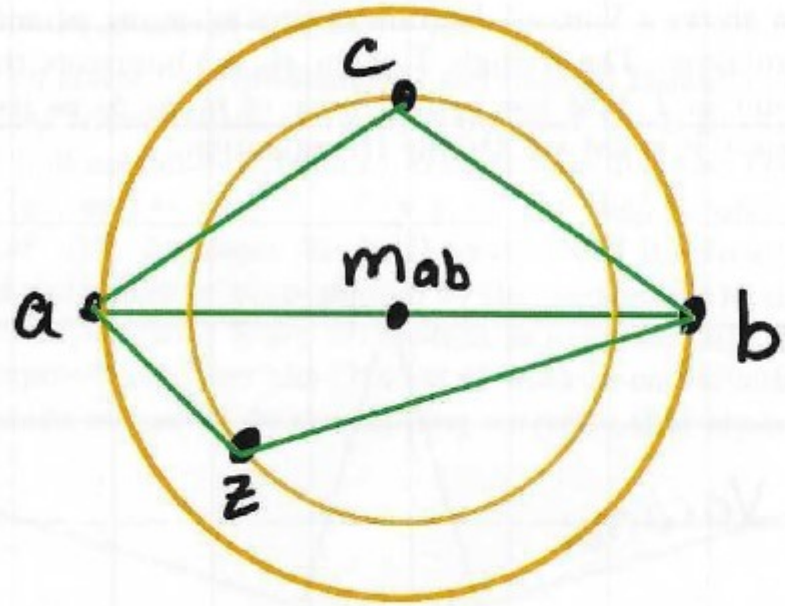
(1) The figure below shows a Voronoi diagram of sites p_1, p_2, p_3, p_4 and the corresponding Delaunay triangulation. The triangle $T = (p_1, p_2, p_3)$ intersects the interior of $V(p_4)$, so there is a point in T that has p_4 and none of p_1, p_2, p_3 as its nearest neighbour among sites. Hence, it is not a Pitteway triangulation.



(2) For a given Delaunay triangulation, for a Delaunay edge ab let m_{ab} be the midpoint of ab and let C_{ab} be the circle that has ab as a diameter chord (that is, its centre is m_{ab} and its radius is $r = d(a,b)/2 = d(m_{ab}, a) = d(m_{ab}, b)$).

Proposition 1. *A Delaunay triangulation is a Pitteway triangulation if and only of the following property holds, which we call the good-midpoint property: for any interior Delaunay edge ab, the circle C_{ab} contains no sites in its interior.*

Proof. ($\Longrightarrow$). Suppose the Delaunay triangulation is a Pitteway triangulation. We show the good-midpoint property holds. Consider an interior Delaunay edge ab, with incident triangles $T_1 = (a, b, c)$ and $T_2 = (a, b, z)$. We show C_{ab} contains no sites in its interior. Suppose it does, for the sake of contradiction. Then there is a closer site to m_{ab} than either a or b. Since the triangulation is a Pitteway triangulation, c and z must be closest sites to m_{ab}. Let $r = d(m_{ab}, a) = d(m_{ab}, b)$ and $r' = d(m_{ab}, c) = d(m_{ab}, z) < r$. Then c, z lie on the r'-radius circle C' centred at m_{ab}. This circle is split into two semi-circles by ab, and c and z lie in opposite semi-circles.



We claim that ab cannot be a Delaunay edge. Why? Any circle with ab as a chord must contain one of the two semi-circles completely inside it, which means c or z must be inside the circle. Hence, ab cannot be a Delaunay edge, which is a contradiction.

($\Longleftarrow$). By the contrapositive. Suppose the Delaunay triangulation is not a Pitteway triangulation. We show the good-midpoint property fails. Some triangle $T = (a, b, c)$ has a "bad" point x where x is in the interior of a Voronoi region $V(p)$ where $p \notin \{a, b, c\}$. Note that p is not in T (since it is a face), so p is outside T.

First, we claim there is a boundary point of T in the interior of $V(p)$. If x is on the boundary we are done. If not then x is in the interior of T. Since p is outside of T but $x \in V(p)$, this Voronoi region (by convexity) crosses the boundary of T. Hence, T contains a boundary point y that is in the interior of $V(p)$. Without loss of generality, say y is on the Delaunay edge ab.

First, observe that ab is an interior Delaunay edge, since it is inside the convex hull of a, c, b, p and so is in the interior of the convex hull of sites. We claim that m_{ab} is strictly closer to p than a or b, failing the good-midpoint property. Recall, that y is strictly closer to p than a, and also $d(m_{ab}, y) + d(y, a) = d(m_{ab}, a)$ since y is on ab. By the triangle inequality,

suppose wlog that y is between a and x.

$$d(m_{ab}, p) \leq d(m_{ab}, y) + d(y, p) < d(m_{ab}, y) + d(y, a) = d(m_{ab}, a) = d(m_{ab}, b).$$

Hence, m_{ab} is strictly closer to p than a or b.

□

Camilo Talero

4: Bounding rectangle

(CGAA Exercise 8.4) Let L be a set of n lines in the plane. Give an $O(n \log n)$ time algorithm to compute an axis-parallel rectangle that contains all the vertices of the arrangement of L in its interior.

Given 2 lines $ax + b, cx + d$ with $a < c$ as x tends to infinity, then eventually $cx + d > ax + b$ for all large enough x.

More generally for a set of lines, for large enough x the line with largest slope will upper bound all other lines, followed by the line with second largest slope and so on...

Thus we can assign an order to the points based on the slope, similar to what was done in class on Nov1.

The rightmost intersection between two lines must occur between 2 consecutive lines in this ordering. In other words the rightmost intersection occurs between some l_i and l_{i+1} where they are the i^{th} and $(i + 1)^{th}$ steepest lines in the set.

Proof:

Up to a translation, the rightmost point is at $x = 0$ and thus the equations of the 2 lines intersecting at that point are ax, bx for some $a < b \in \mathbb{R}$. Assume there is some α, β such that $\alpha x + \beta$ is a line in L with $a < \alpha < b$. The intersection $ax = \alpha x + \beta$ occurs at $x = -\frac{\beta}{\alpha - a}$, the denominator is positive, so β has to be positive, otherwise it would be to the right of the rightmost point. The intersection $bx = \alpha x + \beta$ occurs at $x = -\frac{\beta}{\alpha - b}$. The denominator is negative so β has to be negative, otherwise the intersection occurs to the right of the rightmost point. So the only possible choice for β is 0. So we get that either $\alpha x = ax$ or $\alpha x = bx$. So clearly there cannot be any lines with slopes between a, b.

Thus grab all lines an sort them by slope $O(n \log n)$. Compute the pair wise intersections of all consecutive lines $O(n)$. Find the point with largest x. A vertical line passing through this point is, by construction to the right of all points in the arrangement.

Rotate the line set by 90 degrees $O(n)$ repeat the above $O(n \log n)$. We have found a line such that all points in the current arrangement are to the left of this line, which is equivalent to saying all points are below the line in the original arrangement.

Do so two more times $O(n \log n)$. We have found 4 lines that bound the set. Compute their intersection $O(1)$ and rotate the 4 resulting intersection points by 90 degrees again. The rectangle defined by the 4 resulting points is, by construction, the axis aligned bounding rectangle of the vertices in the arrangement.

The algorithm is dominated by the 4 sorts of the lines, so it's $O(n \log n)$.